Sensitivity Analysis in Multiconductor Transmission Line Networks

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Abstract: - The paper deals with a technique of sensitivity calculation in linear hybrid networks containing elements with both lumped and distributed parameters when the multiconductor transmission lines (MTL) are considered as the distributed parts. The technique utilizes the modified nodal analysis (MNA) method to do overall description of the network enabling determining sensitivities with respect to both lumped and distributed parameters. As the first step the frequency–domain sensitivities formulae are derived then the numerical inversion of Laplace transforms (NILT) is applied to get the time-domain sensitivities. In this paper an innovative method for the calculation of the sensitivities just with respect to distributed MTL parameters is presented.

Key-Words: - Sensitivity analysis, Hybrid network, Multiconductor transmission line, Modified nodal analysis, Chain matrix, Frequency–domain sensitivity, Time-domain sensitivity, Laplace transform, Numerical inversion

1 Introduction

The sensitivity analysis of transmission line structures plays the important role in the solution of signal integrity problems in modern mixed digital/analog systems. It can concern simulation, modeling and optimization of high-speed circuit boards, integrated circuits interconnects, computer buses performance etc. [1].

The paper deals with the technique of time-domain sensitivity calculation in the hybrid networks containing elements with both lumped and distributed parameters. Herein the solution is restricted to linear networks that contain uniform lossy multiconductor transmission lines (MTL) as the distributed-parameter parts. The procedure utilizes the modified nodal analysis (MNA) equation method to perform an overall network description in the frequency domain [2,3]. Unlike a modal analysis method describing behaviour of the MTL parts [2], the MTLs' chain matrices are used in this paper [3]. The advantage of such approach lies not only in easier and more compact matrix-form description but mainly also in the possibility to extend the solution against nonuniform MTLs if necessary. In principle the sensitivity analysis can be performed with respect to parameters of lumped elements, the primary parameters of MTLs, and also the MTLs' physical parameters. The paper brings a new approach just into the solution of last two cases.

To obtain the time-domain sensitivities from those in frequency domain the procedure of numerical inversion of Laplace tarnsforms is finally applied. For this purpose the method based on the FFT and quotient-difference algorithm [4] is used. The complete solution is carried out in Matlab language environment that is very suitable just at solving matrix-described problems numerically.

2 Problem Formulation

Consider a linear network containing a section with lumped-parameter elements and P distributed–parametr subnetworks that are formed by lossy multiconductor transmission lines, see Fig.1.

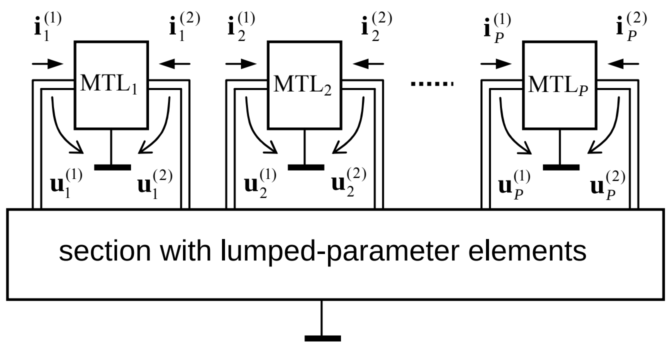

Fig.1 Linear network containing MTL subnetworks

The aim is to formulate equations enabling calculation of sensitivities with respect to parameters both lumped and distributed elements. In this paper we will suppose just uniform MTLs and zero initial voltage and current distributions along MTLs' wires, even if the method can further be generalized.

3 Network MNA Matrix Equation

The equations describing considered network in the time domain can be formulated using the modified nodal analysis equation method [2,3]

$$\mathbf{C}_M \frac{d\mathbf{v}_M(t)}{dt} + \mathbf{G}_M \mathbf{v}_M(t) + \sum_{k=1}^{P} \mathbf{D}_k \mathbf{i}_k(t) = \mathbf{i}_M(t) , \qquad (1)$$

where $\mathbf{C}_M$ and $\mathbf{G}_M$ are $N \times N$ constant matrices with entries determined by the lumped memory and memoryless components, respectively, $\mathbf{v}_M(t)$ is the $N \times 1$ vector of node voltages appended by currents of independent voltage sources and inductors, $\mathbf{i}_M(t)$ is the $N \times 1$ vector of source waveforms, $\mathbf{i}_k(t)$ is the $n_k \times 1$ vector of currents entering the *k*-th MTL, and $\mathbf{D}_k$ is the $N \times n_k$ selector matrix with entries $d_{i,j} \in \{0,1\}$ mapping the vector $\mathbf{i}_k(t)$ into the node space of the network. Applying Laplace transformation the frequency-domain MNA equation has the form

$$[\mathbf{G}_M + s\mathbf{C}_M]\mathbf{V}_M(s) + \sum_{k=1}^{P} \mathbf{D}_k \mathbf{I}_k(s) = \mathbf{I}_M(s) + \mathbf{C}_M \mathbf{v}_M(0). \quad (2)$$

The MTLs consist of $m_k = n_k/2$ active conductors and can be regarded as $2m_k$ –ports. Then the $\mathbf{I}_k(s)$ in (2) is assembled to contain vectors of currents entering input and output ports as $\mathbf{I}_k(s) = [\mathbf{I}_k^{(1)}(s), \mathbf{I}_k^{(2)}(s)]^T$. Supposing only MTLs zero initial conditions then the admittance equation of *k–th* MTL_k can be formulated as

$$\mathbf{I}_k(s) = \mathbf{Y}_k(s)\mathbf{V}_k(s), \quad (3)$$

where $\mathbf{V}_k(s) = [\mathbf{V}_k^{(1)}(s), \mathbf{V}_k^{(2)}(s)]^T$ is assembled to contain vectors of voltages occuring on input and output ports.

