4th WSEAS Int. Conf. on NON-LINEAR ANALYSIS, NON-LINEAR SYSTEMS and CHAOS, Sofia, Bulgaria, October 27-29, 2005 (pp1-7)

On special types of two- and three-parametric bifurcations in piecewise–smooth dynamical systems

VIKTOR AVRUTIN and MICHAEL SCHANZ
Institute of Parallel and Distributed Systems, University of Stuttgart
Universitätstrasse 38, 70569 Stuttgart, GERMANY

Abstract: - The aim of this paper is to present a brief overview about a special kind of two-parametric (or co-dimension two) bifurcations in piecewise–smooth dynamical systems. The characteristic property of these bifurcations is, that at the bifurcation point in a 2D parameter space an infinite number of bifurcation curves intersect. Several types of these bifurcations are discussed. Additionally, a new type of three parametric (or co-dimension three) bifurcations is reported.

Key–Words: - multi-parametric bifurcations, co-dimension two and three bifurcations, big bang bifurcations, period increment, period adding, piecewise smooth systems

1 Introduction

Piecewise–smooth dynamical systems are in the meanwhile well-known to be important both from the theoretical as well as from the applications' point of view [1, 2]. Investigating 1D parameter spaces of these systems, one observes often bifurcation scenarios, which are very difficult to explain [3]. It turns out, that in many cases these scenarios represent 1D traces or sections of multi-parametric bifurcations, which can be adequately described only by investigating the coresponding high-dimensional parameter spaces. Until now there are many open questions related to multi-parametric bifurcations in piecewise–smooth systems [4]. In this work we present an overview about a special kind of two-parametric bifurcations and introduce a new type of three-parametric bifurcations, which can be observed in these systems. In this work, we use scalar piecewise–smooth maps, because they are more simple to investigate and allow much more analytical calculations to unearth the basic principles and mechanisms of the occurring bifurcation phenomena. However, these bifurcation phenomena, are observed also in many other dynamical systems, especially in high-dimensional ones and in dynamical systems continuous in time.

2 Basic definitions

Let us firstly summarize the basic definitions and notions we use in the following. According to standard definitions, a *bifurcation* is a qualitative change of the asymptotic dynamics of a system, induced by an infinitesimal variation of the system parameters. Many bifurcations (usually denoted as *local*) can be described using the linear stability analysis, whereas for other bifurcations (usually denoted as *global*) this is not possible. A special kind of global bifurcations, which occurs in piecewise–smooth dynamical systems, are *border–collision bifurcations* (see for instance [5, 6, 7, 8]. For an actual overview see [4] and references therein). We denote as the *domain* of a bifurcation the set of parameter values, for which this bifurcation occurs. Typically, a bifurcation's domain represents an $(n-1)$-dimensional subspace of the n-dimensional parameter space. In this case, the bifurcation is called *one-parametric* often denoted also as *codimension one* bifurcation [9]. Especially, in one-dimensional parameter spaces, the domains of one-parametric bifurcations are points (usually denoted as bifurcation points). Obviously, in this case there are exactly two different asymptotic dynamics in in the vicinity of the bifurcation point. In two-dimensional parameter spaces the domains of one-dimensional bifurcations are represented by curves, in three-dimensional parameter spaces by surfaces, etc.. In the case, that the dimension of the bifurcation's domain is less than $n-1$, the bifurcation is denoted as *multi-parametric*. More precisely, if the domain of a bifurcation has the dimension m, then the bifurcation is $(n-m)$-parametric or has the codimension $(n-m)$. Obviously, such bifurcations can be observed only by investigation of n-dimensional parameter spaces with $n \geq 2$. A typical property of these bifurcations is, that in their vicinity may exist more than two different asymptotic dynamics. However, this condition is neither necessary nor sufficient for multi-parametric bifurcations.

