A Novel Optimal Routing and A Fault-Tolerant Routing for the Incrementally Extensible Hypercube Network

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Abstract: - The incrementally extensible hypercube (IEH) network, a variant of the hypercube, is defined for every positive number and has been widely discussed. However, some properties of the IEH network are incorrect and have been misused. Moreover, no deterministic fault-tolerant routing has been proposed. In this paper, we first simplify the construction of an IEH network. Then we point out the mistakes in the topology of the number of edges, the diameter, and the shortest paths. We present the correct results and propose a novel optimal routing algorithm. Moreover, a deterministic fault-tolerant routing algorithm for the IEH network is proposed.

Key-Words: - Hypercubes; Incrementally Extensible Hypercube networks; Hamming distance; Optimal routing; Fault-tolerant routing.

1 Introduction

With the rapid growth of the amount of information, interconnection networks with parallel processing have become more important than before. Among various efficient interconnection networks, the hypercube with low latency, high regularity, and good fault-tolerance has attracted many researchers [1, 8]. However, the number of nodes in a hypercube is restricted to powers of two. Many variants without this restriction have been widely studied [1, 2, 4, 5, 7, 9].

In 1992, Sur and Srimani first proposed the incrementally extensible hypercube (IEH) network as a variant of the hypercube [10]. Unlike a hypercube restricted to 2^n nodes, the IEH network can be constructed for an arbitrary number of nodes. An IEH network consists of different size of hypercubes, called *subcubes*, connected by *Inter-Cube* (IC) edges, and thus preserves several advantages of the hypercube such as a low diameter, good fault-tolerance, a simple routing algorithm, and good regularity. There are many extensive researches in the IEH network [3, 6, 11, 12]. However, some proposed properties, such as the diameter and the number of edges, of the IEH network are unsatisfied or incorrect, and have been misused for a long time [10]. Besides, the claimed shortest routing is not optimal and the proposed fault-tolerant routing is non-deterministic [11, 12].

In this paper, we first simplify the construction of an IEH network. Then we point out the pitfalls of the properties of IEH networks and present the correct ones. Moreover, we propose a novel optimal routing and a deterministic fault-tolerant routing for the IEH network. The rest of this paper is organized as follows. In Section 2, we introduce the construction of IEH networks and discuss their topologies and properties. We propose a novel optimal routing for the IEH network in Section 3 and a deterministic fault-tolerant routing for the IEH network in Section 4. Finally, the conclusion of this paper is presented in Section 5.

2 IEH Networks

In this section, we describe a simplified construction of the incrementally extensible hypercube (IEH) network. In this paper, we consider $2^n < N < 2^{n+1}$ and let $N = (c_n, c_{n-1}, \cdots, c_0)_2$ be the binary representation of N where $c_n = 1$ and $c_i \in \{0,1\}$ for other i. Then the IEH network $G(N)$ with N nodes can be recursively constructed by the following algorithm [11].

The CONSTRUCTION Algorithm
Input: a positive integer N.
Output: the IEH network $G(N)$.

1. For each $c_i = 1$, construct a subcube H_i.
2. Label each node in H_i as an $(n+1)$-bit binary

representation $11\cdots10b_{i-1}\cdots b_0$ where the last i bits are the same as an i-cube, and the prefix is $n-i$ 1's followed by a single zero.

3. Find the minimum i such that $c_i = 1$. Set $j = i$ and $G_j = H_i$.
4. Recursive construction.
   Set $i = i + 1$.
   while $i \le n$
   if $c_i = 1$ **then**
   Connect the node $11\cdots1b_j b_{j-1}\cdots b_0$ in G_j to the following node by a *1-IC edge*:
   $$\overbrace{11\cdots1}^{n-i}0\overbrace{11\cdots1}^{i-j-1}b_j b_{j-1}\cdots b_0,$$
   and to the following $i-j-1$ nodes in H_i each by a *2-IC edge*:
   $$\overbrace{11\cdots1}^{n-i}0\overbrace{1\cdots101\cdots1}^{i-j-1}b_j b_{j-1}\cdots b_0,$$
   where there is a single 0 among the middle $i-j-1$ bits.
   Set $j = i$ and let G_j be the composed graph obtained in the above step.
   endif
   $i = i + 1$.
   endwhile
5. Finally the IEH network $G(N)$ is obtained.