After substituting (3) into (2) the resultant MNA matrix equation is of the form

$$\mathbf{V}_M(s) = \mathbf{Y}_M^{-1}(s)\left[\mathbf{I}_M(s) + \mathbf{C}_M \mathbf{v}_M(0)\right], \quad (4)$$

where

$$\mathbf{Y}_M(s) = \mathbf{G}_M + s\mathbf{C}_M + \sum_{k=1}^{P} \mathbf{D}_k \mathbf{Y}_k(s) \mathbf{D}_k^T. \quad (5)$$

Herein the s–domain solution is prepared making for the derivation of sensitivities in the frequency domain.

4 Frequency-Domain Sensitivity

Let us consider a parameter γ that a sensitivity will be considered with respect to. Further think of the (4) in the form

$$\mathbf{Y}_M(s)\mathbf{V}_M(s) = \mathbf{I}_M(s) + \mathbf{C}_M \mathbf{v}_M(0) \quad (6)$$

and perform the differentiation w. r. to γ. We get

$$\frac{\partial \mathbf{Y}_M(s)}{\partial \gamma}\mathbf{V}_M(s) + \mathbf{Y}_M(s)\frac{\partial \mathbf{V}_M(s)}{\partial \gamma} = \frac{\partial \mathbf{C}_M}{\partial \gamma}\mathbf{v}_M(0), \quad (7)$$

where $\partial \mathbf{I}_M(s)/\partial \gamma = \mathbf{0}$ and $\partial \mathbf{v}_M(0)/\partial \gamma = \mathbf{0}$ were taken into account as neither sensitivities w. r. to the values of independent current/voltage sources or initial conditions of memory elements are considered here. From (7) we can write

$$\frac{\partial \mathbf{V}_M(s)}{\partial \gamma} = \mathbf{Y}_M^{-1}(s)\left(\frac{\partial \mathbf{C}_M}{\partial \gamma}\mathbf{v}_M(0) - \frac{\partial \mathbf{Y}_M(s)}{\partial \gamma}\mathbf{V}_M(s)\right). \quad (8)$$

Now we will split the solution into three different cases.

4.1 Lumped Parameter Sensitivity

Let γ is a lumped parameter of some element of the network. It means that it is certainly contained in $\mathbf{C}_M$ or $\mathbf{G}_M$ matrix according to the element type. Taking into account (5) we can generally write

$$\frac{\partial \mathbf{V}_M(s)}{\partial \gamma} = \mathbf{Y}_M^{-1}(s)\left(\frac{\partial \mathbf{C}_M}{\partial \gamma}\left(\mathbf{v}_M(0) - s\mathbf{V}_M(s)\right) - \frac{\partial \mathbf{G}_M(s)}{\partial \gamma}\mathbf{V}_M(s)\right). \quad (9)$$

In case of $\gamma \equiv c_M$ as a memory-element parameter, then

$$\frac{\partial \mathbf{V}_M(s)}{\partial c_M} = \mathbf{Y}_M^{-1}(s)\frac{\partial \mathbf{C}_M}{\partial c_M}\left(\mathbf{v}_M(0) - s\mathbf{V}_M(s)\right), \quad (10)$$

while for $\gamma \equiv g_M$ as a memoryless-element one, then

$$\frac{\partial \mathbf{V}_M(s)}{\partial g_M} = -\mathbf{Y}_M^{-1}(s)\frac{\partial \mathbf{G}_M(s)}{\partial g_M}\mathbf{V}_M(s). \quad (11)$$

As is pointed out in [2] if the element is connected between nodes *i* and *j*, the matrix derivatives in (10) or (11) are replaced by the matrix $(\mathbf{e}_i - \mathbf{e}_j) \cdot (\mathbf{e}_i - \mathbf{e}_j)^T$, with $\mathbf{e}_i$ defined as $N \times 1$ column vector with the value 1 at *i*-th position, and with zeros elsewhere. If *j* denotes the reference node then the respective matrix is $\mathbf{e}_i \cdot \mathbf{e}_i^T$.

4.2 MTL Parameter Sensitivity

Let us consider γ to be some parameter of the *k*-th MTL_k such as a component of per-unit-length matrices $\mathbf{R}_0$, $\mathbf{L}_0$, $\mathbf{G}_0$ and $\mathbf{C}_0$ (or some geometrical parameter affecting these matrices) or the MTL's length *l*. Such parameter γ is implicated inside the MTL admittance matrix $\mathbf{Y}_k(s)$ in (5). Then coming from (8) and taking (5) into account we can write

$$\frac{\partial \mathbf{V}_M(s)}{\partial \gamma} = -\mathbf{Y}_M^{-1}(s)\mathbf{D}_k\frac{\partial \mathbf{Y}_k(s)}{\partial \gamma}\mathbf{D}_k^T\mathbf{V}_M(s), \quad (12)$$

because all the $\partial \mathbf{Y}_j(s)/\partial \gamma = \mathbf{0}$, $j \neq k$.

