Characterizing the Result of the Division of Fuzzy Relations

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Abstract: - The role and properties of the division operator are well known in the framework of queries addressed to regular relational databases. However, Boolean queries may turn out to be too restrictive to answer some user needs and it is desirable to consider extended queries by introducing preferences inside selection conditions. In this paper, the extension of the division operator is investigated in the context of graded relations, i.e., whose tuples are weighted. Several interpretations of the division are possible and they mainly depend on the roles of the grades attached to tuples of input relations. Their properties are examined in the perspective of a characterization of the result obtained as a quotient, similarly to that obtained for integers.

Key-Words: - Relational databases, flexible queries, relational division, quotient

1 Introduction

The database domain is an important field of research and development and many works aim at enriching database management systems (DBMSs) capabilities. The research reported in this paper is intended for allowing the expression of flexible queries, i.e., where preferences intervene in selection conditions instead of Boolean ones. This view is illustrated by the query: "find the affordable restaurants located close to the seashore". In such a situation, discrimination among restaurants has to take into account both the price of the menu(s) and the location of the restaurants (and optionally levels of importance attached to these criteria).

Several works devoted to the expression and the interpretation of fuzzy queries in the relational framework [4] have been undertaken (in particular [1, 7, 9]). Selection, projection, Cartesian product, join as well as set-oriented operations have been studied in order to take into account levels of preference. On the contrary, the division operation has not been so much investigated [2, 3, 5, 6, 8, 10, 11] and different extensions have been proposed with various motivations and contexts, in particular depending on the nature of the relations involved and the meaning of degrees associated with tuples.

In the remainder of this paper, the division of fuzzy relations is investigated. The principal objective is to discuss the properties of the result delivered by a division operation. Indeed, this result depends on the approach adopted for the extension of the division as it is mentioned in the works reported in [2, 3, 6]. We would like to determine if the result obtained is a quotient in the sense of the properties which hold when the division of two integers is performed (which turn out to hold with the division of regular relations). The key point behind this is of a semantic nature, because a negative answer would mean that the term division is inappropriate.

The paper is organized as follows. In section 2, the definition of the division of regular relations is recalled as well as the two characteristic properties of a quotient. The principle for adapting the division to fuzzy relations, which relies on the notion of a degree of inclusion (instead of a usual Boolean inclusion) is described in section 3. The next three sections are devoted to the study of the properties of the result of the division of fuzzy relations according to various approaches of extension. Section 7 concludes the paper. The major results obtained are recalled and some perspectives for future works are outlined.

2 Some Reminders about the Division

The relational division, i.e., the division of relation r whose schema is R(A, X) by relation s whose schema is S(B) where A et B are compatible sets of attributes (i.e., defined on the same domains of values) is defined as:

$$\text{div}(r, s, \{A\}, \{B\}) = \{x \mid (x \in \text{domain}(X)) \wedge (s \subseteq K_r(x))\} \quad (1)$$

where $K_r(x) = \{a \mid \langle a, x\rangle \in r\}$. In other words, an element x belongs to the result of the division of r by s if and only if it is associated in r with *at least all* the values a appearing in s. The justification of the term "division" assigned to this operation relies on the fact that a property similar to that of the quotient of integers holds. Indeed, the resulting relation t obtained with this definition has the double characteristic of a quotient, namely:

$$\text{prod}(t, s) \subseteq r \tag{2a}$$

$$\forall t1, (t1 \supset t) \Rightarrow (\text{prod}(t1, s) \not\subseteq r) \tag{2b}$$

with prod(t, s) denoting the Cartesian product of relations t and s.

Proof. Case 1. Neither the result of the division, nor the divisor relation is empty. Let x be an element of t and a be an element of s. Let us suppose that <x, a> does not belong to r, then x would not be associated with all the values of s and it would not be in the result of the division of r by s, hence inclusion (2a) holds. Now, let us consider relation $t1 = t \cup \{y\}$ ($y \notin t$). The Cartesian product of t1 and s contains a tuple <y, b> which does not belong to r, otherwise y would be associated with any value a of s and it would have been in t. It follows that property (2b) holds.

Case 2. The result of the division is empty but the divisor is not empty. Property (2a) holds since the Cartesian product of t and s is empty and then included in any relation. No element x is associated with all the elements of s and if y is added to t, property (2b) does not hold since the Cartesian product of {y} with s involves elements which are not in r.

