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Charge-5 Soliton Collisions on T_2

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Abstract

We consider topological charge-5 soliton collisions in the nonlinear $O(3)$ sigma model on a flat torus T_2. Only for tightly localised configurations at the initial time do we observe dual-polygon scattering.

Key-Words: Soliton, scattering, topological charge, sigma model.

1 Introduction

The nonlinear $O(3)$ or CP^1 sigma model in two spatial dimensions possesses soliton solutions (or energy lumps) which are extensively studied for their various applications in field theory, condensed matter and other branches of physics [1].

A most interesting charasteristic of topological defects of the sigma type, like vortices, strings, skyrmions, *etc.*, is their dual-polygon scattering: when N identical solitons are placed at the vertices of a regular polygon and sent with equal speed to the centre of the polygon, they scatter off and move towards the vertices of the dual polygon, *ie*, the scattering angle is π/N [2, 3]. Such behaviour may be explained in terms of the symmetry of the initial configuration under the transformations of the dihedral group D_N, when the solitons dwell in an isotropic manifold as S_2, in the planar case.

Limiting ourselves to the homotopy class $N = 5$, in the present paper we continue our investigation [4] of lump scattering in the $O(3)$ model on a flat torus T_2. Because the latter does not possess the D_5 symmetry of the initial soliton state, dual scattering does not have to take place in general. We are interested on how the anisotropy of the torus affects the scattering angle.

Since the full time-dependent $O(3)$ model has no known analytical solutions, our study relies on the numerical simulations of the time evolution.

2 The Model

The non-linear $O(3)$ sigma model in (2+1) dimensions involves three real scalar fields $\vec{\phi}(x^\mu) \equiv <\phi_1, \phi_2, \phi_3>$ with the constraint that for all $x^\mu \equiv (x^0, x^1, x^2) = (t, x, y)$ the fields lie on the unit sphere S_2: $\vec{\phi}.\vec{\phi} = 1$. Subject to this constraint the lagrangian density and the equations of motion are

$$\mathcal{L} = \frac{1}{4}(\partial_\mu \vec{\phi}).(\partial^\mu \vec{\phi}), \quad (1)$$

$$\partial^\mu \partial_\mu \vec{\phi} + (\partial^\mu \vec{\phi}.\partial_\mu \vec{\phi})\vec{\phi} = \vec{0}. \quad (2)$$

For any value of t, the fields $\vec{\phi}$ are harmonic maps $T_2 \mapsto S_2$ satisfying the periodic boundary conditions

$$\vec{\phi}(x + mL, y + nL) = \vec{\phi}(x, y), \quad (3)$$

where $m, n = 0, 1, 2, ...$, and the period L denotes the size of the square torus.

We conveniently use the CP^1 formulation of the model, consisting of one independent complex field W related to $\vec{\phi}$ via the stereographic projection $W = \frac{\phi_1 + i\phi_2}{1 - \phi_3}$. Introducing complex coordinates $z = x_1 + ix_2$ on the torus the boundary conditions (3) take the form

$$W(z + mL + inL) = W(z). \quad (4)$$

The <u>static</u> N-soliton configurations (instantons) are order-N elliptic functions which we write in terms the Weierstrass' σ function as

$$W = \lambda \prod_{j=1}^{N} \frac{\sigma(z - a_j)}{\sigma(z - b_j)}, \quad \sum_{j=1}^{N} a_j = \sum_{j=1}^{N} b_j. \quad (5)$$

The integer N represents the topological charge when $N > 1$, λ is related to the size of the solitons and the zeros a_j and poles b_j determine the positions of the lumps. Amongst other applications, the fields (5) were used in [5] within a model quantum field theory.

The static energy density associated with (1) is

$$E = 2\frac{|\partial_z W|^2 + |\partial_{\bar{z}} W|^2}{(1 + |W|^2)^2}, \quad (6)$$

plots of which produce the familiar CP^1 lumps of energy localised in space. The energy E is related to the the soliton number N by the Bogomolnyi bound

$$E \geq 2\pi|N|. \quad (7)$$

The instanton solutions correspond to the equality in (7): solutions carrying $N > 0$ ($N < 0$) imply $\partial_{\bar{z}} W = 0$ ($\partial_z W = 0$), the Cauchy-Riemann conditions for W being an analytic function of z ($\bar{z}$).

As per the numerical set-up, fields of the form (5) serve as initial conditions for our numerical simulation, where the time dependence is introduced via the Lorentz boosting. Since during the simulations the field W may become arbitrarily large, we run our simulations in the $O(3)$ formulation (2).