Until now, the most of known results are related to the case $n = 2$. In this case the domain of a two-parametric bifurcation is given by a point, where some number of one-parametric bifurcation curves intersect each other. An example for this phenomenon is the Hopf-Hopf bifurcation, which represents the intersection of two Hopf bifurcation curves. One can say, that the Hopf-Hopf bifurcation is induced by two Hopf bifurcation occurring at the same point in the parameter space. Several other two-parametric bifurcations, induced by local bifurcations, are known for instance, the cusps, induced by two saddle-node bifurcations or the Hopf-pitchfork bifurcation. In piecewise–smooth dynamical systems multi-parametric bifurcations can be induced by border–collision bifurcations as well.

A special type of two-parametric bifurcations, denoted as *big bang bifurcation*, is given by the case, that at some point in the parameter space an infinite number of bifurcation curves intersect [10]. These bifurcations can be induced by both local as well as global bifurcations. The big bang bifurcations occurring in piecewise–smooth dynamical systems, are typically induced by border–collision bifurcations. However we show also an example, where a big bang bifurcation is induced by simple local bifurcations.

Remarkably, the used definition of the big bang bifurcation does not specify, which kind of asymptotic dynamics can be observed in the vicinity of the bifurcation point. A more precise classification of big bang bifurcations can be obtained according to the one-parametric bifurcation scenario taking place along the border of an infinite small convex open vicinity of the bifurcation point. Until now, we have found some dynamical systems, which show along this border such bifurcation scenarios, as period doubling, period increment and period adding [11]. According to these scenarios, we denote the corresponding two-parametric bifurcations as period doubling, period increment and period adding big bang bifurcations.

3 Two-parametric bifurcations

3.1 Period doubling big bang bifurcation

One of the most well-known bifurcation scenarios is the period doubling scenario, which represents an infinite sequence of flip bifurcations, where

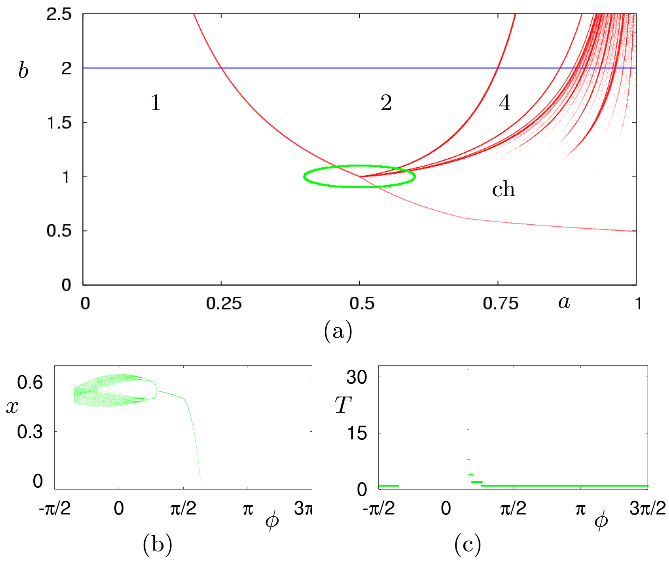

Figure 1: *Power law map (1): (a) Period doubling big bang bifurcation at $a = 1/2$, $b = 1$. Bifurcation diagram (b) and period diagram (c) of the period doubling scenario along the circle around the bifurcation point marked in (a). Numbers in (a) and in all following figures mark the areas in the 2D parameter space leading to corresponding periods, 'ch' marks the area with chaotic behavior.*

the period after the nth bifurcation is given by $p_n = p_0 2^n$, $n = 1, 2, \ldots$. It turnes out, that there exists a two-parametric bifurcation (which we denote as a period–doubling big bang bifurcation), that shows this scenario along the border of an infinite small convex open vicinity of the bifurcation point. In order to demonstrate this, let us consider the *power low* map, defined by

$$x_{n+1} = a2^b \left((1/2)^b - |x_n - 1/2|^b \right) \qquad (1)$$

on the interval $[0, 1]$. Hereby the parameter a determines the value of the system function at its maximum for $x = 1/2$, whereas the parameter b determines, whether the maximum is smooth (for $b > 1$) or not (otherwise). In the special case $b = 2$ the map (1) is identical with the *logistic map*

$$x_{n+1} = 4ax_n(1 - x_n) \qquad (2)$$

and for $b = 1$ with the *tent map*

$$x_{n+1} = \begin{cases} 2ax & \text{if} \quad x < 1/2 \\ 2a(1-x) & \text{if} \quad x \geq 1/2 \end{cases} \qquad (3)$$

