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Cellular Automata in Non-Euclidean Spaces

SHUICHI YUKITA
Faculty of Computer and Information Sciences
Hosei University
3-7-2 Kajino-cho Koganei-shi Tokyo 184-8584
JAPAN

Abstract: Classical results on the surjectivity and injectivity of parallel maps are shown to be extendible to the cases with non-Euclidean cell spaces of particular types. Also shown are obstructions to extendibility, which may shed light on the nature of classical results such as the Garden-of-Eden theorem.

Key–Words: Cellular automata, Non Euclidean cell spaces, the Garden-of-Eden theorem

1 Introduction

A cellular automaton is a network of identical, uniformly interconnected and synchronously clocked finite state machines. Cellular automata provide simple and powerful models for parallel computation and natural phenomena, which interests researchers from computer science, biology, physics, and other mathematical science fields.

The network topology of a cellular automaton is usually assumed to be a lattice in Euclidean *n*-space. This is considered to be enough since most applications in the above mentioned fields seem to fit in this setting. However, when we study crystal growth or physical phenomena in a curved surface/space, we are naturally led to the study of cellular automata with other network topologies such as fractals and Cayley graphs, which we will call non-Euclidean cellular automata. At present, attractive applications are rare, which confines the study of non-Euclidean cellular automata to a limited circle of theoretical researchers.

Even in this situation, some of the rich classical Euclidean results are shown to hold in the non-Euclidean framework, which may help to attract researchers' attention. One example is the extension of the *Garden-of-Eden*(GOE) theorem.

The classical GOE theorem is emerged from the problem of self-reproducing machines [1, 2, 3]. It claims that the existence of mutually erasable patterns is equivalent to the existence of a GOE pattern. A GOE pattern is a local configuration which cannot be reproduced in any environments. If a configuration contains a copy of a GOE pattern, the configuration can not be reproduced. Therefore, a *self-reproducing* configuration must not contain any copy of a GOE pattern.

In 1993, Machì and Mignosi proved the GOE theorem for cellular automata on Cayley graphs of non-exponential growth [4]. This was the first nontrivial non-Euclidean result. Since many important classical results rely on the GOE theorem, the extended GOE theorem plays an important role for the development of non-Euclidean theory.

Aside from Machì and Mignosi' success, the author took a different approach. Restricting the cellular spaces to the class of Heisenberg groups, explicit construction of an anisotropic Moore-Myhill tiling was obtained [5], which proved to be effective in the non-Euclidean extension.

Based on these earlier results, the author extended the following classical theories:

- ○ Sato and Honda's dynamical theory [6],
- ○ Maruoka and Kimura's theory of weak and strong properties [7],
- ○ Ito, Osato, and Nasu's theory of linear cellular automata [8].

Unfortunately, the results in these cited papers have been presented in diverse styles with their own particular notation. So, it is worthwhile to provide a unified exposition of these contributions.

Lengthy technical proofs are omitted, which are given in each of the cited papers.

The rest of this paper is organized as follows. Section 2 gives basic definitions. Sections 3–5 describe the author's contributions to the non-Euclidean cellular automata theory. Concluding remarks and references are given in the final section.

2 Cellular Automata on Cayley Graphs

This section gives definitions, fixes notation, and provides basic facts.

Definition 1. Let G be a group. The *Cayley graph of G with respect to a subset N of G* is a directed pseudograph with vertex set G and edge set E, where

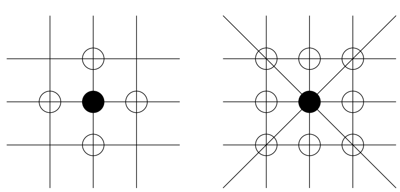
$$E = \{(g_1, g_2) \in G \times G \,|\, g_1 = g_2 h \text{ for } h \in N\}.$$

This graph is denoted by $\Gamma(G,N)$.

Remark 2. The unit e of G is allowed to be included in N, namely loops are allowed in $\Gamma(G,N)$. Some authors adopt different definitions in which N must be a set of generators of G and must not contain e. However, the above form is adequate for our purpose. In what follows, we say simply "graph" instead of "directed pseudograph." In the following examples, though Cayley graphs are directed, they are always drawn as undirected graphs by identifying edges (g_1, g_2) and (g_2, g_1) and omitting loops. This is to avoid unnecessary complexity in the figures.