To compute the sensitivity (12) it is necessary to find the MTL admittance matrix derivatives $\partial \mathbf{Y}_k(s)/\partial \gamma$. The approach shown herein is completely different from that usually used, see e.g. [2]. Namely, instead of the modal analysis technique that requires solving the system of linear equations to find primarily the eigenvectors and eigenvalues sensitivities, the straightforward technique based on the chain/admittance matrix conversion is used in this paper. Firstly we will show how to perform this conversion going out the basic MTL matrix equations.

4.2.1 MTL Matrix Equation Formulation

Suppose a uniform MTL of the length l, with per-unit-length matrices $\mathbf{R}_0$, $\mathbf{L}_0$, $\mathbf{G}_0$ and $\mathbf{C}_0$. In the time–domain the basic MTL matrix equation is of the form [3]

$$\frac{\partial}{\partial x}\begin{bmatrix}\mathbf{v}(x,t)\\ \mathbf{i}(x,t)\end{bmatrix}=\begin{bmatrix}\mathbf{0} & \mathbf{-R}_0\\ \mathbf{-G}_0 & \mathbf{0}\end{bmatrix}\cdot\begin{bmatrix}\mathbf{v}(x,t)\\ \mathbf{i}(x,t)\end{bmatrix}-\begin{bmatrix}\mathbf{0} & \mathbf{L}_0\\ \mathbf{C}_0 & \mathbf{0}\end{bmatrix}\cdot\frac{\partial}{\partial t}\begin{bmatrix}\mathbf{v}(x,t)\\ \mathbf{i}(x,t)\end{bmatrix}. \quad (13)$$

After finishing Laplace transformation and considering zero initial voltage and current distributions along the MTL's wires the equation (13) results in

$$\frac{d}{dx}\begin{bmatrix}\mathbf{V}(x,s)\\ \mathbf{I}(x,s)\end{bmatrix}=\begin{bmatrix}\mathbf{0} & \mathbf{-Z}_0(s)\\ \mathbf{-Y}_0(s) & \mathbf{0}\end{bmatrix}\cdot\begin{bmatrix}\mathbf{V}(x,s)\\ \mathbf{I}(x,s)\end{bmatrix}, \quad (14)$$

where $\mathbf{V}(x,s)=\mathsf{L}\left[\mathbf{v}(x,t)\right]$ and $\mathbf{I}(x,s)=\mathsf{L}\left[\mathbf{i}(x,t)\right]$ are the column vectors of Laplace transforms of voltages and currents at distance x from MTL input $^{(1)}$, respectively, and $\mathbf{0}$ is the zero matrix. Further

$$\mathbf{Z}_0(s)=\mathbf{R}_0+s\mathbf{L}_0 \ , \quad (15)$$

$$\mathbf{Y}_0(s)=\mathbf{G}_0+s\mathbf{C}_0 \quad (16)$$

are series impedance and shunting admittance matrices, respectively. To simplify further solution the (14) will be rewritten into the symbolic form

$$\frac{d}{dx}\mathbf{W}(x,s)=\mathbf{M}(s)\mathbf{W}(x,s)\,. \quad (17)$$

Considering now $\mathbf{W}(0,s)$ as a solution at $x=0$ (MTL's input) the solution at x coordinate can be written as

$$\mathbf{W}(x,s)=\mathbf{\Phi}(x,s)\mathbf{W}(0,s) \ , \quad (18)$$

with $\mathbf{\Phi}(x,s)$ as the integral matrix defined in case of the uniform MTL ($\mathbf{M}(s)\neq f(x)$) by the matrix exponential function

$$\mathbf{\Phi}(x,s)=e^{\mathbf{M}(s)\cdot x} \ . \quad (19)$$

When the coordinate $x=l$ (MTL's output) is considered the integral matrix $\mathbf{\Phi}(l,s)$ acts as the MTL chain matrix $\mathbf{\Phi}(s)$ in terms of multiport theory. We have

$$\mathbf{\Phi}(s)=e^{\mathbf{M}(s)l}=\begin{bmatrix}\mathbf{\Phi}_{11}(s) & \mathbf{\Phi}_{12}(s)\\ \mathbf{\Phi}_{21}(s) & \mathbf{\Phi}_{22}(s)\end{bmatrix}. \quad (20)$$