Case 3. The divisor s is empty. The solution returned by (1) is the (possibly infinite) set of the values in the domain of X. Properties (2a) and (2b) are both satisfied since the Cartesian product of t and s is empty and t cannot be augmented.•

Remark. When the divisor is empty, the theoretical solution of the division is the entire domain of X. In practice, such a solution cannot be computed since the domains of the attributes are not represented (and are thus unknown) in database systems. To overcome this problem, a solution is to adapt the definition of the division by constraining the possible elements of the result to belong to the dividend relation. So, the practical computation of the result can be performed even if the divisor is empty and the definition of the division becomes:

$$\text{div}(r, s, \{A\}, \{B\}) = \{x \mid (x \in \text{proj}(r, X)) \wedge (s \subseteq K_r(x))\} \tag{3}$$

where proj(r, X) stands for the projection of relation r over attribute X defined as:

$$\text{proj}(r, X) = \{x \mid \exists t \wedge (t \in r) \wedge (t.X = x)\} \tag{4}.$$

The characterization of a quotient is changed into:

$$\forall x, (x \in t) \Rightarrow (\text{prod}(\{x\}, s)) \subseteq r) \tag{5a}$$

$$\forall t1, (t1 = t \cup \{x\}) \wedge (x \in \text{proj}(r, X)) \Rightarrow (\text{prod}(\{x\}, s) \not\subseteq r)) \tag{5b}.$$

Expressions (5a) and (5b) express the fact that the relation (t) resulting from the division is a quotient, i.e., *the largest relation* over X whose Cartesian product with the divisor returns a result smaller than or equal to the dividend (according to regular set inclusion).

Example 1. Let us take a database involving the two relations order (o) and product (p) with respective schemas O(#p, store, qty) and P(#p, price). Tuples <n, s, q> of o and <n, pr> of p state that the product whose number is n has been ordered to store s in quantity q and that its price is pr. The query aiming at retrieving the stores which have been ordered all the products priced under $127 in a quantity greater than 35, can be expressed thanks to a division as:

div(o-g35, p-u127, {#p}, {#p})

where relation o-g35 corresponds to pairs (n, s) such that product n has been ordered to store s in a quantity over 35 and relation p-u127 gathers products whose price is under $127. From the following extensions of relations o and p:

o

#p	store	qty
15	32	50
12	32	68
34	32	49
26	32	78
26	7	120
78	7	30
12	7	96

p

#p	price
15	102
4	200
12	87
26	59
78	345
34	258

the relations o-g35 and p-u127 obtained are:

o-g35

np	store
15	32
12	32
34	32
26	32
26	7
12	7

p-u127

np
15
12
26

whose division using formula (3) leads to a result made of the single element {32}. It can easily be checked that this result satisfies expressions (2a) and (2b), or alternatively (5a) and (5b).♦

3 Principle for Adapting the Division to Fuzzy Relations

2.1 Fuzzy queries and relations

The context considered now is that of flexible queries where conditions call on preferences instead of Boolean criteria. The answer to such a query is made of a set of elements rank-ordered according to their accordance with the preferences. From now on, predicates of flexible queries are assumed to be modeled by fuzzy sets [1] and fuzzy relations are used instead of regular ones.

Formally, a fuzzy relation is defined as a fuzzy subset of the Cartesian product of domains of values. Hence, a fuzzy relation r whose schema is R(A, B, C) is made of a set of weighted triples denoted by $\mu_r(t)/t$, where t = <a, b, c> and $\mu_r(t)$ stands for the membership degree of t in relation r, i.e., its compatibility with the fuzzy concept associated with this relation. It is worth noticing that a regular relation is just a special case of a fuzzy relation where the degree attached to every tuple equals 1.

3.2 Principle for the extended division

By analogy with a query calling on a division such as that of example 1, one may envisage the query aiming at the determination of *the extent* to which any store has been ordered all the *fairly cheap* products in a *high* quantity, which is expressed thanks to a division of fuzzy relations, namely:

div(hq-o, fcp-p, {np}, {np})

where the degree attached to any tuple of hq-o (resp. fcp-p) expresses the compatibility of the quantity (resp. price) with high (resp. fairly cheap).