The period–doubling big bang bifurcation occurs in system (1) at the point $a = 1/2$, $b = 1$. In this case the co-dimension two bifurcation is induced by local bifurcations, namely a transcritical one and an infinite number of flip bifurcations. Note, that the horizontal line $b = 2$ in Fig. 1 corresponds to the period–doubling scenario of the logistic map. Decreasing the value of b and varying a, one observes, how this scenario shrinks to one point

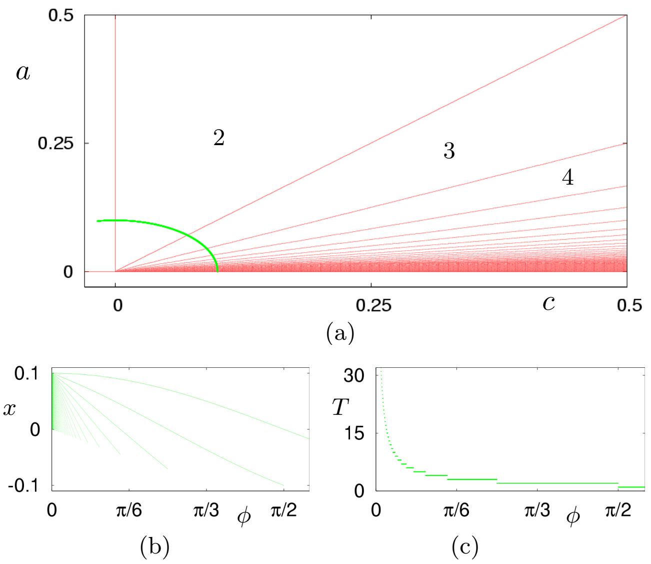

Figure 2: *Piecewise-linear map (4): case $b = 0$: (a) Period increment big bang bifurcation at $a = 0$, $b = 0$, $c = 0$. Bifurcation diagram (b) and period diagram (c) of the period increment scenario along the circle around the bifurcation point marked in (a).*

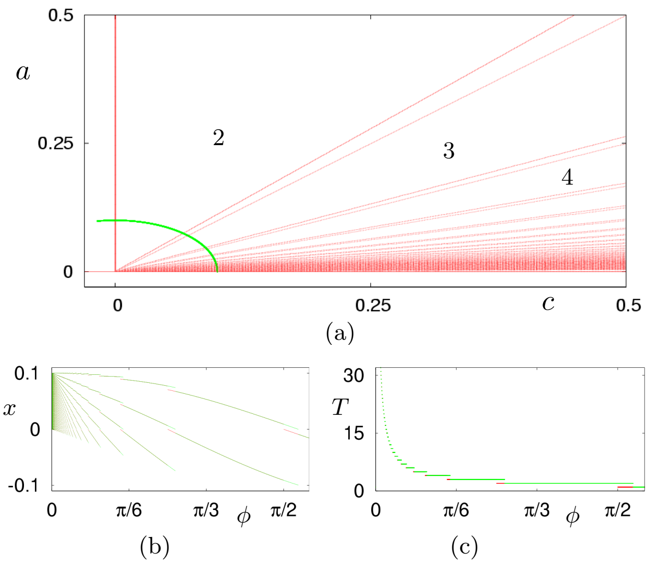

Figure 3: *Piecewise–linear map (4): case $-1 < b < 0$: (a) Period increment big bang bifurcation with coexisting attractors at $a = 0$, $b = -0.1$, $c = 0$. Bifurcation diagram (b) and period diagram (c) of the period increment scenario along the curve around the bifurcation point marked in (a).*

at the line $b = 1$. We remark additionally, that the big bang bifurcation occures in system (1) at that value of the parameter b, where the system function undergoes the transition from smooth to non-smooth. This seems to be typical for these bifurcations. As a practical example let us refer to the impact oscillator described in [12], where a big bang bifurcation occurs at the point, where the system undergoes a transition from impactless motion to the one with impacts.

