Signal Energy-Metric Approach to Stability Analysis of Linear and Non-Linear Causal Systems

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Abstract. This paper deals with internal stability problems of a class of finite dimensional causal systems. Asymptotic stability as well as stability in the sense of Liapunov is analyzed by a new approach based on an abstract energy concept induced by the output signal power. The resulting metric-energy function determines both, the structure of a proper system representation as well as the corresponding system state space topology. Several examples are shown for illustration of fundamental ideas and basic attributes of the proposed method.

Key-Words. Signal power, signal energy, metric, structure, state, nonlinear system, internal, external, representation

1 Introduction

Almost in any field of science and technology some sort of stability problem can appear. Instability is certainly the most important phenomena which should be avoided before any other aspect of reality will be attacked. Two typical situations should be distinguished in dynamical systems theory, if a stability problem has to be solved. The first one arises if the energy function of a given system is known in mathematical form and can be explicitly used to describe the time evolution of internal system energy $E[x(t)]$. In such situations some form of the *energy non-increasing test* [1]:

$$E(x) > 0\,, \quad \frac{\mathrm{d}E(x)}{\mathrm{d}t} \le 0 \qquad (1)$$

can be used.

On the other hand, there are certainly even more real world situations in which some form of energy conservation law is known to play a crucial role, but any mathematical expression for the *system energy is not available.* One standard way to overcome this difficulty is to make some *additional restrictive assumptions,* such as *linearity and time-invariance*, and try to develop some *algebraic stability tests* based on the *explicit knowledge of the solution* of differential or difference equations, describing trajectories of the system.

For continuous-time system representations sets of *necessary and sufficient conditions* for roots s_i:

$$\mathrm{Re}\, s_i < 0, \qquad (2)$$

or for coefficients a_i of the *system characteristic polynomials P(s)* have been obtained.

For the so-called *non-critical cases* A. M. Liapunov has legitimated the *linearization approach* above by his *first method*, also called *Indirect Liapunov`s Method,* in the year 1892. Substantially more appreciated became his *second method* - the famous *Direct Liapunov's Method,* which instead of the physical energy $E$ works with a *set of axiomatically defined scalar functions V* of the state $x(t)$, called *Liapunov`s functions* [2], [3]. The main goal of the paper is to present an alternative method for stability analysis. Instead of Liapunov functions a proper state space metric [4] is introduced and utilized as a basic tool.

2 Internal and external stability

Recall that from general point of view any collection of trajectories constitutes a dynamical system which, in principle, can be described either by its *external behavior*, or by an *internal structure*. In the *input-to-output framework* the external behavior of a continuous-time causal system can be seen as a collection of all *input-output trajectories* satisfying the relation:

$$F(t, y, \dot{y}, ..., y^{(n)}, u, \dot{u}, ...u^{(m)}) = 0, \quad m \le n \qquad (3)$$

The input signals $u(.)$ and output signals $y(.)$, explicitly reflect a *signal orientation property* of causality relation (3) and determine the *external causality structure,* which is important for external stability. Formally, we can write for *an external stability property:*

$$\{(3) \text{ is stable }\} \Leftrightarrow \{\ |u(t)| < \delta \Rightarrow |y(t)| < \varepsilon\ \} \qquad (4)$$

In the present paper mainly concepts concerning the internal stability will be examined. In such a case of the *state-to-state framework*, only *an internal causality structure,* reflecting a *time orientation property* of the causality relation and describing a collection of all *state trajectories*, seems to be appropriate:

$$\dot{x}(t) = f\left[x(t)\right],\ x(t_0) \in X \subset R^n \qquad (5)$$

in which *no external signals* are explicitly introduced.