The extension of the division to fuzzy relations is based on the adaptation of formula (3) where on the one hand the regular implication is replaced by a fuzzy one (i.e., an application from $[0, 1]^2$ to [0, 1]), denoted by $\Rightarrow_f$, and the universal quantifier is interpreted by the infimum, on the other hand, the restriction of the calculus to the values present in the dividend accounts for the fact that the divisor is a fuzzy relation. This leads to:

$$\forall x \in \text{proj}(\text{supp}(r, X)),\ \mu_{\text{div}(r,\, s,\, \{A\},\, \{B\})}(x) = d = \deg(s \subseteq K_r(x)) \qquad (6)$$

where supp(r) denotes the support of the fuzzy relation r, i.e., the regular relation $\{t \mid \mu_r(t) > 0\}$, proj(supp(r, X)) represents the domain of X restricted to those values appearing in the dividend (r), $K_r(x)$ is defined as:

$$K_r(x) = \{\mu/a \mid \mu/\langle x, a\rangle \in r\},$$

and $\deg(E \subseteq F)$ denotes the degree of inclusion of E in F. Several types of degrees of inclusion exist depending on the approach adopted. The logical one is based on:

$$E \subseteq F \Leftrightarrow \forall x \in X, (x \in E) \Rightarrow (x \in F) \qquad (7)$$

which leads to:

$$\deg(E \subseteq F) = \inf_x \mu_E(x) \Rightarrow_f \mu_F(x) \qquad (8)$$

where $\Rightarrow_f$ is a fuzzy implication. Another one is founded on cardinalities of fuzzy sets, namely:

$$E \subseteq F \Leftrightarrow \text{card}(E \cap F) = \text{card}(E) \qquad (9)$$

which leads to a degree of inclusion expressing a ratio of cardinalities:

$$\deg(E \subseteq F) = \text{card}((E \cap F) / \text{card}(E) \qquad (10).$$

A third one stems from the equivalence:

$$E \subseteq F \Leftrightarrow ((E - F) = \emptyset) \qquad (11)$$

which gives birth to:

$$\deg(E \subseteq F) = 1 - h(E - F) \qquad (12)$$

where h(E) denotes the height of the fuzzy set E, i.e., the highest degree of its elements.

In the next sections, the properties of the result of extended division is studied with respect to the satisfaction of properties (5a) and (5b).

4 Using R-implications for the Division of Fuzzy Relations

4.1 Some reminders on R-implications

An R-implication, denoted by $\Rightarrow_{R\text{-}i}$, is defined as:

$$p \Rightarrow_{R\text{-}i} q = \sup_{[0,1]} \{u \mid \top(p, u) \leq q\} \qquad (13)$$

where $\top(a, b)$ stands for a triangular norm (operator associative, commutative, monotonic, whose neutral element is 1) extending the conjunction, or also:

$$p \Rightarrow_{R-i} q = \begin{vmatrix} 1 & \text{if } p \leq q \\ f(p, q) & \text{otherwise} \end{vmatrix}$$

where f(p, q) expresses a partial satisfaction (a value less than 1) when the threshold p is not reached by the conclusion part q. The minimal element of R-implications, called Gödel implication, is obtained by choosing $\top(a, b) = \min(a, b)$ in formula (13). It is defined as:

$$p \Rightarrow_{Gö} q = \begin{vmatrix} 1 & \text{if } p \leq q \\ q & \text{otherwise} \end{vmatrix}$$

Other representatives of R-implications are Goguen implication:

$$p \Rightarrow_{Gg} q = \begin{vmatrix} 1 & \text{if } p \leq q \\ q/p & \text{otherwise} \end{vmatrix}$$

obtained with $\top(a, b) = a \times b$, and Lukasiewicz implication:

$$p \Rightarrow_{Lu} q = \begin{vmatrix} 1 & \text{if } p \leq q \\ 1 - p + q & \text{otherwise} \end{vmatrix}$$

obtained with $\top(a, b) = \max(a + b - 1, 0)$.

4.2 R-implication and division of fuzzy relations

Combining expression (6) and the choice of an R-implication in expression (8), the following definition of the division of fuzzy relations is obtained:

$$\forall x \in \text{proj}(\text{supp}(r, X)),\ \mu_{\text{div}(r, s, \{A\}, \{B\})}(x) = d = \inf_s (\mu_s(a) \Rightarrow_{R\text{-}i} \mu_r(a, x)) \quad (14).$$

Assessing the fact that the result is a quotient entails an adaptation of the double characterization conveyed by formulas (5a) and (5b) in order to take into account that the relations are fuzzy, which yields:

$$\forall x, (x \in \text{proj}(\text{supp}(r), \{X\}) \wedge \mu_t(x) = d)) \Rightarrow (\text{prod}(\{d/\langle x\rangle\}, s)) \subseteq r \quad (15a)$$

$$\forall x, (x \in \text{proj}(\text{supp}(r), \{X\}) \wedge \mu_t(x) = d) \Rightarrow (\forall d1 > d, \text{prod}(\{d1/\langle x\rangle\}, s) \not\subseteq r) \quad (15b).$$