3.2 Period increment big bang bifurcation

Another bifurcation scenario, which can be often observed, especially when dealing with piecewise–smooth dynamical systems, is the period increment scenario. It represents an infinite sequence of bifurcations, where the period after the nth bifurcation is given by $p_n = p_0 + n\Delta p$, $n = 1, 2, \ldots$. Typically, the bifurcations leading to this scenario are border–collision bifurcations. Remarkably, there exists a two-parametric bifurcation (denoted as period increment big bang bifurcation), which shows such a scenario along the border of the vicinity of the bifurcation point. One of the most simple dynamical systems, demonstrating this bifurcation, is the following *piecewise–linear map*:

$$x_{n+1} = \begin{cases} bx + c & \text{if} \quad x < 0 \\ x - a & \text{if} \quad x \geq 0 \end{cases} \tag{4}$$

with $a > 0$, $|b| < 1$, $c > 0$. The main advantage of system (4) is, that the n-periodic limit cycles and their stability areas can be calculated analytically for arbitrary n. As shown in [13], in the case $b = 0$ the structure of the $[a \times c]$-plane is completely determined by a single period adding big bang bifurcation at $a = 0$, $c = 0$. Additionally map (4) clarifies, why the existence of an infinite number of different asymptotic dynamics in the vicinity of a point in parameter space is not sufficient for a multi-parametric bifurcation. Namely, in any vicinity of each point on the cumulation line $a = 0$ there is an infinite number of different limit cycles, but only the point $a = 0$, $c = 0$ represents a big bang bifurcation.

3.3 Period increment big bang bifurcation with coexistence of attractors

A typical phenomenon related to the period increment scenario, is the coexistence of subsequent limit cycles. As in the case of the usual period increment scenario, there exists a sequence of limit cycles with periods $p_0 + n\Delta p$, $n = 1, 2, \ldots$. However, the bifurcation scenario represents in this case a sequence of pairs of bifurcations. At the first bifurcation of the nth pair the limit cycle with period $p_0 + n\Delta p$ emerges, but the previous limit cycle with period $p_0 + (n-1)\Delta p$ is not affected by this bifurcation. This limit cycle is destroyed at the second bifurcation of the nth pair, and until the first bifurcation of the $(n+1)$th pair the limit cycle with period $p_0 + n\Delta p$ represents the only attractor.

The big bang bifurcation showing the described scenario occures in system (4) for $-1 < b < 0$. As in the case $b = 0$ this bifurcation dominates the complete parameter plane $[a \times c]$.

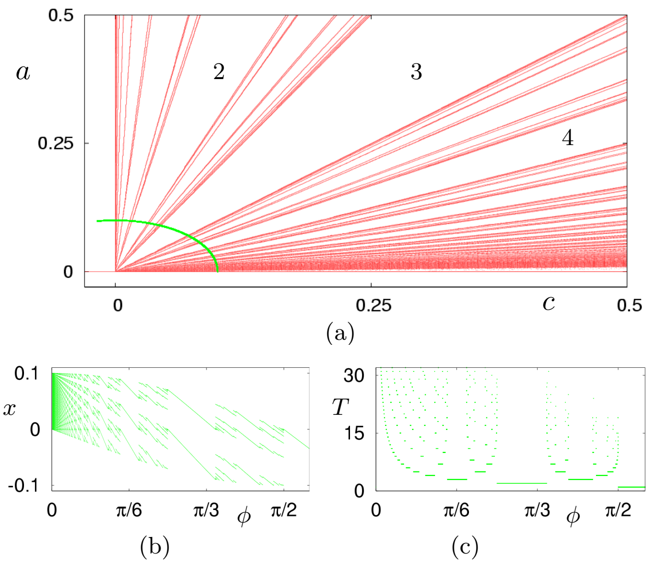

Figure 4: *Piecewise-linear map (4): case $0 < b < 1$: (a) Period adding big bang bifurcation at $a = 0$, $b = 0.5$, $c = 0$. Bifurcation diagram (b) and period diagram (c) of the period increment scenario along the curve around the bifurcation point marked in (a).*