Note that all IC edges can be partitioned into two types: 1-IC edges and 2-IC edges according to the Hamming distance between their endpoints. Moreover, including the cube-edges of each subcube, each edge of an IEH network connects nodes with Hamming distance of 1 or 2. In particular, each node of an IEH network has at most one 1-IC edge to each subcube. Figure 1 shows the IEH network $G(11)$ where the dashed lines denote 1-IC edges, the bold solid lines denote 2-IC edges, and the other solid lines denote the cube-edges.

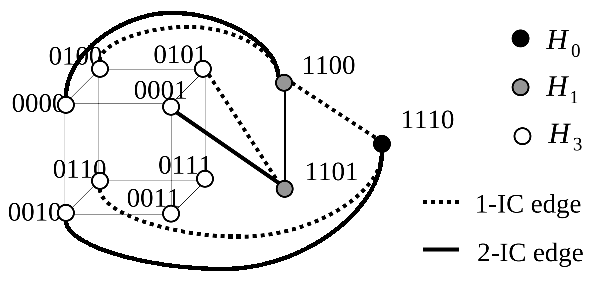

Figure 1. The IEH network $G(11)$.

In the following, we consider the properties of the IEH network. Firstly, we count the number of IC edges from a node v in H_j to subcube H_i for $i > j$. In [10], the authors claimed that there are $i - j$ IC edges if $i - j > 1$. However, it is incorrect since in $G(11)$ there are only two IC edges from the node 1110 in H_0 to subcube H_3 shown in Figure 1. We correct the statement as follows.

Lemma 1. *Let v be a node in H_j of an IEH network $G(N)$. Then there are $i-k$ IC edges from node v to H_i where k is the largest number such that $j \le k < i$ and $c_k = 1$.*

Proof. By the construction of the IEH network. □

The near regularity of the IEH network has been implicitly derived by its connectivity and its maximum degree [10-12]. We present a lemma of the minimum degree such that the near regularity can be directly obtained.

Lemma 2. (See [11]) *The node connectivity of the IEH network $G(N)$ is n.*

Lemma 3. (See [11]) *The maximum degree of the IEH network $G(N)$ is $n+1$.*

Lemma 4. *The minimum degree of the IEH network $G(N)$ is n.*

Proof. Let $n = j_k > j_{k-1} > \cdots > j_2 > j_1 \ge 0$ for all $c_{j_i} = 1$. Then every node v in subcube H_{j_i} has j_i cube-edges, at most one backward IC edge connecting to $G_{j_{i-1}}$, and other $n - j_i$ forward IC edges where there are $j_{i+1} - j_i$ IC edges connecting to $H_{j_{i+1}}$, $j_{i+2} - j_{i+1}$ IC edges to $H_{j_{i+2}}$, and so on by Lemma 1. Hence the minimum degree of the IEH network $G(N)$ is n. □

Theorem 5. *The IEH network $G(N)$ is almost regular.*

Proof. By Lemmas 3 and 4. □

Next, we consider the diameter of the IEH network, which denotes the worst delay of transmission in a network. The diameter of the IEH network $G(N)$, denoted by $D(G(N))$, is claimed as $n+1$ in [11]. However, it is incorrect, since the diameter of $G(11)$ is 3, not 4, shown in Figure 1. We correct the diameter of the IEH network as follows.

Theorem 6. *The diameter of IEH network $G(N)$ is either n or $n+1$.*

Proof. Since each node in $G(N)$ is labeled by $n+1$ bits, the Hamming distance between any two nodes is at most $n+1$. Since every edge in the IEH network changes either one or two bits, we can obtain $D(G(N)) \le n+1$. On the other hand, the diameter of subcube H_n is n, and thus $D(G(N)) \ge n$. Note that $D(G(11)) = 3 = n$ and $D(G(6)) = 3 = n+1$. Hence the diameter of the IEH network $G(N)$ is either n or $n+1$. □

3 A Novel Optimal Routing

In this section, we discuss the problem of optimal routings in the IEH network. A shortest routing is claimed and proposed in [12]. However, the proposed routing is not shortest. Consider the transmission from node 1110 to the node 0001 in the IEH network $G(13)$ shown in Figure 2. Based on the proposed routing algorithm [12], the routing path with transmission delay 4 is as follows.

