Notes on Linear Discrete-Space Systems and Undergraduate Education

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Abstract: - The cornerstone of the theory of discrete-space linear systems is the idea that every such system has an input-output map H that can be represented by a convolution sum or the familiar generalization of a convolution sum. This thinking involves an oversight which, for the case of bounded inputs mapped continuously into bounded outputs, was recently corrected by adding an additional term to the representation. Here we give a necessary and sufficient condition under which the additional term vanishes. The condition provides the basis for some related material given concerning engineering education and introductory courses in the area of signals and systems.

Key-Words: - linear systems, multidimensional systems, shift-varying systems, myopic maps.

1 Introduction

In the signal-processing literature, $x(\alpha)$ typically denotes a function. In the following we distinguish between a function x and $x(\alpha)$, the latter meaning the value of x at the point (or time) α. Sometimes a function x is denoted by $x(\cdot)$, and also we use Hx to mean $H(x)$. This notation is often useful in studies of systems in which signals are transformed into other signals.

The cornerstone of the theory of discrete-time single-input single-output linear systems is the idea that every such system has an input-output map H that can be represented by an expression of the form

$$(Hx)(n) = \sum_{m=-\infty}^{\infty} h(n,m)x(m) \tag{1}$$

in which x is the input and h is the system function associated with H in a certain familiar way. It is widely known that this, and a corresponding representation for time-invariant systems in which $h(n,m)$ is replaced with $h(n-m)$, are discussed in many books. Almost always it is emphasized that these representations hold *for all* linear input-output maps H. In [1] we direct attention to the fact that such statements are in error and we give a correct representation in which an additional term is added to the right side of (1).[1] This writer does not claim that H's for which the additional term is needed are necessarily of importance in applications, but he does feel that their existence shows that the analytical ideas in the books are flawed.[2]

More specifically, it is shown in [1] that

$$(Hx)(n) = \sum_{m=-\infty}^{\infty} h(n,m)x(m) + \lim_{k\to\infty}(HE_k x)(n)$$

for each n, in which h has the same meaning as in (1), and $E_k x$ denotes the function given by $(E_k x)(m) = x(m)$ for $|m| > k$ and $(E_k x)(m) = 0$ otherwise. This holds whenever the input set is the set of bounded functions, the outputs are bounded, and H is continuous (with respect to the usual sup norm). In particular, we see that in this setting, an H has a representation of the form given by (1) if and only if

$$\lim_{k\to\infty}(HE_k x)(n) = 0 \tag{2}$$

for all x and n. Since this is typically a very reasonable condition for a system map H to satisfy, it is clear that the H's that cannot be represented using just (1) are rather special.

The main result in [1] is actually more general in that H's are addressed for which inputs and outputs depend on an arbitrary finite number of variables. This case is of interest in connection with, for example, image processing. Also considered are H's for which inputs and outputs are defined on just the nonnegative integers because that case too arises often in applications. In that setting the situation with regard to the need for an additional term in the representation is different: no additional term is needed for causal maps H. Here, in Section 2.2, we show that the key condition (2) can be replaced with a condition

[1] It appears that as early as 1932 Banach was aware of the lack of existence of generalized convolution-sum representations for certain linear system maps (see [2, pp. 158, 159]). For further material concerning discrete-time systems, in the context of the theory of conjugate spaces, see e.g., [3, p. 228, Table 1 and p. 229, Exercise 9]. Also, in [4, p. 1159] attention is directed to material in [5, p. 58] which shows that certain time-invariant H's do not have convolution representations.

[2] As described in Section 2.3, the oversight in the books is due to the lack of validity of the interchange of the order of performing a certain infinite sum and then applying $(H\cdot)(n)$.

of the form

$$(Hx)(n) = \lim_{k\to\infty} (HQ_k x)(n)$$

for all x and n, where Q_k is a simple windowing map. This condition appears to be more palatable to students in introductory courses in the area of signals and systems. Section 2.3 contains some related comments concerning earlier work and engineering education. Throughout Section 2, we address the case in which inputs and outputs depend on an arbitrary finite number of variables.