Definition 1:(Internal stability of an equilibrium state) The *equilibrium state* x^* of the internal system representation (5), defined by the relation:

$$f(x^*) = 0 \qquad (6)$$

is:

- *Stable* (in the sense of Liapunov – SSL) if, for each $\varepsilon > 0$, there is $\delta = \delta(\varepsilon) > 0$ such that:

$$\left\| x(t_0) - x^* \right\| < \delta \Rightarrow \left\| x(t) - x^* \right\| < \varepsilon,\ \ \forall t \geq t_0 \qquad (7)$$

- *Unstable* if it is not stable (in the SSL)
- *Asymptotically stable* if it is stable (in the SSL) and δ can be chosen such that:

$$\left\| x(t_0) - x^* \right\| < \delta \Rightarrow \lim_{t\to\infty} x(t) = x^* \qquad (8)$$

Theorem 1: (Sufficient stability conditions [1]) Let $x^* = 0$ be an *equilibrium state of the system representation (5) and* $D \subset R^n$ *be a domain in the state space* X *containing* $x^* = 0$. *Let* $V: D \to R$ *be a continuously differentiable function, such that*

$$x^* = 0 \ \Rightarrow V(x^*) = 0 \text{ and } \dot{V}(x) > 0 \text{ in } D - \{x^*\} \qquad (9)$$

$$\dot{V}(x) < 0 \ \text{ in } \ D \qquad (10)$$

then, x^* =0 *is stable in the SSL. Moreover, if*

$$\dot{V}(x) < 0 \ \text{ in } \ D - \{x^*\},\ \ x^* = 0 \qquad (11)$$

then $x^* = 0$ *is asymptotically stable.*

Remark 1: Stability conditions Theorem 1 are due to original work of A.M. Liapunov. It has been proven later by N.N.Krasovski [1], that the *condition (11) for asymptotic stability can be replaced by the condition (10),* if certain additional conditions are fulfilled.

The main advantage of the Liapunov's approach above is its generality. It applies for time-varying linear and nonlinear systems as well. Notice that the stability conditions are *only sufficient*. Its main drawback is lack of any systematic and universally applicable technique for *generation of the Liapunov functions V(x)* having the required properties.

2 Signal power balance relation and energy-metric approach

As an alternative to the method of Liapunov functions above a conceptually different approach can be based on the idea that, in fact, *it is not the physical energy by itself*, but only a *measure of distance from the system equilibrium to the actual state x(t)*, what is needed for stability analysis. Thus, instead of the physical energy a *metric* $\rho\left[x(t),\ x^*\right]$ will be defined in a proper way, and for an *abstract energy E(x)* we then put formally:

$$E(x) = \frac{1}{2}\rho^2\left[x(t),\ x^*\right] \qquad (12)$$

Within the state space paradigm the concept of an abstract energy seems to be one of the most natural means describing the *internal system topology*. A measure of distance of actual state from an equilibrium point or, more generally from an invariant set can be thought as a measure of energy accumulated in the state space of the given system. To avoid confusion an abstract system energy concept and the concept of signal power for both the continuous- and discrete-time system representations will be defined first.

We start with a *natural assumption that every real signal must be generated by a realizable system.* Let such a system, called *signal generating system* (SGS), be given in the form:

$$\begin{aligned} \Re\{S\}:\ & \dot{x}(t) = A\,x(t) + B\,u(t), \qquad x(t_0) = x^0, \\ & y(t) = C\,x(t), \end{aligned} \qquad (13)$$

It seems natural to suppose that *every real system* has to satisfy *some form of energy conservation law*. Let the *immediate value of the output signal power and corresponding value of the system energy, accumulated in the state x(t)* be defined by:

$$P(t) = \left\|y(t)\right\|^2,\ E(t) = \delta\left\|x(t)\right\|^2,\ \frac{\mathrm{d}E(x)}{\mathrm{d}t} = -P(t),\ \delta > 0 \qquad (14)$$

Putting $u(t) = 0,\ \forall t \geq t_0$ and computing the *derivative of the energy function* $E(t)$ along the equivalent representation of the given SGS we get the *signal power balance relation*:

$$\frac{\mathrm{d}E(x)}{\mathrm{d}t} = \delta\, x^T(t)[A + A^T]\,x(t) = -\,y^2(t) \qquad (15)$$

and, by integration, the *energy conservation principle for a proper chosen equivalent representation.* After some manipulations also a *special form* of the well known *Lyapunov's equation*, expressing in fact the signal power balance, could be obtained.