It is also necessary to specify the Cartesian product of fuzzy relations as well as the inclusion used in the previous two expressions. In the usual case, the Cartesian product of r and s is defined as:

$$r \times s = \{uv \mid (u \in r) \textbf{ and } (v \in s)\},$$

which can be extended to two fuzzy relations r and s as follows:

$$\mu_{r \times s}(uv) = \top(\mu_r(u), \mu_s(v)).$$

The inclusion is extended in a straightforward manner to fuzzy relations by:

$$r \subseteq s \Leftrightarrow \forall x \in X, \mu_r(x) \leq \mu_s(x))\}.$$

As the inequality $\top(p, (p \Rightarrow_{R\text{-}i} q)) \leq q$ holds, if one uses the norm which generates the R-implication in the Cartesian product, the following inclusion holds:

$$\text{prod}(\text{div}(r, s, \{A\}, \{B\}), s) \subseteq r.$$

Moreover, let x be an element of the result t of the division and let b be the value of s for which:

$$\mu_t(x) = \inf_s \mu_s(a) \Rightarrow_{R\text{-}i} \mu_r(a, x) = (\mu_s(b) \Rightarrow_{R\text{-}i} \mu_r(b, x)).$$

Then, the degree of <x, b> in $(t \times s)$ is:

$$\top(\mu_t(x), \mu_s(b)) = \top((\mu_s(b) \Rightarrow_{R\text{-}i} \mu_r(b, x)), (\mu_s(b)) \leq \mu_r(b, x).$$

But it is known that:

$$\forall u, (u > \mu_t(x)) \Rightarrow (\top(u, \mu_s(b)) > \mu_r(b, x)),$$

from which it can be deduced that the degree assigned to x is maximal and property (15b) holds.

Finally, it appears that the division of fuzzy relations based on R-implications delivers a quotient provided that the Cartesian product used for the characterization makes use of the norm which serves for generating the R-implication.

Example 2. Let us consider the two following fuzzy relations r and s whose respective schemas are R(A, X) and S(B):

r

A	X	μ
a1	x	0.7
a2	x	0.4
a3	x	1
a1	y	1
a2	y	0.6
a3	y	0.2

s

B	μ
a1	0.8
a2	0.5
a3	0.3

The result t of the division of r and s is successively computed with different R-implications in expression (14), using the previous extensions of the relations. With Gödel implication, the result t is:

$$\mu_t(x) = \inf(0.8 \Rightarrow_{Gö} 0.7, 0.5 \Rightarrow_{Gö} 0.4, 0.3 \Rightarrow_{Gö} 1) = \inf(0.7, 0.4, 1) = 0.4,$$

$$\mu_t(y) = \inf(0.8 \Rightarrow_{Gö} 1, 0.5 \Rightarrow_{Gö} 0.6, 0.3 \Rightarrow_{Gö} 0.2)$$

$= \inf(1, 1, 0.2) = 0.2.$

When performing the Cartesian product of t and s with the norm "minimum", one gets the relation:

{0.4/<a1, x>, 0.4/<a2, x>, 0.3/<a3, x>,
0.2/<a1, y>, 0.2/<a2, y>, 0.2/<a3, y>}

which is strictly included in r (formula (15a) holds). It is easy to check that formula (15b) holds as well, because of the presence of the tuples <a2, x> and <a3, y> whose grades equal those of r.

Similarly, with Goguen implication, the result of the division of r by s is:

$$\mu_t(x) = \min(0.8 \Rightarrow_{Gg} 0.7, 0.5 \Rightarrow_{Gg} 0.4, 0.3 \Rightarrow_{Gg} 1) = \min(7/8, 4/5, 1) = 0.8,$$

$$\mu_t(y) = \min(0.8 \Rightarrow_{Gg} 1, 0.5 \Rightarrow_{Gg} 0.6, 0.3 \Rightarrow_{Gg} 0.2) = \min(1, 1, 2/3) = 2/3.$$

The Cartesian product (using the norm "product") of t with the divisor s delivers the relation:

{0.64/<a1, x>, 0.4/<a2, x>, 0.24/<a3, x>,
(1.6/3)/<a1, y>, (1/3)/<a2, y>, 0.2/<a3, y>}

here again included in the dividend r. Moreover, it involves the tuples <a2, x> and <a3, y> with the same degree as in r, which guarantees that formula (15b) is satisfied.♦