2 The Equivalent Condition

2.1. Preliminaries

Let D denote either $\mathcal{Z}^d$ or $\mathcal{Z}_+^d$ where d is a positive integer, $\mathcal{Z}$ is the set of all integers, and $\mathcal{Z}_+$ is the set of nonnegative integers. We use $\ell_\infty(D)$ to denote the normed linear space of bounded complex-valued functions x defined on D with the norm $\|\cdot\|$ given by $\|x\| = \sup_{\alpha \in D} |x(\alpha)|$.

For each positive integer k, let c_k stand for the discrete hypercube $\{\alpha \in D : |\alpha_j| \le k \quad \forall j\}$ (α_j is the jth component of α), and let $\ell_1(D)$ denote the set of complex-valued maps g on D such that

$$\sup_k \sum_{\beta \in c_k} |g(\beta)| < \infty.$$

For each $g \in \ell_1(D)$ the sum $\sum_{\beta \in c_k} g(\beta)$ converges to a finite limit as $k \to \infty$, and we denote this limit by $\sum_{\beta \in D} g(\beta)$.

2.2. Main Result

Define maps Q_k and E_k from $\ell_\infty(D)$ into itself by $(Q_k x)(\alpha) = x(\alpha),\ \alpha \in c_k$ and $(Q_k x)(\alpha) = 0$ otherwise, and $(E_k x)(\alpha) = x(\alpha),\ \alpha \notin c_k$ and $(E_k x)(\alpha) = 0$ otherwise.

Let $\mathcal{H}$ be the set of all continuous linear maps from $\ell_\infty(D)$ into itself. For each $H \in \mathcal{H}$ define h on $D \times D$ by $h(\cdot, \beta) = H\delta_\beta$ for $\beta \in D$, where $(\delta_\beta)(\alpha) = 1$ for $\alpha = \beta$ and $(\delta_\beta)(\alpha)$ is zero otherwise. Of course $h(\cdot, \beta)$ is the response of H to a unit "impulse" occurring at $\alpha = \beta$.

We say that $H \in \mathcal{H}$ belongs to $\mathcal{H}_0$ if $h(\alpha, \cdot) \in \ell_1(D)$ for each $\alpha \in D$ and

$$(Hx)(\alpha) = \sum_{\beta \in D} h(\alpha, \beta) x(\beta), \quad \alpha \in D \qquad (3)$$

for $x \in \ell_\infty(D)$. As indicated in the Introduction, it is known that $\mathcal{H}_0$ is a proper subset of $\mathcal{H}$.

Theorem : An element of $\mathcal{H}$ belongs to $\mathcal{H}_0$ if and only if

$$(Hx)(\alpha) = \lim_{k\to\infty} (HQ_k x)(\alpha), \quad \alpha \in D \qquad (4)$$

for all $x \in \ell_\infty(D)$.

Proof:

We will use the following result (see [1]).

Lemma: Let $H \in \mathcal{H}$, $\alpha \in D$, and $x \in \ell_\infty(D)$. Then

(i) g defined on D by $g(\beta) = h(\alpha, \beta) x(\beta)$ belongs to $\ell_1(D)$.

(ii) $\lim_{k\to\infty} (HE_k x)(\alpha)$ exists and is finite.

(iii) We have

$$(Hx)(\alpha) = \sum_{\beta \in D} h(\alpha, \beta) x(\beta) + \lim_{k\to\infty} (HE_k x)(\alpha).$$

Returning to the proof of the theorem, let $H \in \mathcal{H}$ satisfy (4). Using (i) of the lemma, and with $x(\beta) = 1$ for all β, $h(\alpha, \cdot) \in \ell_1(D)$ for each α. By (iii) of the lemma,

$$(Hx)(\alpha) = \lim_{k\to\infty} (HQ_k x)(\alpha)$$

$$= \lim_{k\to\infty} \sum_{\beta \in D} h(\alpha, \beta)(Q_k x)(\beta) + \lim_{k\to\infty} \lim_{p\to\infty} (HE_p Q_k x)(\alpha)$$