1110 → 0110 → 0111 → 0101 → 0001.

However, there is a shorter path with transmission delay 3 as follows.

1110 → 1000 → 0000 → 0001.

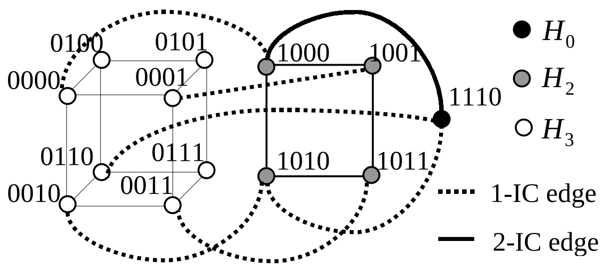

Figure 2. The IEH network $G(13)$.

Next, we cite the following definition to locate the subcube that a node belongs to.

Definition 7. (See [12]) *Let $\rho(v)$ be the length of the 1-prefix in the binary representation of the node v.*

Lemma 8. (See [12]) *A node v in an IEH network $G(N)$ must belong to the subcube $H_{n-\rho(v)}$.*

In the following, we propose a novel optimal routing in the IEH network. Let s be the source and d be the destination of a transmission in an IEH network $G(N)$. Without loss of generality, assume that $\rho(s) \ge \rho(d)$. The main idea of the optimal routing is to route along 2-IC edges sequentially if possible.

The OPTIMAL ROUTING algorithm

Input: The IEH network $G(N)$ with $N = (c_n, c_{n-1}, \cdots, c_0)_2$, source s, destination d.

Output: An optimal routing between s and d.

Step 1. Set $h = s \oplus d = h_n h_{n-1} \cdots h_0$. /* Compute the Hamming distance h between s and d. */

Step 2. Find j_i such that $j_i = n - \rho(s)$. /* Find the subcube that the node s belongs to. */

Step 3.

while $h \ne 0$

if $\rho(s) \ne \rho(d)$ **then**

if $h_{j_i} = 1$ and at least one $h_t = 1$ for $t < j_i$ **then**

Consider the following three cases.

(i) /* Find the backward 2-IC edge, if exist. */ For $j_i > l > j_{i-1}$, if $h_l = 1$ and the node z which is different from s on the bits j_i and l exists, then transfer the message to z by the 2-IC edge. Set $s = z$ and return to Step 1.

(ii) /* Find the backward 1-IC edge, if exist. */ If the above l does not exist and if the node z which is different from s on the bit j_i exists, then transfer the message to z by the 1-IC edge. Set $s = z$ and return to Step 1.

(iii) /* Find a cube-edge. */ Let l be the minimum number such that $h_l = 1$ and $j_i > l \ge 0$. Then transfer the message to the node z which is different from s on the bit l by the cube-edge. Set $s = z$ and return to Step 1.

else /* Find a forward edge. */

Consider the following two cases.

(i) If there exists a maximum m such that $h_m = 1$, for $j_{i+1} > m > j_i$, then transfer the message to the node z which is different from s on the bits j_{i+1} and m by the 2-IC edge. Set $s = z$ and return to Step 1.

(ii) If the above m does not exist, then transfer the message to the node z which is different from s on the bit j_{i+1} by the

1-IC edge. Set $s = z$ and return to Step 1.

else /* s and d are in the same subcube H_{j_i} . */

The shortest path is the same as that in the subcube.

endwhile

Theorem 9. Our routing algorithm is optimal.