Because of the MTL reciprocity $det\mathbf{\Phi}(s)=1$ holds valid, and its homogeneity leads to additional identity between its submatrices as

$$\mathbf{\Phi}_{22}(s)=\mathbf{\Phi}_{11}^T(s) \ . \quad (21)$$

Thus after denoting

$$\mathbf{W}(0,s)=\mathbf{W}^{(1)}(s)=[\mathbf{V}^{(1)}(s),\mathbf{I}^{(1)}(s)]^T\,, \quad (22)$$

$$\mathbf{W}(l,s)=\mathbf{W}^{(2)}(s)=[\mathbf{V}^{(2)}(s),-\mathbf{I}^{(2)}(s)]^T\,, \quad (23)$$

the considered MTL can be described by the equation

$$\begin{bmatrix}\mathbf{V}^{(2)}(s)\\ -\mathbf{I}^{(2)}(s)\end{bmatrix}=\begin{bmatrix}\mathbf{\Phi}_{11}(s) & \mathbf{\Phi}_{12}(s)\\ \mathbf{\Phi}_{21}(s) & \mathbf{\Phi}_{11}^T(s)\end{bmatrix}\cdot\begin{bmatrix}\mathbf{V}^{(1)}(s)\\ \mathbf{I}^{(1)}(s)\end{bmatrix}. \quad (24)$$

As regards the MTL admittance matrix

$$\mathbf{Y}(s)=\begin{bmatrix}\mathbf{Y}_{11}(s) & \mathbf{Y}_{12}(s)\\ \mathbf{Y}_{21}(s) & \mathbf{Y}_{22}(s)\end{bmatrix}, \quad (25)$$

the reciprocity results in the equality

$$\mathbf{Y}_{21}(s)=\mathbf{Y}_{12}^T(s) \ . \quad (26)$$

Furthermore due to the MTL homogeneity the additional identities for its submatrices are valid as

$$\mathbf{Y}_{12}^T(s)=\mathbf{Y}_{12}(s)\,, \quad (27)$$

$$\mathbf{Y}_{22}(s)=\mathbf{Y}_{11}(s) \ . \quad (28)$$

Therefore the decomposed MTL admittance equation (3) can be expressed (when the index k is omitted) by

$$\begin{bmatrix}\mathbf{I}^{(1)}(s)\\ \mathbf{I}^{(2)}(s)\end{bmatrix}=\begin{bmatrix}\mathbf{Y}_{11}(s) & \mathbf{Y}_{12}(s)\\ \mathbf{Y}_{12}(s) & \mathbf{Y}_{11}(s)\end{bmatrix}\cdot\begin{bmatrix}\mathbf{V}^{(1)}(s)\\ \mathbf{V}^{(2)}(s)\end{bmatrix}. \quad (29)$$

Now the conversion formulae between $\mathbf{\Phi}(s)$ and $\mathbf{Y}(s)$ will be used to help in the finding $\partial\mathbf{Y}(s)/\partial\gamma$ derivative. Following multiport theory and equations (24) to (29) then after simple matrix manipulations the admittance matrix $\mathbf{Y}(s)$ can be expressed by way of

$$\mathbf{Y}(s)=\begin{bmatrix}-\mathbf{\Phi}_{12}^{-1}(s)\mathbf{\Phi}_{11}(s) & \mathbf{\Phi}_{12}^{-1}(s)\\ \mathbf{\Phi}_{12}^{-1}(s) & -\mathbf{\Phi}_{12}^{-1}(s)\mathbf{\Phi}_{11}(s)\end{bmatrix}. \quad (30)$$

Its derivative $\partial\mathbf{Y}(s)/\partial\gamma$ can be computed by assembling derivatives of particular submatrices, that are

$$\frac{\partial\mathbf{Y}_{11}(s)}{\partial\gamma}=-\frac{\partial\mathbf{\Phi}_{12}^{-1}(s)}{\partial\gamma}\mathbf{\Phi}_{11}(s)-\mathbf{\Phi}_{12}^{-1}(s)\frac{\partial\mathbf{\Phi}_{11}(s)}{\partial\gamma}, \quad (31)$$

$$\frac{\partial\mathbf{Y}_{12}(s)}{\partial\gamma}=\frac{\partial\mathbf{\Phi}_{12}^{-1}(s)}{\partial\gamma}\,, \quad (32)$$

and where the inverse matrix differentiation is made up by a rule

$$\frac{\partial\mathbf{\Phi}_{12}^{-1}(s)}{\partial\gamma}=-\mathbf{\Phi}_{12}^{-1}(s)\frac{\partial\mathbf{\Phi}_{12}(s)}{\partial\gamma}\mathbf{\Phi}_{12}^{-1}(s) \ . \quad (33)$$

As is obvious we have to state a chain matrix derivative

$$\frac{\partial\mathbf{\Phi}(s)}{\partial\gamma}=\begin{bmatrix}\dfrac{\partial\mathbf{\Phi}_{11}(s)}{\partial\gamma} & \dfrac{\partial\mathbf{\Phi}_{12}(s)}{\partial\gamma}\\ \dfrac{\partial\mathbf{\Phi}_{21}(s)}{\partial\gamma} & \dfrac{\partial\mathbf{\Phi}_{11}^{\mathrm{T}}(s)}{\partial\gamma}\end{bmatrix}=\frac{\partial}{\partial\gamma}e^{\mathbf{M}(s)l} \ . \quad (34)$$

Further solution will be split into two different cases.