For a square torus the function σ possesses a simple Laurent expansion of the form

$$\sigma(u) = \sum_{j=0}^{\infty} G_j u^{4j+1}, \qquad G_j \equiv G_j(L) \in \Re, \quad (8)$$

where it is sufficient to compute the series (8) up to G_5 as our coefficients for $j > 5$

are negligibly small; we have

$$\left.\begin{array}{rcl} G_0 &=& 1 \\ G_1 &=& -0.7878030 \\ G_2 &=& -0.221654845 \\ G_3 &=& 9.36193 \times 10^{-3} \\ G_4 &=& 7.20830 \times 10^{-5} \\ G_5 &=& 2.37710 \times 10^{-5} \end{array}\right\}.$$

We have used the fourth-order Runge-Kutta method of simulating the time dependence and approximated the spatial derivatives by finite differences. The Laplacian has been evaluated using the standard nine-point formula.

The discrete model has been evolved on a $n_x \times n_y = 200 \times 200$ square periodic lattice with spatial and time steps δx=δy=0.02 and δt=0.005, respectively. The vertices of our fundamental period cell are then $(0,0)$, $(0,L)$, (L,L), $(L,0)$; $L = n_x \times \delta x = 4$.

3 Scattering

Consider a configuration of five tightly localised solitons, described by the elliptic function (5) with $N = 5$, $\lambda = (1,0)$ $a_1 = (2.85, 2)$, $b_1 = (1.45, 1.95)$ and

$$\begin{array}{rcl} a_j &=& (a_1 - C)\exp(i\eta_j) + C \\ b_j &=& (b_1 - C)\exp(i\eta_j) + C \\ \eta_j &=& (j-1)\frac{2\pi}{5},\ j = 2,3,4,5, \end{array} \qquad (9)$$

where $C \equiv (L/2, L/2)$ is the centre of the lattice. The energy distribution (6) for this configuration is depicted in figure (1), top-left.

Note that the choice (9) automatically satisfies the selection rule $\sum_{j=1}^{3} a_j = \sum_{j=1}^{3} b_j$ between the zeroes and poles of W.

Now, it is well-known [6] that the planar sigma model is scale invariant, so its lumps can shrink or expand indefinitely as the time elapses. As usual, we stabilise the solitons by adding to the pure lagrangian (1) a Skyrme-like term [7]

$$-\frac{\theta_1}{4}[(\partial^\mu\vec{\phi}.\partial_\mu\vec{\phi})^2 - (\partial^\mu\vec{\phi}.\partial^\nu\vec{\phi})(\partial_\mu\vec{\phi}.\partial_\nu\vec{\phi})], \qquad (10)$$

where $\theta_1 = 0.001$ in all our simulations. The resulting lagrangian, a planar analogue of the Skyrme model of nuclear physics [8], has been extensively studied; in particular, recently in connection with Skyrme crystals [9] and because of its similarities with the structure of fullerene shells in carbon chemistry [10].

The evolution of the configuration (9) is shown in figure 1 through various snapshots at different times. The identical skyrmions are boosted to the centre of the grid where they collide at a zero impact parameter; the lumps coalesce in a ringish structure, re-arrange themselves and come out moving towards the vertices of the dual-pentagon, scattering at an angle of $\pi/5$ radians. The latter can be best appreciated in figure 2 (top) where we have superimposed the states $t = 0$ and $t = 1.5$.

Note that because of the periodic boundary conditions we have a periodic tiling of the plane with a fundamental cell $L \times L$ hosting the solitons but there are further solitons from other tiles interacting with the original ones, a situation that might affect their scattering behaviour. There is also some overlapping between the solitons; we may reduce the possibility of its occurring by placing the lumps far from the borders of the fundamental square –hence resembling a configuration on $\Re_2$ as in figure (2)-top. But as the lumps are more widely separated, closer to the borders, the anisotropy of T_2 might have a telling effect in the dynamics.

In figure 2 (bottom, t=0) we have a configuration of widely separated lumps, corresponding to $a_1 = (3.55, 2)$, $b_1 = (1.45, 1.95)$, $\lambda = (1.5, 0)$. The initial five lumps are the ones positioned farther apart from the centre; they do not look perfectly identical as they begin to feel one another through the wrap around of their tails. After the collision we have the configuration labelled $t = 3$ in the lower part of figure (2), where we observe that the five structures move along a line which does not form 36° with the initial direction of motion.

4 Conclusions

We have studied head-on collisions between CP^1 solitons on a flat torus in the homotopy class $N = 5$. As in the usual case on the compactified plane $\Re_2 \cup \{\infty\} \approx S_2$, the lumps scatter off along a line that forms $\pi/5$ radians with the initial direction of motion, but only when the lumps start close to each other, away from the borders of the fundamental square.

On the other hand, when the initial solitons are placed farther from one another the anisotropy of the torus cannot be ignored, and the solitons do not show dual-pentagon scattering.

We hope to report soon on scattering results in other topological classes.