Hence, in case of zero input $u(t)=0, \forall t \geq t_0$ the *total energy accumulated in the system* in time t_0 must be equal to the *amount of energy dissipated* on the interval $[t_0;\infty)$ *by the output*:

$$E(t_0)=\int_{t_0}^{\infty}\|y(t)\|^2 dt \tag{16}$$

It is worthwhile to note that in general case *the minimality of system representation* is equivalent to observability of (A, C) and controlability of (A, B), but for zero input only the observability is necessary. Thus the given representation must be in the *state equivalence relation* with a *structurally observable* representation called *observability normal form.* On the other hand, from the *energy conservation principle* in form of the Eqns.(14), (15) it follows, that another special form of a *structurally dissipative* state equivalent system representation (Fig. 1) called *dissipation normal form must exist* and can be specified by the triplex of matrices (A, B, C) as follows:

$$A=\begin{pmatrix} -\alpha_1, & \alpha_2, & 0, & 0, & \cdots, & 0, & 0 \\ -\alpha_2, & 0, & \alpha_3, & 0, & \cdots, & 0, & 0 \\ 0, & -\alpha_3, & 0, & \alpha_4 & \cdots, & 0, & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots, & \vdots & \vdots \\ 0, & 0, & 0, & 0, & -\alpha_{n-1}, & 0, & \alpha_n \\ 0, & 0, & 0, & 0, & \cdots 0, & -\alpha_n & 0 \end{pmatrix}, \quad C^T=\begin{bmatrix} \gamma \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \quad B=\begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \vdots \\ \beta_{n-1} \\ \beta_n \end{bmatrix} \tag{17}$$

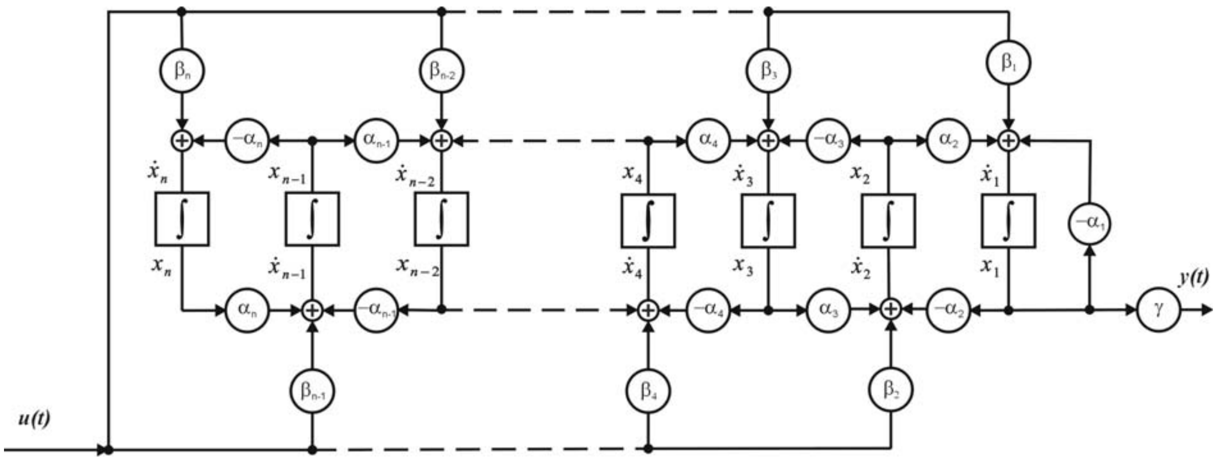

Fig. 1: Structure of continuous-time system representation in the dissipation normal form

It is easy to show that the set of *real basic design parameters* α_i , γ , β_i must satisfy the following fundamental *consistency conditions:*

1. $\forall i,\ i \in \{1,2,...,n\}: 0 < \alpha_i < \infty \Leftrightarrow$ *structural asymptotic stability* (18)
2. $\forall i,\ i \in \{2,3,...,n\}: 0 \neq \alpha_i, \gamma \neq 0, \exists i: \beta_i \neq 0 \Leftrightarrow$ *structural minimality* (19)