4.2.2 MTL Length Sensitivity

Let us suppose $\gamma \equiv l$. Because the MTL's length l does not affect any per-unit-length matrices, the matrix $\mathbf{M}(s)$ is a constant matrix. Therefore (34) results in a standard matrix exponential derivative as

$$\frac{\partial \mathbf{\Phi}(s)}{\partial l} = \mathbf{M}(s)\mathbf{\Phi}(s) = \mathbf{\Phi}(s)\mathbf{M}(s) . \qquad (35)$$

The matrix product (35) is commutative. Substituting for $\mathbf{M}(s)$, see (14) and (17), and performing necessary multiplications, we can write

$$\frac{\partial \mathbf{\Phi}(s)}{\partial l} = -\begin{bmatrix} \mathbf{\Phi}_{12}(s)\mathbf{Y}_0(s) & \mathbf{\Phi}_{11}(s)\mathbf{Z}_0(s) \\ \mathbf{\Phi}_{11}^T(s)\mathbf{Y}_0(s) & \mathbf{\Phi}_{21}(s)\mathbf{Z}_0(s) \end{bmatrix}. \qquad (36)$$

Comparing (36) and (34) we know respective derivatives which are needed in (31) to (33). After finishing the substitutions and arrangements, and taking into account both the commutative property of (35) and also the (30), the derivatives (31) and (32) result in

$$\frac{\partial \mathbf{Y}_{11}(s)}{\partial l} = -\mathbf{Y}_{12}(s)\mathbf{Z}_0(s)\mathbf{Y}_{12}(s) , \qquad (37)$$

$$\frac{\partial \mathbf{Y}_{12}(s)}{\partial l} = -\mathbf{Y}_{11}(s)\mathbf{Z}_0(s)\mathbf{Y}_{12}(s) \ . \qquad (38)$$

We have just got the admittance matrix derivative

$$\frac{\partial \mathbf{Y}_k(s)}{\partial l} = -\begin{bmatrix} \mathbf{Y}_{12}(s)\mathbf{Z}_0(s)\mathbf{Y}_{12}(s) & \mathbf{Y}_{11}(s)\mathbf{Z}_0(s)\mathbf{Y}_{12}(s) \\ \mathbf{Y}_{11}(s)\mathbf{Z}_0(s)\mathbf{Y}_{12}(s) & \mathbf{Y}_{12}(s)\mathbf{Z}_0(s)\mathbf{Y}_{12}(s) \end{bmatrix}, \qquad (39)$$

that is needed for $\partial \mathbf{V}_M(s)/\partial l$ sensitivity, see (12).

4.2.3 MTL Primary Parameter Sensitivity

The primary parameters are given electrical parameters of the MTL, and the per-unit-lenght matrices define the matrix $\mathbf{M}(s)$, see (14) to (17). Therefore, the derivative (34) must first be performed with respect to the matrix $\mathbf{M}(s)$, while l is a constant. The procedure is more complicated compared to (35). The way how to do it can be based on the infinite series expansion of the matrix exponential function. Because of

$$e^{\mathbf{M}(s)l} = \sum_{k=0}^{\infty} \frac{l^k}{k!}\mathbf{M}^k(s) \ , \qquad (40)$$

we can proceed as

$$\frac{\partial \mathbf{\Phi}(\mathrm{s})}{\partial \gamma} = \sum_{k=0}^{\infty} \frac{l^k}{k!}\frac{\partial \mathbf{M}^k(s)}{\partial \gamma} . \qquad (41)$$

However, neither the above derivatives can be done by rules for non-matrix variables because the commutative law is not generally valid for products of matrices. For example, taking into account a square of the matrix, an equation is in effect as

$$\frac{\partial \mathbf{M}^2(\mathrm{s})}{\partial \gamma} = \frac{\partial\left[\mathbf{M}(\mathrm{s})\mathbf{M}(\mathrm{s})\right]}{\partial \gamma} = \frac{\partial \mathbf{M}(\mathrm{s})}{\partial \gamma}\mathbf{M}(\mathrm{s}) + \mathbf{M}(\mathrm{s})\frac{\partial \mathbf{M}(\mathrm{s})}{\partial \gamma} . \qquad (42)$$

Generalizing this rule on a derivative of the *k*–th power of the matrix we have

$$\frac{\partial \mathbf{M}^k(\mathrm{s})}{\partial \gamma} = \sum_{j=1}^{k} \mathbf{M}^{j-1}(\mathrm{s})\frac{\partial \mathbf{M}(\mathrm{s})}{\partial \gamma}\mathbf{M}^{k-j}(\mathrm{s}) \ , \qquad (43)$$

where $\mathbf{M}^0(\mathrm{s}){=}\mathbf{E}$ is the identity matrix. However, using (43) would be computationally ineffective because we need derivatives of all the matrix powers in (41). After all we can proceed by two equivalent ways as

$$\frac{\partial \mathbf{M}^k(\mathrm{s})}{\partial \gamma} = \frac{\partial\left[\mathbf{M}(\mathrm{s})\mathbf{M}^{k-1}(\mathrm{s})\right]}{\partial \gamma} = \frac{\partial \mathbf{M}(\mathrm{s})}{\partial \gamma}\mathbf{M}^{k-1}(\mathrm{s}) + \mathbf{M}(\mathrm{s})\frac{\partial \mathbf{M}^{k-1}(\mathrm{s})}{\partial \gamma}, \qquad (44)$$

$$\frac{\partial \mathbf{M}^k(\mathrm{s})}{\partial \gamma} = \frac{\partial\left[\mathbf{M}^{k-1}(\mathrm{s})\mathbf{M}(\mathrm{s})\right]}{\partial \gamma} = \frac{\partial \mathbf{M}^{k-1}(\mathrm{s})}{\partial \gamma}\mathbf{M}(\mathrm{s}) + \mathbf{M}^{k-1}(\mathrm{s})\frac{\partial \mathbf{M}(\mathrm{s})}{\partial \gamma} . \qquad (45)$$