(for each α and each x), in which the iterated limit on the extreme right side is zero, because $\lim_{p\to\infty} (HE_p Q_k x)(\alpha)$ is zero for each k. Also,

$$\lim_{k\to\infty} \sum_{\beta \in D} h(\alpha, \beta)(Q_k x)(\beta) = \lim_{k\to\infty} \sum_{\beta \in c_k} h(\alpha, \beta) x(\beta)$$

$$= \sum_{\beta \in D} h(\alpha, \beta) x(\beta) \qquad (5)$$

in which we have used the observation that each q_α defined on D by $q_\alpha(\beta) = h(\alpha, \beta) x(\beta)$ belongs to $\ell_1(D)$. This shows that $H \in \mathcal{H}_0$. Conversely, if $H \in \mathcal{H}_0$, (5) with the order of the three terms reversed holds for all α and x, showing that H satisfies (4). This completes the proof.

2.3. Comments

It is clear that $h(\alpha, \beta) = h(\alpha - \beta, 0)$ for shift-invariant maps $H \in \mathcal{H}$. If H_1 and H_2 are shift-invariant elements of $\mathcal{H}$ that do not belong to $\mathcal{H}_0$, it can happen that $H_1 H_2 \neq H_2 H_1$ (i.e., that H_1 and H_2 do not commute; see [6]).

Condition (4) has the interpretation that H is "myopic" in the sense that for each x and each α the value of $(Hx)(\alpha)$ must be (roughly speaking) relatively independent of the values of x at points remote from α. In this connection, it is shown in [4] that discrete-time single-input single-output shift-invariant causal maps have convolution-sum representations if and only if they possess fading memory. Fading memory in the sense of [4] is a comparatively more complicated concept. However, our theorem shows that for the maps considered in (4), and assuming they are continuous, the fading memory condition is equivalent to the condition that (4) is met. And for these causal maps, Q_k in (4) can be replaced with $Q_{k,\alpha}$ defined for $-k < \alpha$ by $(Q_{k,\alpha} x)(\beta) = x(\beta),\ \beta \in \{-k, -k+1, \ldots, \alpha\}$ and $(Q_{k,\alpha} x)(\beta) = 0$ otherwise. We use the term "myopic" in our interpretation of (4) because the term "fading-memory" is inappropriate when applied to noncausal systems, in that noncausal systems may anticipate as well as remember.

Using a version of the dominated convergence theorem, it is not difficult to modify the proof of our theorem to show that Q_k in (4) can be defined instead by

$$(Q_k x)(\alpha) = q_k(\alpha)x(\alpha), \quad \alpha \in D$$

where the functions q_k belong to $\ell_\infty(D)$ with unit norm and satisfy $\lim_{k\to\infty} q_k(\alpha) = 1$ for each α, as well as $\|E_p q_k\| \to 0$ as $p \to \infty$.

For any $H \in \mathcal{H} \cap \mathcal{H}_0$ one has

$$\sum_{\beta \in D} |h(\alpha, \beta)| \le \|H\|$$

for all α, where $\|H\|$ is the induced norm of H. This (essentially well-known fact) follows from the inequality $\|Hx\| \le \|H\|$ for $\|x\| = 1$ and a simple argument by contradiction.

We note that (4) is met if $D = \mathcal{Z}_+^d$ and H is "ξ-anticipative," by which we mean that there is a nonnegative number ξ such that for each α we have $(Hx)(\alpha)$ independent of $x(\beta)$ for $\beta_j > \alpha_j + \xi \quad \forall j$. In particular, (4) is satisfied when $D = \mathcal{Z}_+^d$ and H is causal (i.e., is 0-anticipative). However, with $D = \mathcal{Z}_+^d$ and H not necessarily causal, it can happen that (4) is not met, even with $d = 1$.