In discrete-time case we proceed conceptually *by exactly the same way* as before. The *signal generating system* (SGS) is now represented by:

$$\Re\{S\}:\ x(k+1)=Ax(k)+Bu(k), \qquad x(k_0)=x^0, \qquad y(k)=Cx(k), \tag{20}$$

and the *immediate value of the output signal power and corresponding value of the system energy, accumulated in the state,* be defined by:

$$P(k)=\|y(k)\|,^2\ E(k)=\delta\|x(k)\|,^2\ P(k)=-\Delta E(k)$$

Putting $u(k)=0: \forall k \geq 0$ and computing the *difference of the energy function* $E(k)$ along any trajectory of the system representation, we get *the signal power balance relation*:

$$\Delta E[x(k)]=\delta x^T(k)[A^T A+I]x(k)=-y^2(k) \tag{22}$$

After some manipulations a *special form* of discrete-time *Lyapunov's equation*, expressing in fact *the signal energy conservation principle*, could be obtained. Assuming $u(k)=0, \forall k \geq 0$, *the energy accumulated in the system* in time $k=0$ is equal to *the sum of energy quanta dissipated* at the interval $[0;\infty)$ *by the output signal*, given by:

$$E(k=0)=\sum_{k=0}^{\infty}\|y(k)\|^2 \tag{23}$$

Again, exactly as in the continuous-time version above, the system representation must be in *state equivalence relation* with a special *structurally observable representation* called *observability normal form.* On the other hand, from the *energy conservation principle* in form of the Eqns.(8), (9) it follows, that another special form of *structurally dissipative state equivalent system representation* called *discrete-time* *dissipation normal form must exist* and can be specified by the triplex (A, B, C) according the Eqn. (25).

It is easy to show that the set of *real basic* (direct) *design parameters* δ_i and the set of *real complementary* (feed-back) *parameters* Δ_i must satisfy the following *consistency conditions*:

$$0 < \delta_i \le 1, \quad \delta_i^2 + \Delta_i^2 = 1,\ i \in \{1, 2, ..., n\}, \quad \delta_n = \gamma, \tag{24}$$

having two important *consequences*:

$$1.\ \forall i, \quad i \in \{1, 2, ..., n\}: \ |\Delta_i| < 1 \quad \Leftrightarrow$$

structural asymptotic stability (26a)

$$2.\ \forall i: 0 < \delta_i \le 1, \ \gamma \ne 0, \ \beta_n \ne 0 \quad \Leftrightarrow$$

structural minimality (26b)

The derived *structure of the discrete-time system representation in dissipation normal form* corresponding to the Eqns.(25) is shown at the Fig. 2.

$$A = \begin{bmatrix} -\Delta_{n-1}\cdot\Delta_n & \delta_{n-1} & 0 & \cdots & 0 & 0 & 0 \\ -\Delta_{n-2}\cdot\delta_{n-1}\cdot\Delta_n & -\Delta_{n-2}\cdot\Delta_{n-1} & \delta_{n-2} & \vdots & \vdots & \vdots & \vdots \\ -\Delta_{n-3}\cdot\delta_{n-2}\cdot\delta_{n-4}\cdot\Delta_3 & \vdots & \vdots & \ddots & \delta_3 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & -\Delta_2\cdot\Delta_3 & \delta_2 & 0 \\ \vdots & \vdots & \vdots & \vdots & -\Delta_1\cdot\delta_2\cdot\Delta_3 & -\Delta_1\Delta_2 & \delta_1 \\ \delta_1\cdot\delta_2\cdots\delta_{n-1}\cdot\Delta_n & \cdots & \cdots & \cdots & \delta_1\cdot\delta_2\cdot\Delta_3 & \delta_1\cdot\Delta_2 & \Delta_1 \end{bmatrix}, \quad C^T = \begin{bmatrix} \gamma \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \quad B = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \vdots \\ \beta_{n-1} \\ \beta_n \end{bmatrix} \tag{25}$$