Proof. According to our routing algorithm, consider the node s in the subcube H_i. For two specific bits, if s does not have any 2-IC edge which can change those bits, then nor does any node in H_j for $j > i$ by the construction of the IEH network. Hence the proposed routing algorithm is optimal. □

4 A Fault-Tolerant Routing

In this section, we discuss the fault-tolerant routing in the IEH network with a single faulty node. Sur and Srimani proposed several node disjoint paths such that we can choose one of them as the fault-tolerant routing. In the following, we propose a deterministic fault-tolerant routing algorithm.

Let f be the faulty node in the IEH network. A node is called a *suitable node* if it reduces the Hamming distance between the destination and the present node. The concept of a deterministic fault-tolerant routing algorithm is based on the idea of detouring the faulty node in the original shortest path by finding another suitable node if possible. The fault-tolerant routing algorithm is as follows.

The FAULT-TOLERANT ROUTING algorithm

Input: N, source s, destination d, faulty node f.

Output: A fault-tolerant routing path.

Follow the optimal routing algorithm until the next node is the faulty node. Let s' be the present node. Without loss of generality, assume that $\rho(s') \ge \rho(d)$.

Case 1. $\rho(f) \ge \rho(s') \ge \rho(d)$. /* The optimal routing uses a backward edge from s' to f. */

Consider that the node f does not exist, and continue to find the next node.

Case 2. $\rho(s') > \rho(f) > \rho(d)$. /* s', d and f are in distinct subcubes. */

Consider the following two situations.

(i) If the optimal routing uses a 2-IC edge from s' to f, then we can find another 2-IC edge or a 1-IC edge to a suitable node (See Figure 3).

(ii) If the optimal routing uses a 1-IC edge from s' to f, then continue to find a maximum m such that $h_m = 1$, for $j_{i+2} > m > j_{i+1}$ (See Figure 4).

Case 3. $\rho(s') > \rho(f) = \rho(d)$. /* d and f are in the same subcube other than s'. */

Consider the following two situations.

(i) Choose a suitable node if exists in the present subcube (See Figure 5).

(ii) If there is no suitable node, then choose a node that increases the Hamming distance (See Figure 6).

Case 4. $\rho(s') = \rho(f) = \rho(d)$. /* s', d, and f are in the same subcube. */

The fault-tolerant routing is the same as that in the subcube.

Figure 3 shows the shortest path and the fault-tolerant routing from the node 1110 to the node 0000 in the IEH network $G(13)$ with the faulty node 1000.

1110 → 1000 → 0000
➘ 1010 ➔ 0010 ➔ 0000

Figure 3. The shortest path routing (→) and the fault-tolerant routing (➔).

Figure 4 shows the shortest path and the fault-tolerant routing from the node 1110 to the node 0010 in the IEH network $G(13)$ with the faulty node 1010.

1110 → 1010 → 0010
➘ 0110 ➔ 0010

Figure 4. The shortest path routing (→) and the fault-tolerant routing (➔).

Figure 5 shows the shortest path and the fault-tolerant routing from the node 1110 to the node 0001 in the IEH network $G(13)$ with the faulty node 0010.

1110 → 1010 → 0010 → 0011
➘ 1011 ➔ 0011

Figure 5. The shortest path routing (→) and the fault-tolerant routing (➔).

Figure 6 shows the shortest path and the fault-tolerant routing from the node 1001 to the node 0101 in the IEH network $G(13)$ with the faulty node 0001.

The proposed fault-tolerant routing does work even though the faulty node is unknown.

1001 → 0001 → 0101
↘ 1000 ➔ 0000 ➔ 0100 ➔ 0101

Figure 6. The shortest path routing (→) and the fault-tolerant routing (➔).

Theorem 10. The fault diameter is at most 2 more steps than the shortest path.

Proof. As detouring the faulty node, if a suitable node is used, then the fault-tolerant routing is at most one more step than the shortest path, and at most 2 more steps otherwise. □

5 Conclusion

In this paper, we first simplify the construction of the IEH network. Then we point out the mistakes in the previous papers and correct the topology of the number of edges, the diameter. We also propose a optimal routing algorithm and a deterministic fault-tolerant routing algorithm for the IEH network.

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