Both above equations can be treated as the recurrence ones, when starting with $k = 2$. For a computation the derivative $\partial \mathbf{M}(s)/\partial \gamma$ is prepared and the matrix $\mathbf{M}(s)$ is successively multiplied by itself in a respective loop. The experience have shown that a sufficient number of terms in (41) is about one hundred.

Now we will split the solution into four different cases depending on γ parameter. Considering the series impedance $\mathbf{Z}_0(s)$ and the shunting admittance $\mathbf{Y}_0(s)$ definition formulae (15) and (16), respectively, the derivatives $\partial \mathbf{M}(s)/\partial \gamma$ are given by block matrices as

$$\gamma \equiv R_{ij} \in \mathbf{R}_0 \quad \Rightarrow \quad \frac{\partial \mathbf{M}(s)}{\partial R_{ij}} = \begin{bmatrix} \mathbf{0} & -\dfrac{\partial \mathbf{R}_0}{\partial R_{ij}} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} , \qquad (46)$$

$$\gamma \equiv L_{ij} \in \mathbf{L}_0 \quad \Rightarrow \quad \frac{\partial \mathbf{M}(s)}{\partial L_{ij}} = \begin{bmatrix} \mathbf{0} & -s\dfrac{\partial \mathbf{L}_0}{\partial L_{ij}} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} , \qquad (47)$$

$$\gamma \equiv G_{ij} \in \mathbf{G}_0 \quad \Rightarrow \quad \frac{\partial \mathbf{M}(s)}{\partial G_{ij}} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ -\dfrac{\partial \mathbf{G}_0}{\partial G_{ij}} & \mathbf{0} \end{bmatrix} , \qquad (48)$$

$$\gamma \equiv C_{ij} \in \mathbf{C}_0 \quad \Rightarrow \quad \frac{\partial \mathbf{M}(s)}{\partial C_{ij}} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ -s\dfrac{\partial \mathbf{C}_0}{\partial C_{ij}} & \mathbf{0} \end{bmatrix} . \qquad (49)$$

Keeping earlier introduced formalism we can define the $m \times 1$ column vector $\mathbf{e}_i$ containing the only nonzero value 1 at *i*-th position, and zeros elsewhere. Now when supposing γ as the *i*–th diagonal parameter of a per-unit-length matrix, the derivatives inside (46) – (49) can be expressed by $\mathbf{e}_i \cdot \mathbf{e}_i^T$ matrix. For γ being a parameter occuring in pairs at nondiagonal $(i,j)-th$ and $(j,i)-th$ positions, the derivatives result in $\mathbf{e}_i \cdot \mathbf{e}_j^T + \mathbf{e}_j \cdot \mathbf{e}_i^T$ matrix, in case (46), (47), or else $-(\mathbf{e}_i \cdot \mathbf{e}_j^T + \mathbf{e}_j \cdot \mathbf{e}_i^T)$ matrix, for (48), (49). Finally, the equations (31) – (33) are used to get $\partial \mathbf{Y}_k / \partial \gamma$ needed for $\partial \mathbf{V}_M(s)/\partial \gamma$ sensitivity, see (12).

4.2.4 MTL Physical Parameter Sensitivity

Finally we can suppose γ as a general MTL's physical parameter, e.g. width of the line wires, spacing between them, material properties, etc. [2]. In contrast to the extra physical parameter - the length l, those considered here can affect values of all the per-unit-length matrices $\mathbf{R}_0$, $\mathbf{L}_0$, $\mathbf{G}_0$ and $\mathbf{C}_0$ in general. Now to determine $\partial \mathbf{V}_M(s)/\partial \gamma$ sensitivity, see (12), the chain rule can be used for a computation of the MTL admittance matrix derivative as

$$\frac{\partial \mathbf{Y}_k}{\partial \gamma} = \sum_{i=1}^{m} \sum_{j=i}^{m} \left(\frac{\partial \mathbf{Y}_k}{\partial R_{ij}} \frac{\partial R_{ij}}{\partial \gamma} + \frac{\partial \mathbf{Y}_k}{\partial L_{ij}} \frac{\partial L_{ij}}{\partial \gamma} + \frac{\partial \mathbf{Y}_k}{\partial G_{ij}} \frac{\partial G_{ij}}{\partial \gamma} + \frac{\partial \mathbf{Y}_k}{\partial C_{ij}} \frac{\partial C_{ij}}{\partial \gamma} \right), \quad (50)$$

where $R_{ij} \in \mathbf{R}_0$, $L_{ij} \in \mathbf{L}_0$, $G_{ij} \in \mathbf{G}_0$ and $C_{ij} \in \mathbf{C}_0$, and m denotes the order of the per-unit-length matrix.