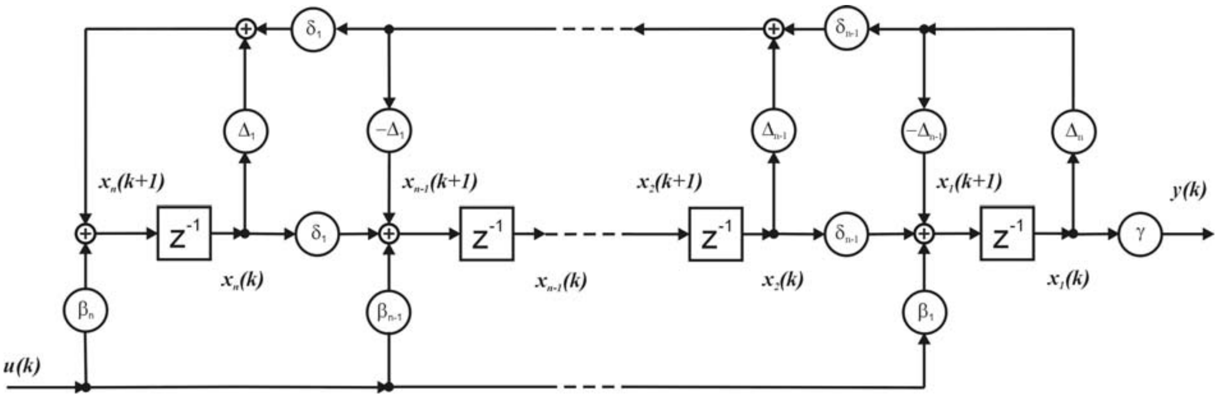

Fig. 2:Structure of discrete-time system representation in dissipation normal form

3 Examples

<u>*Example 1*</u>. (Stability analysis of a linear system)

Let the representation (13) is given for $n = 4$, the input signal $u(t) = 0$, for $t \ge t_0$, and the corresponding characteristic polynomial has the following general form:

$$P_n(s) = \det[sI - A] = \\ = s^n + a_1 s^{n-1} + a_2 s^{n-2} + ... + a_{n-1}s + a_n \tag{27}$$

Let the *parameters* $a_1, a_2, ..., a_n$ of $P_n(s)$ are considered as *unknown* and the *region of asymptotic stability in a parameter space* has to be specified. Assume that

$$\det A \ne 0 \iff \exists A^{-1} \tag{28}$$

The condition (28) is *necessary and sufficient for existence of the unique equilibrium state $x^* = 0$,* and for n = 4 it follows from the Eqn.(17)

$$\det A = \alpha_2^2 \alpha_4^2 \ne 0 \Leftrightarrow \alpha_2 \ne 0, \alpha_4 \ne 0 \tag{29}$$

where

$$A = \begin{bmatrix} -\alpha_1, & \alpha_2, & 0, & 0 \\ -\alpha_2 & 0 & \alpha_3 & 0 \\ 0 & -\alpha_3 & 0 & \alpha_4 \\ 0 & 0 & -\alpha_4 & 0 \end{bmatrix} \tag{30}$$

Hence the parameters $a_1, \ldots, a_4$ of the characteristic polynomial are explicitly expressed by

$$\begin{aligned} a_1 &= \alpha_1, \\ a_2 &= \alpha_2^2 + \alpha_3^2 + \alpha_4^2, \\ a_3 &= \alpha_1(\alpha_3^2 + \alpha_4^2), \\ a_4 &= \alpha_2^2 \alpha_4^2 \end{aligned} \tag{31}$$

It follows for $a_i, i \in \{1, 2, 3, 4\}$ that

$$\alpha_i \in R \Leftrightarrow x_i(t) \in R \tag{32}$$

i.e. for all state variables x_i^2 is non-negative.