3.4 Period adding big bang bifurcation

Another bifurcation scenario, which can be often observed, in piecewise–smooth dynamical systems, is the period adding. In this case there exists an infinite number of limit cycles with different periods, whereby between the parameter intervals leading to limit cycles with periods p_1 and p_2 there exists a parameter interval leading to a limit cycle with period p_1+p_2. The mechanism leading to this scenario and its relationship to the well-known devil's staircases and Farey trees are described in [14]. Therein it is also shown, that there exists an unified framework, namely an *infinite adding scheme*, which allows the calculation of all periods existing between p_1 and p_2. However, it is more important, that there exists a two-parametric bifurcation showing such a scenario along the border of the vicinity of the bifurcation point. This period adding big bang bifurcation can be observed in system (4) for $b > 0$, as shown in Fig 4.

Note, that the structures of the 2D parameter spaces of the dynamical systems discussed so far are relative simple. As a more complex example one can consider the piecewise-quadratic map, investigated in [14]:

$$x_{n+1} = \begin{cases} \beta x_n(1-x_n) & \text{if} \quad x_n < 1/2 \\ 1-\alpha x_n(1-x_n) & \text{if} \quad x_n \geq 1/2 \end{cases} \qquad (5)$$

The structure of the parameter space $[\alpha \times \beta] = [0,4]^2$ is dominated here by an infinite number of

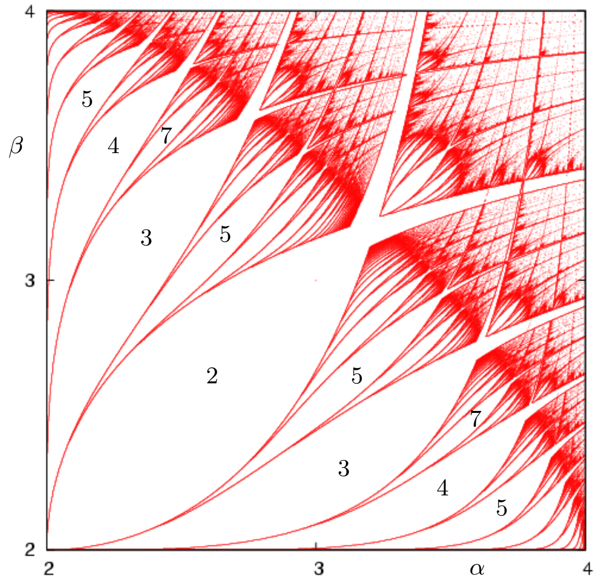

Figure 5: *Piecewise-quadratic map (5): Period adding big bang bifurcation at $\alpha = \beta = 2$ and an infinite number of further big bang bifurcations. In the middle of the figure the area leading to period 2 can be observed. This area has a characteristic shape consisting of a main body and two longish bands. Within this area, at $\alpha = \beta = 3$, a cusp occurs, and at the boundary, between the two longish bands occurs the next period adding big bang bifurcation. The areas leading to all other periods, have the same properties.*

period adding big bang bifurcations and an infinite number of cusp catastrophes. The interaction between these two types of two-parametric bifurcations lead to a striking self-similarity of the parameter space of map (5) (see Fig. 5).

4 Three-parametric bifurcations

Until now we have described some types of two-parametric bifurcations, observed by the investigation of piecewise-smooth maps. In 2D parameter spaces the domains of these bifurcations are represent singular points. Of course, in a 3D parameter space, these domains become curves. As one can expect, at these curves some points may exist, where the corresponding bifurcation is three-parametric.

In order to demonstrate this, let us consider the complete 3D parameter space $[a \times b \times c]$ of the map (4), and revise the results discussed so far from this more general perspective. A schematic representation of an area in this space is shown in Fig. 6. The subareas leading to stable limit cycles with periods n and $n+1$ are represented by the polyeders $A_1A_3A_0C_0C_1C_2$ and $A_2A_5A_0C_0C_3C_4$ respectively. As shown in Secs. 3.2, 3.3 and 3.4, below the plane $b = 0$ these polyeders overlap, and hence in the overlapping area $A_2A_3A_0B_0B_2$ the limit cycles with periods n and $n+1$ coexist. By varying parameter b from negative to pos-