The theorem provides conditions under which the conclusion of a familiar short engineering argument of long standing can be justified. That argument, which is often taught to students and which concerns the representation of linear discrete-time or discrete-space shift-invariant systems, proceeds as follows (using our notation). Let H be the input-output map of such a system, and let X be the family of all possible input functions – assumed only to be complex-valued, and defined on $\mathcal{Z}^d$. Typically, $d = 1$. One writes

$$x(\alpha) = \sum_{\beta \in \mathcal{Z}^d} \delta(\alpha - \beta)x(\beta) \tag{6}$$

for any input x, in which δ is the usual discrete unit impulse function. Noting (6), and appealing to the linearity and shift-invariance of H, one is said to have

$$Hx = H \sum_{\beta \in \mathcal{Z}^d} \delta(\cdot - \beta)x(\beta) = \sum_{\beta \in \mathcal{Z}^d} H\delta(\cdot - \beta)x(\beta) \tag{7}$$

in which $H\delta(\cdot - \beta)$ is $h(\cdot - \beta)$, where h is the system's impulse response $H\delta$. Thus, one concludes that

$$(Hx)(\alpha) = \sum_{\beta \in \mathcal{Z}^d} h(\alpha - \beta)x(\beta) \tag{8}$$

for all $x \in X$. In particular, one concludes that the input-output properties of H are completely defined by its impulse response. As indicated in the Introduction, this conclusion is now known to be incorrect. The main problem with the argument just described is that the interchange of the order of summation and operation by H in (7) is in fact not justified by merely the linearity of H. Linearity (in particular, the superposition part of linearity) concerns H operating on finite sums, not infinite sums [3].

[3] It is known [7] that what might be called "infinite superposition" can fail.

The engineering argument we have described is sometimes slightly modified to address system maps H that are not necessarily shift invariant, and the well-known conclusion is that the corresponding representation takes the form

$$(Hx)(\alpha) = \sum_{\beta \in \mathcal{Z}^d} h(\alpha, \beta)x(\beta). \tag{9}$$

This is (8) with $h(\alpha - \beta)$ replaced with some $h(\alpha, \beta)$ and, as is well known, $h(\alpha, \beta)$ is interpreted to be the response of H at the point (or time) α to an impulse applied at the point (or time) β. Our theorem shows that (9) is valid for the important family of $\ell_\infty(D)$ inputs, under the assumption that H satisfies certain continuity and mapping conditions, and a certain key limit condition.

With regard to engineering education, and introductory courses in the area of signals and systems, our theorem provides the following outline of a revised argument that yields (9). We have

$$x(\alpha) = \sum_{\beta \in \mathcal{Z}^d} \delta(\alpha - \beta)x(\beta) \tag{10}$$

and so

$$Hx = H \sum_{\beta \in \mathcal{Z}^d} \delta(\cdot - \beta)x(\beta).$$

Using the linearity of H, and under the additional assumptions that the input set X is $\ell_\infty(D)$, that all outputs belong to $\ell_\infty(D)$, that H is continuous, and that condition (4) is met, the order of performing the summation and operating by H can be shown to be able to be interchanged, yielding

$$Hx = \sum_{\beta \in \mathcal{Z}^d} H\delta(\cdot - \beta)x(\beta).$$

And with h given by $h(\alpha, \beta) = [H\delta(\cdot - \beta)](\alpha)$, this becomes (9). Related outlines have been given concerning shift-invariant H's in continuous-space input settings (see e.g., [8] and [9]).

2.4. Conclusion

The cornerstone of the theory of discrete-space linear systems is the idea that every such system has an input-output map H that can be represented by a convolution sum or the familiar generalization of a convolution sum. This thinking involves an oversight which, for the case of bounded inputs mapped continuously into bounded outputs, was recently corrected by adding an additional term to the representation. We have given a necessary and sufficient condition under which the additional term vanishes. The condition has provided the basis for some related material given concerning engineering education and introductory courses in the area of signals and systems.

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