5 Using S-implications for the Division of Fuzzy Relations

5.1 Some reminders on S-implications

An S-implication, denoted by $\Rightarrow_{S\text{-}i}$, is defined from a triangular co-norm $\perp$ (operator associative, commutative, monotonic, whose neutral element is 0) extending the disjunction by:

$$p \Rightarrow_{S\text{-}i} q = \perp(1 - p, q) \quad (16).$$

There is an infinity of such implications and the most common representatives of this family are Kleene-Dienes implication:

$$p \Rightarrow_{K\text{-}D} q = \max(1 - p, q)$$

obtained with $\perp(a, b) = \max(a, b)$ in (16), Reichenbach implication:

$$p \Rightarrow_{Rb} q = 1 - p + pq$$

obtained with $\perp(a, b) = a + b - ab$, and Lukasiewicz implication:

$$p \Rightarrow_{Lu} q = \min(1, 1 - p + q)$$

obtained with $\perp(a, b) = \min(1, a + b)$.

5.2 S-implication and division

By analogy with formula (14), the definition of the division of fuzzy relations using an S-implication is defined as:

$$\forall x \in \text{proj}(\text{supp}(r, X)), \mu_{div(r, s, \{A\}, \{B\})}(x) = d = \inf_s (\mu_s(a) \Rightarrow_{S\text{-}i} \mu_r(a, x)) \quad (17).$$

The question is to decide whether the result obtained can be characterized as a quotient in general. In others words, it is necessary to identify a norm associated with the Cartesian product in expressions (15a) and (15b) so that they hold. In this paper, we limit ourselves to prove that this is not feasible for the divisions based on Kleene-Dienes implication (which is the minimal S-implication), Reichenbach implication and the maximal S-implication using counter-examples. Of course, Lukasiewicz implication which is both an S-implication and an R-implication is compatible with the satisfaction of properties conveyed by expressions (15a) and (15b) provided that the norm:

$$\top(a, b) = \max(a + b - 1, 0)$$

is used for the Cartesian product.

Let us consider the relations:

r

A	X	μ
a1	x	1
a2	x	0.8

s

B	μ
a1	1
a2	0.5

The result t of the division of r by s with expression (17) using Kleene-Dienes implication is:

$$\mu_t(x) = \inf(1 \Rightarrow_{K\text{-}D} 1, 0.5 \Rightarrow_{K\text{-}D} 0.8) = \inf(1, 0.8) = 0.8.$$

The Cartesian product of t and s with the largest norm (the minimum) returns:

{0.8/<a1, x>, 0.5/<a2, x>}.

However, this relation is not maximal since the product of {0.9/<x>} and s which is:

{0.9/<a1, x>, 0.5/<a2, x>},

is also included in r. It can thus be deduced that in general the division using Kleene-Dienes implication does not deliver a quotient since expression (15b) does not hold.

Similarly, the result t of the division of r by s with expression (17) using Reichenbach implication is:

$$\mu_t(x) = \inf(1 \Rightarrow_{Rb} 1, 0.5 \Rightarrow_{Rb} 0.8)$$
$$= \inf(1, 0.9) = 0.9.$$

The Cartesian product of t and s with the largest norm (the minimum) returns:

{0.9/<a1, x>, 0.5/<a2, x>}.

However, this relation is not maximal since the product of {0.95/<x>} and s which is:

{0.95/<a1, x>, 0.5/<a2, x>},

is also included in r. It can thus be deduced that in general the division using Reichenbach implication does not deliver a quotient since expression (15b) does not hold.

Let us now consider the largest co-norm, known as Weber co-norm, defined as:

$$\perp_W(a, b) = \begin{cases} a & \text{if } b = 0 \\ b & \text{if } a = 0 \\ 1 & \text{otherwise} \end{cases}$$

its associated norm:

$$\top_W(a, b) = \begin{cases} a & \text{if } b = 1 \\ b & \text{if } a = 1 \\ 0 & \text{otherwise} \end{cases}$$

its associated S-implication:

$$p \Rightarrow_W q = \begin{cases} 1 - p & \text{if } q = 0 \\ q & \text{if } p = 1 \\ 1 & \text{otherwise} \end{cases}$$

and the relations:

r

A	X	μ
a1	x	1
a2	x	0.2

s

B	μ
a1	1
a2	0.4

The result t of the division of r by s with expression (17) using the previous implication is:

$$\mu_t(x) = \inf(1 \Rightarrow_W 1, 0.4 \Rightarrow_W 0.2) = \inf(1, 1) = 1.$$

The Cartesian product of t and s with the smallest norm ($\top_W$ above) returns the relation: {1/<a1, x>, 0.4/<a2, x>}, which is not included in the dividend r and expression (15a) does not hold.