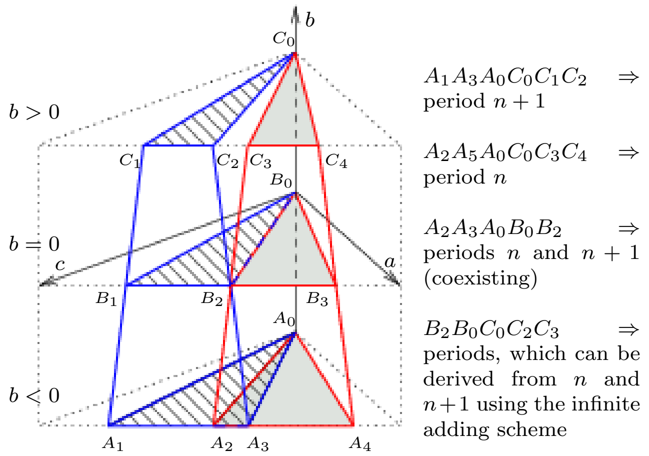

Figure 6: *Piecewise-linear map (4): Schematic representation of the positions in the 3D parameter space corresponding to the two characteristic areas leading to periods n and $n+1$ with respect to each other. The lines A_0C_0, B_0B_1, B_0B_2 and B_0B_3 represent domains of two-parametric bifurcations, the point B_0 represents the domain of a three-parametric one.*

itive values (upwards in Fig. 6), the areas in the plane $[a \times c]$, leading to these limit cycles, shrink, and consequently also the overlapping area. It is clearly shown, that the plane $b = 0$ bounds the overlapping area from above, so that there are no coexisting limit cycles for parameter values $b \geq 0$. For increasing positive values of the parameter b, the abovementioned areas continue to shrink, and the area in between (represented in Fig. 6 by the polyeder $B_2B_0C_0C_2C_3$) increases. As we already know, the limit cycles with high periods, that can be derived from periods n and $n+1$ using the infinite adding scheme, emerge in this area.

Another view on the described phenomenon is presented in Fig. 7. Here the parameters a and c are varied along the curve marked in Figs. 2.(a), 3.(a) and 4.(a), given by a part of a circle ($a = R\sin(\phi)$, $c = R\cos(\phi)$ with $R = 0.1$ and $\phi = 0..\pi/2$). Additionally the parameter b is varied as well. Hence, the bifurcation scenarios presented in Figs. 2.(b), 3.(b) and 4.(b) take place along the horizontal lines marked in Fig. 7. As one can see, along the line $b = 0$, the map (4) shows an infinite sequence of period adding big bang bifurcations. Note, that all these period adding big bang bifurcation points represent the intersections of the curve marked in Fig. 2.(a) with the bifurcation lines originating from the period increment big bang bifurcation point shown in this figure. It can be shown analytically, that the structure presented in Fig. 7 does not depend on the radius R and hence, that for $R \to 0$ all these bifurcations collapse to the origin $a = b = c = 0$.

As a consequence, the point $a = b = c = 0$ is the

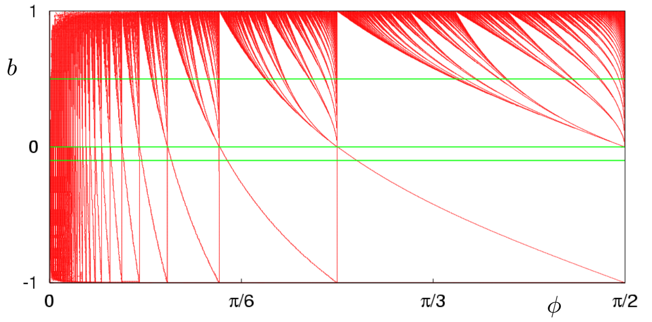

Figure 7: *Piecewise-linear map (4): structure of the cylindric surface around the b-axis of the 3D parameter space $[a \times b \times c]$. The marked horizontal lines correspond to Figs. 2, 3 and 4. Along the line $b = 0$ an infinite sequence of period adding big bang bifurcations can be observed.*

most important point in the 3D parameter space, because all the bifurcations domains originate here. Recall, that each point on the b-axis represents a big bang bifurcation point, whereby the type of this bifurcations changes at the origin, i.e., at $b = 0$. For this reason, the bifurcation occurring at the origin is three-parametric. Recall further, that each point on each of the infinite number of bifurcation lines in the plane $b = 0$ represents also a big bang bifurcation point. This follows directly from the geometrical properties of the presented structure.