5 Time-Domain Sensitivity and Examples

To get time-domain sensitivities a method for numerical inversion of Laplace transforms will be used as

$$\frac{\partial \mathbf{v}_M(t)}{\partial \gamma} = \mathcal{L}^{-1}\left[\frac{\partial \mathbf{V}_M(s)}{\partial \gamma} \right]. \quad (51)$$

In the examples below there will be stated semirelative sensitivities according to the formula

$$\mathbf{S}_\gamma\left(\mathbf{v}_M(t), \gamma\right) = \gamma \frac{\partial \mathbf{v}_M(t)}{\partial \gamma}. \quad (52)$$

As is obvious it is possible to get results for all the nodal voltages and/or branch currents simultaneously, when applying NILT method running on all the vector elements in parallel. Such a procedure has been created in Matlab language environment, see [4].

Now the above discussed techniques will be verified on the network in Fig. 2 which the results obtained by other methods are available for [2]. The MTLs differ in their lengths as $l_1 = 0.05m$, $l_2 = 0.04m$, $l_3 = 0.03m$, while their per-unit-length matrices are identical and equal to

$$\mathbf{R}_0 = \begin{bmatrix} 75 & 15 \\ 15 & 75 \end{bmatrix} \frac{\Omega}{m}, \qquad \mathbf{L}_0 = \begin{bmatrix} 494.6 & 63.3 \\ 63.3 & 494.6 \end{bmatrix} \frac{nH}{m},$$

$$\mathbf{G}_0 = \begin{bmatrix} 0.1 & -0.01 \\ -0.01 & 0.1 \end{bmatrix} \frac{S}{m}, \quad \mathbf{C}_0 = \begin{bmatrix} 62.8 & -4.9 \\ -4.9 & 62.8 \end{bmatrix} \frac{pF}{m}.$$

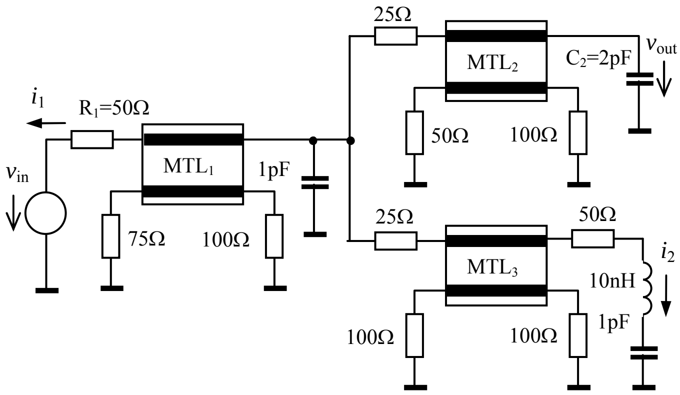

Fig. 2 Hybrid linear network with three MTLs

A 1 V pulse with 1.5 ns rise/fall times and 7.5 ns width acts on the input, with Laplace transform

$$V_{in}(s) = \frac{1 - e^{-1.5\cdot 10^{-9}s} - e^{-6\cdot 10^{-9}s} + e^{-7.5\cdot 10^{-9}}}{1.5e^{-9}s^2}. \quad (53)$$

The waveforms $v_{in}(t)$ & $v_{out}(t)$, i.e. two components of

$$\mathbf{v}_M(t) = \mathcal{L}^{-1}\left[\mathbf{V}_M(s)\right], \quad (54)$$

see (4), obtained via the NILT [4] are shown in Fig.3

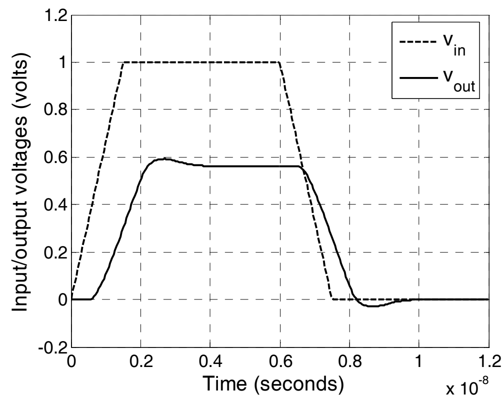

Fig. 3 Input and output voltages

The further waveforms are the semirelative sensitivities of $v_{out}(t)$ in (52), with respect to various parameters γ.