5.3 Difference and S-implication based division

If expression (12) is used for defining an extended division, one has the following definition of the division of fuzzy relations:

$$\forall x \in \text{proj}(\text{supp}(r, X)), \mu_{div(r, s, \{A\}, \{B\})}(x) = d = 1 - h(s - K_r(x)) \quad (18).$$

If the following definition of the difference is taken:

$$\mu_{E-F}(x) = \top(\mu_E(x), 1 - \mu_F(x)),$$

we get:

$$\begin{aligned} d &= 1 - h(s - K_r(x)) \\ &= 1 - \sup_s \top(\mu_s(a), 1 - \mu_{K_r(x)}(a)) \\ &= 1 - \sup_s \top(\mu_s(a), 1 - \mu_r(a, x)) \\ &= \inf_s 1 - \top(\mu_s(a), 1 - \mu_r(a, x)) \\ &= \inf_s \perp(1 - \mu_s(a), \mu_r(a, x)) \\ &= \inf_s (\mu_s(a) \Rightarrow_{S\text{-}i} \mu_r(a, x)). \end{aligned}$$

So, it turns out that the approach for the division of fuzzy relations based on the difference of fuzzy sets is not original since it is already captured by the logical view using S-implications.

6 Using a Degree of Inclusion Based on cardinalities for the Division of Fuzzy Relations

If formula (10) is used as the basis for extending the division, the definition hereafter is obtained:

$$\mu_{div(r, s, \{A\}, \{B\})}(x) = \text{card}(s \cap K_r(x)) / \text{card}(s) = \Sigma_s \top(\mu_s(a), \mu_r(a, x)) / \Sigma_s \mu_s(a) \quad (19).$$

Here again, the result obtained is not a quotient. The reason lies in the fact that expression (15a) does not hold. This is illustrated by the following example where the smallest Cartesian product of the smallest result of the division and the divisor turns out to exceed the dividend.

Example 4. Let us consider the extensions:

r

A	X	μ
a1	x	0.4
a2	x	1
a3	x	1

s

B	μ
a1	1
a2	1
a3	1

Using Weber norm in expression (19), the division of r by s yields:

$$t = ((\top_W(0.4, 1) + \top_W(1, 1) + \top_W(1, 1))/(1 + 1 + 1))/x$$
$$= (2.4/3)/x = 0.8/x.$$

The Cartesian product of t with s based on Weber norm as well returns the relation:

$$\{0.8/\langle a1, x\rangle, 0.8/\langle a2, x\rangle, 0.8/\langle a3, x\rangle\}$$

which is not included in the dividend r.

7 Conclusion

The topic of this paper is the extension of the division to fuzzy relations. The key point dealt with concerns the properties of the result delivered by different approaches to the extended division. More precisely, we are interested in assessing whether the result is a quotient or not, i.e., the largest fuzzy relation which, once composed with the divisor, does not exceed the dividend. Such a property is a characteristic of the result of the division of integers and justifies the appropriateness of the term division.

Starting with the definition of the division of regular relations which calls on an inclusion, three main lines of extension are envisaged depending on the replacement of the inclusion by a degree of inclusion based on: i) an R-implication, ii) an S-implication or iii) a ratio of cardinalities. It turns out that the only sound extension is the one based on R-implications which delivers a quotient. None of the most used S-implications (except Lukasiewicz implication which is both an R and an S-implication) is satisfactory, nor the approach founded on the use of a degree of inclusion expressing a ratio of cardinalities, whatever the norm used to compute the ratio.

This work opens a number of perspectives. In particular, it would be interesting to have a better characterization of the status of S-implications. One may conjecture that none of them is acceptable except for those which are also R-implications (e.g., Lukasiewicz).

The division considered so far can be called a non fuzzy one since only the operand relations are fuzzy. An orthogonal approach for extending the division would be to soften the universal quantifier so as to define a truly fuzzy division based on the fuzzy linguistic quantifier "almost all". The question would then be to determine under which assumptions the result returned by such an approximate division is a quotient.

The same type of question would arise if the operand are no longer relations, but multi-relations, or even fuzzy multi-relations.

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