Thus for the Eucleidian metric $\rho = \rho_2$ we get

$$E\left[x(t)\right] = \tfrac{1}{2}\rho^2\left[x(t),0\right] = \tfrac{1}{2}\|x(t)\|^2 = \tfrac{1}{2}\sum_{i=1}^{n} x_i^2(t) \qquad (33)$$

and consequently it holds:

1^o $E(x) = 0 \Leftrightarrow x(t) = x^*$, $(x^* = 0)$

2^o $x_i(t) \in R \Leftrightarrow x_i^2(t) \geq 0 \Rightarrow E(x) > 0 \Leftrightarrow x(t) \neq x^*$

In order to use energy nonincreasing test (1) we have to compute the derivative of the output signal energy function $E(x)$ along the system representation (13), given by the matrix (30) in the following explicit form:

$$\Re(S): \quad \begin{aligned} \dot{x}_1(t) &= -\alpha_1 x_1(t) + \alpha_2 x_2(t) \\ \dot{x}_2(t) &= \alpha_2 x_1(t) + \alpha_3 x_3(t) \\ \dot{x}_3(t) &= -\alpha_3 x_2(t) + \alpha_4 x_4(t) \\ \dot{x}_4(t) &= -\alpha_4 x_3(t) \end{aligned} \qquad (34)$$

$$y(t) = \gamma\, x_1(t) \qquad (35)$$

We get

$$\left.\frac{\mathrm{d}E(t)}{\mathrm{d}t}\right|_{\Re\{s\}} = -\alpha_1 x_1^2(t) = -\frac{\alpha_1}{\gamma^2}.y^2(t) \qquad (36)$$

where γ is a real power scaling parameter

$$0 < \gamma < \infty \qquad (37)$$

Thus, the *signal energy conservation principle* in form of (15) holds (for $\delta = ½$, $\gamma \neq 0$) iff:

$$P(t) = y^2(t) \quad \Leftrightarrow \quad \alpha_1 = \gamma^2 > 0 \qquad (38)$$

Remark 2: Notice, that α_3 is the only element of the matrix A which can be *arbitrary from the stability analysis* point of view. If we put $\alpha_3 = 0$, then the state variables x_i , $i = 3,4$ become *unobservable by the output y* ; thus *only the first isolated subsystem* with the state variables x_i , $i = 1,2$ which is *observable, will be asymptotic stable*, while the *second one will oscilate on the constant energy level*, corresponding to initial conditions with the *frequency* given by the parameter α_4. As a result the *whole system* is *stable in the sense of Liapunov, but not asymptotically.*

From the equation (31) it follows that in such a case the characteristic polynomial takes the form:

$$P(s) = (s^2 + \alpha_1 s + \alpha_2^2)(s^2 + \alpha_4^2),\ \alpha_1 > 0 \qquad (39a)$$

Hence we have:

$$\mathrm{Re}\, s_1 < 0,\ \mathrm{Re}\, s_2 < 0,\ \mathrm{Re}\, s_3 = 0,\ \mathrm{Re}\, s_4 = 0 \qquad (39b)$$

Remark 3: It is easy to prove in general that for asymptotic stability the conditions mentioned above are *necessary but not sufficient.* If, in addition, the couple (A,C) has the well known *observability property,* then the resulting conditions will be *necessary and sufficient for asymptotic stability,* too.

Example 2. (Asymptotic stability analysis)

Let $n = 4$, the matrix A is given by the eqn. (30) as before and the matrix C is defined by $C = [\ \gamma, 0, 0, 0\ ]$. Then the *observability matrix* H_o is defined by

$$H_0 = \left[C^T; A^T C^T; (A^T)^2 C^T; (A^T)^3 C^T\right] \qquad (40)$$

and the *necessary and sufficient observability conditions* have the following form:

$$\det H_o \neq 0 \Leftrightarrow \alpha_2 \neq 0, \alpha_3 \neq 0, \alpha_4 \neq 0, \gamma \neq 0. \qquad (41)$$

From the Eqns. (41) and (38) the set of *necessary and sufficient conditions of asymptotic stability* results

$$\alpha_1 > 0,\ \alpha_2 \neq 0, \alpha_3 \neq 0, \alpha_4 \neq 0 \qquad (42)$$

Example 3. (Relation to Hurwitz stability criterion)