Three-parametric bifurcations of the described type occur in several dynamical systems, therefore they have to be investigated in more detail. We state, that a bifurcation of this type has two characteristic manifolds in the 3D parameter space, a 1D and a 2D one. Each point on the 1D characteristic manifold except the three-parametric bifurcation point itself, represents a point of a two-parametric bifurcation, which is a period adding big bang on the one side of the 2D characteristic manifold and a period increment big bang with coexisting attractors on the other side. Additionally, in the 2D characteristic manifold the three-parametric bifurcation represents a pure period increment big bang bifurcation. Hereby, each of the infinite number of bifurcation curves, generated by this pure period increment big bang bifurcation represents a domain of period adding big bang bifurcations. In the case of system (4), the 1D manifold is given by the axis b and the 2D manifold by the plane $b = 0$.

Another example for the described three-parametric bifurcations is demonstrated by the map

$$f(x) = \begin{cases} \alpha x + \mu & \text{if} \quad x < 0 \\ \beta x + \mu + 1 & \text{if} \quad x \geq 0 \end{cases} \tag{6}$$

equivalent to the one investigated in [3]. In the cited work it is shown, that map (6) shows a broad spectrum of dynamic behaviors, both chaotic as

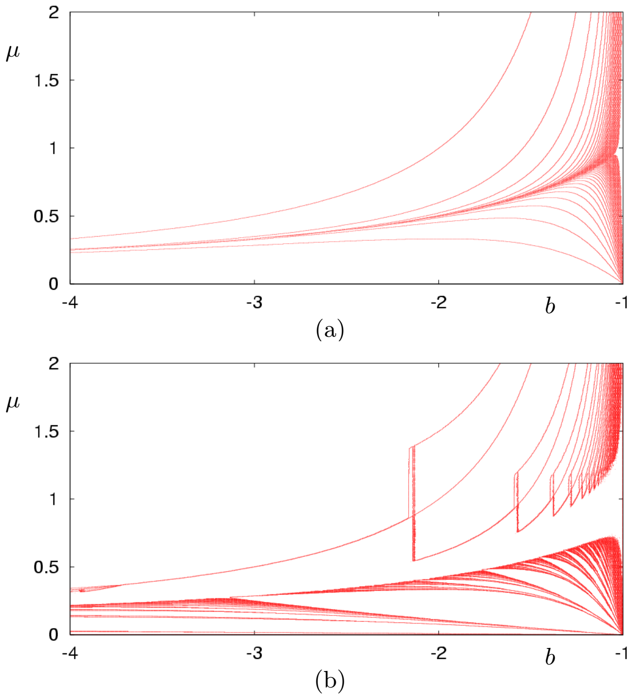

Figure 8: *Piecewise-linear map (6): (a) Plane* $a = 0$*, representing the common 2D manifold of the three-parametric bifurcations at* $\alpha = 0$*,* $\beta = -1$*,* $\mu = 0$ *and* $\alpha = 0$*,* $\beta = -1$*,* $\mu \to \infty$*. In this plane both bifurcations are represented by period increment big bang bifurcations. (b) Plane* $a = 0.1$*. The first three-parametric bifurcation leads in the plane* $a = 0.1$ *to the period adding big bang bifurcation at* $\beta = -1$*,* $\mu = 0$*, whereas the second one leads in this plane to the period increment big bang bifurcation with coexisting attractors at* $\beta = -1$*,* $\mu \to \infty$*.*