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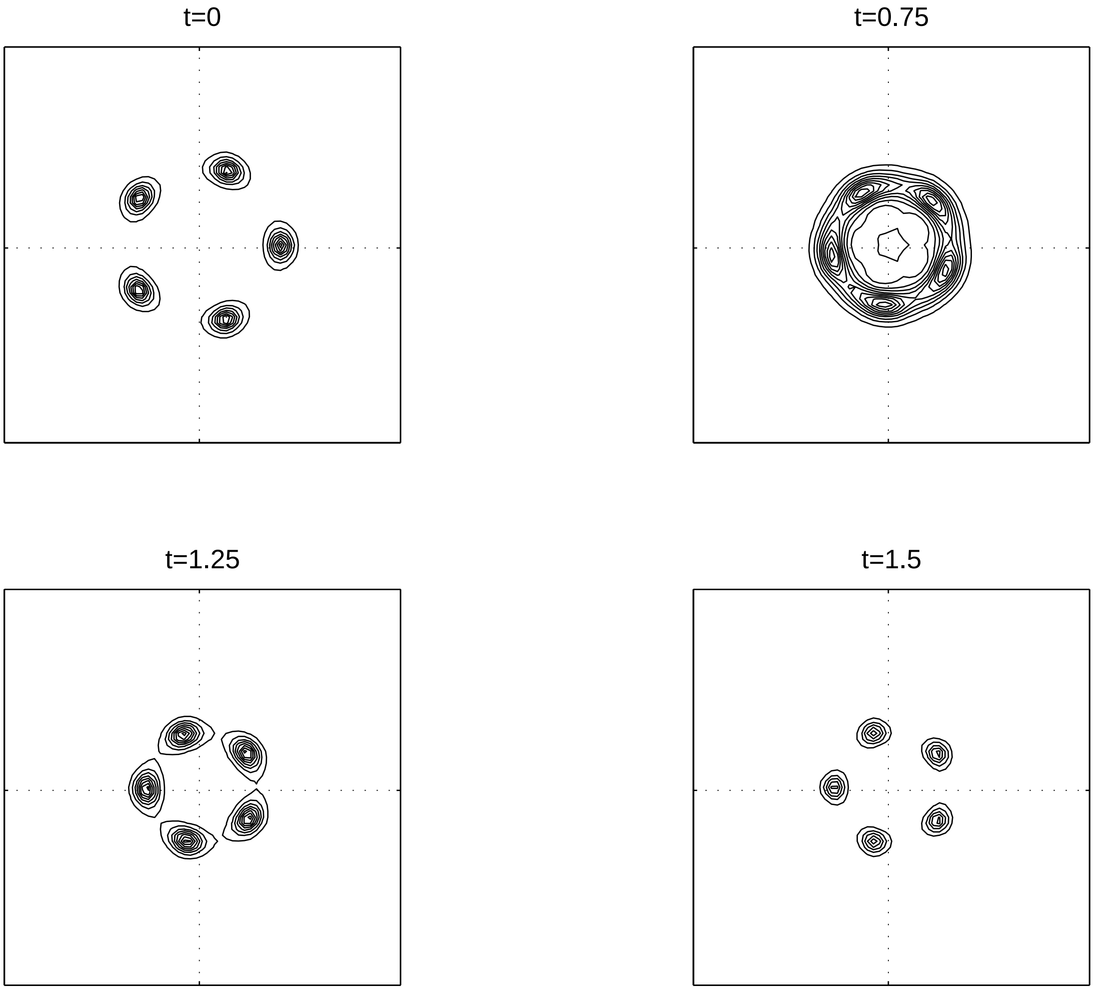
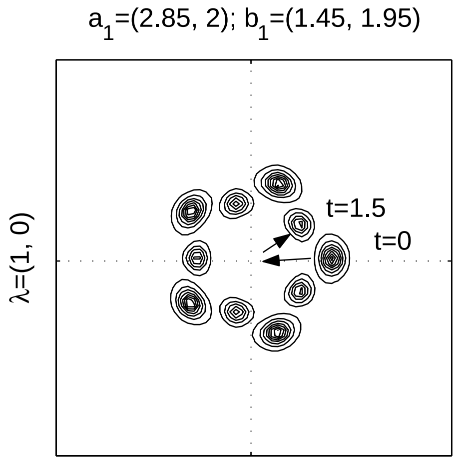


Figure 1: Charge $N = 5$ lumps close to each other at $t = 0$. They show dual-polygon scattering.



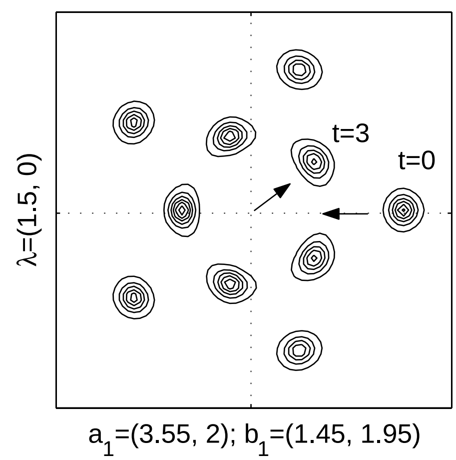


Figure 2: *Top*: Superimposed configurations $t = 0$ and $t = 1.5$ from figure (1). We observe $\pi/5$ scattering. *Bottom*: Charge $N = 5$ solitons starting far from the centre ($t = 0$) do not scatter at 36 degrees ($t = 3$).