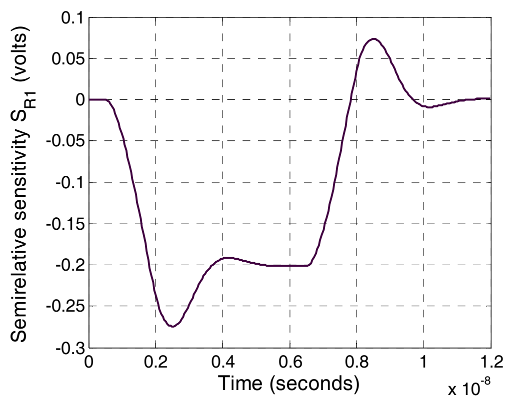

Fig. 4 Semirelative sensitivity v_{out} w. r. to R_1

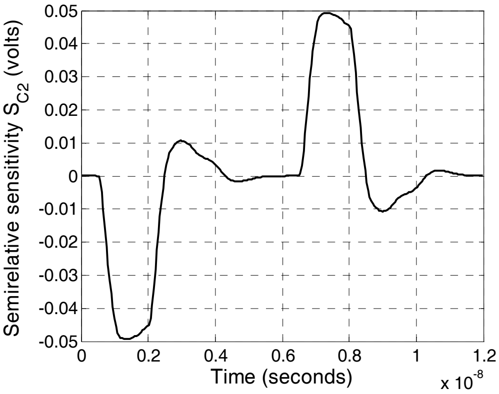

Fig. 5 Semirelative sensitivity v_{out} w. r. to C_2

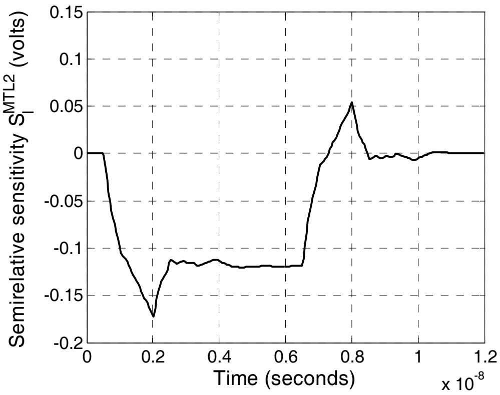

Fig. 6 Semirelative sensitivity v_{out} w. r. to l of MTL_2

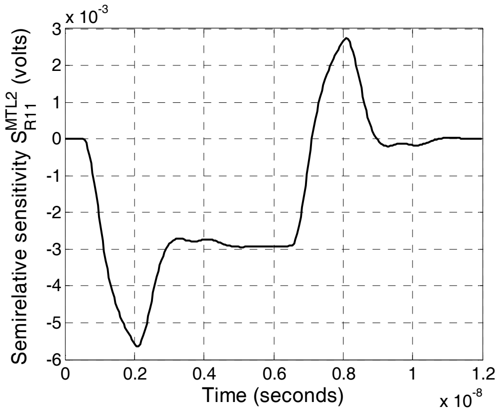

Fig. 7 Semirelative sensitivity v_{out} w. r. to R_{11} of MTL_2

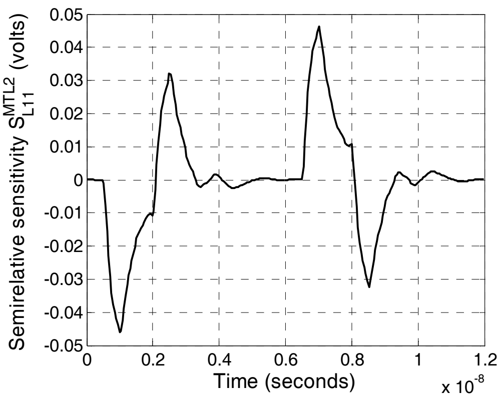

Fig. 8 Semirelative sensitivity v_{out} w. r. to L_{11} of MTL_2

6 Conclusion

The results are practically the same as those published in [2] with only one difference in scaling that was chosen by other way, and without Fig.6 that is not shown there. Therefore above results were also verified by perturbing γ and computing derivatives only approximately as

$$\frac{\partial \mathbf{v}_M(t)}{\partial \gamma} \approx \frac{\Delta \mathbf{v}_M(t)}{\Delta \gamma} = \frac{\mathcal{L}^{-1}\left[\mathbf{V}_M(s,\gamma_2)\right] - \mathcal{L}^{-1}\left[\mathbf{V}_M(s,\gamma_1)\right]}{\gamma_2 - \gamma_1}. \quad (55)$$

Herein the central difference $\Delta\gamma$ was always chosen as 0.1% of the nominal value $\gamma = (\gamma_1 + \gamma_2)/2$. Also these approximate results are very close to ones obtained by the methods under consideration. Namely, for lumped-parameter sensitivities the average RMS error is some 10^{-9}, for distributed-parameter ones then about 10^{-8}. That is why the results based on (55) are not shown graphically as they follow Figs. 4 – 8. The CPU times with the PC 2GHz/256MB are as follows: $\approx 0.7s$ for Figs. 4 & 5, $\approx 0.8s$ for Fig. 6 and $\approx 2.1s$ for Figs. 7 & 8, computing 256 time points in each graph. It should be noticed that all 15 nodal voltages and 2 branch currents were computed simultaneously during above times, and that all computations were realized in Matlab, ver. 7.0.

The method for MTL primary parameter sensitivities calculation promisses further chances to be generalized and improved. It is given by the fact that the MTL chain matrix can also be calculated for nonuniform cases using relatively easy technique. By this the method will enable determining sensitivities with respect to local parameter changes along the MTL structures. Besides techniques how to compute a derivative of the matrix exponential function (40) more effectively should be looked for to acceletate resultant CPU time more and more.

References:

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