If needed, we can determine the set of parameters α_i , $i = 1, 2, 3, 4$ from the Eqn. (31). Then we get:

$$\begin{aligned} \alpha_1 &= a_1 = \Delta_1, \\ \alpha_2 &= \sqrt{\frac{a_1 a_3 - a_3}{a_1}} = \sqrt{\frac{\Delta_2}{\Delta_1}} \\ \alpha_3 &= \sqrt{\frac{a_1 a_2 a_3 - a_3^2 - a_1^2 a_4}{(a_1 a_2 - a_3) a_1}} = \sqrt{\frac{\Delta_3}{\Delta_2 \Delta_1}} \\ \alpha_4 &= \sqrt{\frac{a_1 a_4}{a_1 a_2 - a_3}} = \sqrt{\frac{\Delta_4 \Delta_1}{\Delta_2 \Delta_3}} \end{aligned} \qquad (43)$$

where the new parameters Δ_k , k= 1, 2, … can be properly expressed as *diagonal minors* of the well known *Hurwitz determinant.* It is very easy to derive the general expression for any order n > 3 in the form:

$$\alpha_k = \sqrt{\frac{\Delta_k \Delta_{k-3}}{\Delta_{k-2}\Delta_{k-1}}}\ ,\quad k = 4, 5, 6, \ldots, n \qquad (44)$$

Using the expressions (43), (44) together with the requirement $\alpha_k \in R$, the following set of *equivalent necessary and sufficient conditions of the asymptotic stability* can be obtained:

$$\begin{aligned} \alpha_1 \in R,\ \alpha_1 > 0 &\quad \Leftrightarrow \quad \Delta_1 > 0 \\ \alpha_2 \in R,\ \alpha_2 \neq 0 &\quad \Leftrightarrow \quad \frac{\Delta_2}{\Delta_1} > 0 \\ \alpha_3 \in R,\ \alpha_3 \neq 0 &\quad \Leftrightarrow \quad \frac{\Delta_3}{\Delta_1 \Delta_2} > 0 \\ \alpha_4 \in R,\ \alpha_4 \neq 0 &\quad \Leftrightarrow \quad \frac{\Delta_1 \Delta_4}{\Delta_2 \Delta_3} > 0 \end{aligned} \qquad (45)$$

The resulting conditions (45) are obviously *equivalent to the set of the well known Hurwitz conditions:*

$$\Delta_k > 0\ ,\ \text{k} = 1, 2, \ldots, \text{n} \qquad (46)$$

It means that linear *algebraic methods* for stability analysis can be seen as a *special case* of methods based on the proposed energy-metric approach.

Example 3. (Non-linear stability analysis)

Let us consider a simple *non-linear system given* by the following input-output representation :

$$\ddot{y}(t) + \varepsilon\left[\alpha - \beta y^2(t)\right]\dot{y}(t) + a_2 y(t) = u(t) \quad (47)$$

If the matrix C is defined by $C = [\gamma, 0]$, and the chosen *structure of the matrix A(x)* is defined by

$$A(x_1, x_2) = \begin{bmatrix} -\varepsilon\left[\alpha - \frac{1}{3}\beta x_1^2\right], & \sqrt{a_2} \\ -\sqrt{a_2}\,, & 0 \end{bmatrix} \quad (48)$$

then the system representation is *locally observable if*

$$\gamma \neq 0, \; a_2 > 0 \quad (49)$$

and the *signal energy conservation principle* gives

$$\left.\frac{\mathrm{d}E(t)}{\mathrm{d}t}\right|_{\Re(s)} = -P \leq 0, \; P = \varepsilon\left[\alpha - \tfrac{1}{3}\beta x_1^2\right]x_1^2 \quad (50)$$

It follows that the *unique equilibrium state* $x^* = 0$ is *asymptotically stable in the region* $D \subset X \subset R^2$

$$D = \left\{x_1, x_2 : |x_1| < \sqrt{\frac{3\alpha}{\beta}} \text{ and } x_1^2 + x_2^2 < \frac{3\alpha}{\beta}\right\} \quad (51)$$

if $\varepsilon > 0, \; \alpha > 0, \; \beta > 0, \; a_2 > 0$.