well as periodic. The periodic behaviors of this map can be explained easily, taking into account, that the map undergoes at least two three-parametric bifurcations of the type described above. The first bifurcation occurs at $\alpha = 0$, $\beta = -1$, $\mu = 0$, and the second one at $\alpha = 0$, $\beta = -1$, $\mu \to \infty$. Remarkably, both bifurcations have the same 2D characteristic manifold, given by the plane $\alpha = 0$ (shown in Fig. 8.(a)), but the "direction" of their 1D characteristic manifolds are different. Therefore, for $\alpha > 0$ the two-parametric bifurcation at $\beta = -1$, $\mu = 0$ is a period adding big bang bifurcation, and the one at $\beta = -1$, $\mu \to \infty$ is a period increment one (Fig. 8.(b)). For $\alpha < 0$ the situation is vice versa, i.e. the two-parametric bifurcation at $\beta = -1$, $\mu = 0$ is a period increment big bang bifurcation, and the one at $\beta = -1$, $\mu \to \infty$ is a period adding one.

5 Summary

In this paper we discussed special types of two-parametric bifurcations occurring in piecewise-smooth dynamical systems. These bifurcations (big bang bifurcations) occur at the points in a 2D parameter space, where an infinite number of bifurcation curves intersect. It is shown, that big bang bifurcations can be induced by both local as well as global bifurcations. Additionally, a special type of three-parametric bifurcations is described.

6 Acknowledgment

We would like to thank Prof. Soumitro Banerjee from the Indian Institute of Technology for very helpful hints and discussions and Prof. Paul Levi from the University of Stuttgart for his support of our work.

References

[1] S. Banerjee and G.C. Verghese, editors. *Nonlinear Phenomena in Power Electronics — Attractors, Bifurcations, Chaos, and Nonlinear Control.* IEEE Press, 2001.

[2] Zh. T. Zhusubaliyev and E. Mosekilde. *Bifurcations and Chaos in piecewise-smooth dynamical systems*, volume 44 of *Nonlinear Science A*. World Scientific, 2003.

[3] P. Jain and S. Banerjee. Border collision bifurcations in one-dimensional discontinuous maps. *Int. J. Bif. & Chaos*, 13(11):3341–3352, 2003.

[4] P. Kowalczyk, M. di Bernardo, A. R. Champneys, S. J. Hogan, M. Homer, Yu. A. Kuznetsov, A. Nordmark, and P. Piiroinen. Two-parameter nonsmooth grazing bifurcations of limit cycles: classification and open problems. Preprint 2005.08 of Bristol Centre for Applied Nonlinear Mathematics, 2005.

[5] H. E. Nusse and J. A. Yorke. Border-Collision Bifurcations including 'period two to period three' Bifurcation for Piecewise Smooth Systems. *Physica*, D57:39–57, 1992.

[6] H. E. Nusse, E. Ott, and J. A. Yorke. Border-collision bifurcations: An explanation for observed bifurcation phenomena. *Phys. Rev.*, E49(2):1073–1076, 1994.

[7] W. Chin, E. Ott, H. E. Nusse, and C. Grebogi. Grazing bifurcations in impact oscillators. *Phys. Rev.*, E50(6):4427–4444, 1994.

[8] M. di Bernardo, C. J. Budd, and A. R. Champneys. Grazing and Border-Collisions in Piecewise-Smooth Systems: A Unified Analytical Framework. *Phys. Rev. Lett.*, 86(12):2554–2556, 2001.

[9] Y. Kuznetsov. *Elements of Applied Bifurcation Theory.* Springer, 2004. third edn.

[10] V. Avrutin, G. Wackenhut, and M. Schanz. On dynamical systems with piecewise defined system functions. In *Proc. of Int. Conf. "'Tools for Mathematical Modelling"' MATHTOOLS'99*, 1999.

[11] V. Avrutin. *On behavior of dynamical systems with piecewise-smooth system function.* PhD thesis, University of Stuttgart, 2004. (in german).

[12] F. Peterka. Bifurcations and transition phenomena in an impact oscillator. *Chaos, Solitons & Fractals*, 7(10):1635–1647, 1996.

[13] V. Avrutin and M. Schanz. On multi-parametric bifurcations in a scalar piecewise-linear map. *submitted to Nonlinearity*, 2005.

[14] V. Avrutin and M. Schanz. Self-similar structures in 2D parameter space and the concept of big bang bifurcations. *submitted to Phys. Rev. E*, 2005.