Example 4. (Relation to Direct Method of Liapunov)

Let us consider the same *non-linear system given* by

$$\ddot{y}(t) + \varepsilon\left[\alpha - \beta y^2(t)\right]\dot{y}(t) + a_2 y(t) = u(t) \quad (52)$$

but instead of the *matrix structure* $A(x)$ the state $x(t)$ is defined by $x_1 = y, x_2 = dy/dt$.

Then the corresponding system representation is *structurally observable* with the *observability matrix*

$$H_o = I \quad (53)$$

and from the *signal energy conservation principle*

$$\left.\frac{\mathrm{d}V(t)}{\mathrm{d}t}\right|_{\Re(s)} = -P \leq 0, \; P = \varepsilon\left[\alpha - \tfrac{1}{3}\beta x_1^2\right]x_1^2 \quad (54)$$

a *unique Liapunov function* $V(x)$ can be determined by *isometric transformations of the energy function (12)*

$$E(x) = \frac{1}{2}\rho^2\left[\mathrm{x(t)}, 0\right] \quad (55)$$

and for $\alpha = \beta = a_2 = 1$ we get

$$V(x) = \tfrac{1}{2}\Big[\tfrac{1}{9}\varepsilon^2 x_1^6 - \tfrac{2}{3}\varepsilon^2 x_1^4 + (1+\varepsilon^2)x_1^2 - \\ - \tfrac{2}{3}\varepsilon x_1^3 x_2 + 2\varepsilon x_1 x_2 + x_2^2\Big] \quad (56)$$

Example 5. (Estimation of domain of attraction)

From the Eqns. (51) and (56) we directly get the set

$$D = \left\{x_1, x_2 : |x_1| < \sqrt{\frac{3\alpha}{\beta}}, V[x] < \frac{3\alpha}{\beta}\right\} \quad (57)$$

representing *region of the state space X* for which the property of *asymptotic stability is warranted* by $V(x)$, iff it holds: $\varepsilon > 0, \; \alpha > 0, \; \beta > 0, \; a_2 > 0$. Moreover

$$\beta \to 0 \Leftrightarrow D \to X = R^2 \quad (58)$$

and *global asymptotic stability follows.*

Example 6. (Generation of Liapunov functions)

Let a *non-linear system* is *given* by the representation

$$y^{(4)}(t) + a_1 y^{(3)}(t) + a_2\ddot{y}(t) + a_3\dot{y}(t) + a_4 y(t) = 0 \quad (59)$$

gained *by an approximative linearization* procedure and the state variables are defined by

$$x_1 = y, \quad x_2 = \dot{y}, \quad x_3 = \ddot{y}, \quad x_4 = y^{(3)} \quad (60)$$

then the *observability matrix* is given by $H_o = I$, while the *observability matrix* H_o of the state equivalent representation (30) is *triangular and invertible*. It is easy to show that the Liapunov function V is given by

$$V\left[x(t)\right] = \tfrac{1}{2}x^T(t)\left[H_0^T.H_0\right]^{-1}.x(t) \quad (61)$$

and for (59), (60) it can be explicitly expressed by

$$V = \frac{1}{2}\Bigg[x_1^2 + \left(\frac{\alpha_1}{\alpha_2^2}x_1 + \frac{1}{\alpha_2^2}x_2\right)^2 + \\ + \left(\frac{\alpha_2^2}{\alpha_3^2}x_1 + \frac{\alpha_1}{\alpha_2^2\alpha_3^2}x_2 + \frac{1}{\alpha_2^2\alpha_3^2}x_3\right)^2 + \;.\;.\;.\Bigg] \quad (62)$$

4 Conclusions

In the contribution a new unifying and constructive approach to linear and non-linear stability problems, based on a metric - energy concept of the system state space, has been presented.

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