Example 3. Let $G = \boldsymbol{Z} \times \boldsymbol{Z}$ be the direct product of the infinite cyclic group $\boldsymbol{Z}$ with itself, or in other words, 2-dimensional Euclidean lattice. Let N be the set $\{(0,0),(0,1),(1,0),(-1,0),(0,-1)\} \subset G$. This is a cell space with the so called *Moore's neighborhood*. See Figure 1(a).

Example 4. Let G be as above and let N be the set $\{(0,0),(0,\pm1),(\pm1,0),(\pm1,\pm1)\} \subset G$. This is a cell space with the so called *von Neumann's neighborhood*. See Figure 1(b).



(a) Moore (b) Von Neumann

Figure 1: Choice of N

Let us see other examples of hyperbolic nature.

Example 5. Let G be the free group with two generators a and b. We take $N = \{a, b, a^{-1}, b^{-1}\}$. In this case, we obtain a fractal image as in Figure 2. For clear view it contains only five level recursive constructions.

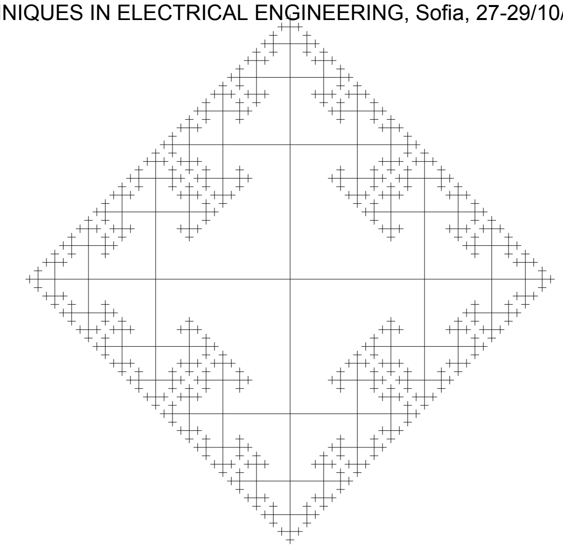


Figure 2: A Cayley graph of a free group

Example 6. Let G be the Fuchsian group with the presentation

$$\langle a_1, b_1, \ldots, a_n, b_n \,|\, a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_n b_n a_n^{-1} b_n^{-1} = e \rangle,$$

where $n > 1$ is an integer and e denotes the identity element of G. We take

$$N = \{a_1, a_1^{-1}, \ldots, a_n, a_n^{-1}, b_1, b_1^{-1}, \ldots, b_n, b_n^{-1}\}.$$

See Figure 3 with $n = 2$. Notice that there are circuits in the graph. The hashed area is to indicate the existence of a circuit.

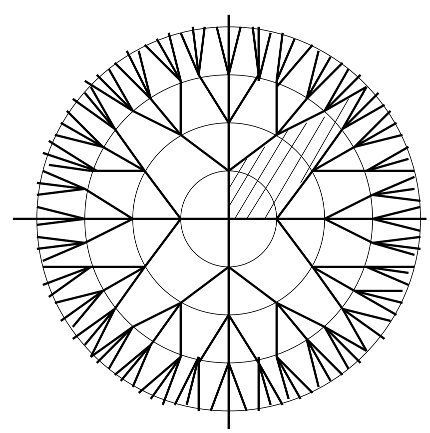


Figure 3: A Cayley graph of a Fuchsian group

So far no conditions were imposed on N. However, to attain meaningful results, we assume throughout this paper that G is finitely generated and that N includes e and generates G.

Next, we define cellular automata on groups or cellular automata on Cayley graphs. We will use these two terms interchangeably.

Definition 7. Let G be a finitely generated group. Let N be a finite subset of G that generates G. Let Q be a finite set called the set of *states*. A *local map* with *support* N is a map $\sigma : Q^N \to Q$. A map $x : G \to Q$ is called a *configuration*. Let C denote the set of all configurations, that is, Q^G, with the product topology. By Tichonov's theorem, this space is compact. The *shift* s_g induced by $g \in G$ is a map $C \to C$ such that for any $x \in C$,

$$[s_g(x)](h) = x(g^{-1}h) \quad \text{for all } h \in G.$$

The *parallel map* T_σ induced by σ and N is a map $C \to C$ such that

$$(T_\sigma(x))(g) = \sigma(s_g^{-1}(x)|_N) \quad \text{for all } x \in C,\ g \in G,$$

where $s_g^{-1}(x)|_N$ denotes the restriction of $s_g^{-1}(x)$ to N. The 4-tuple (G,Q,N,σ) is called a *cellular automaton.* The pair (C,T_σ) forms a discrete dynamical system and is also called a cellular automaton.

Definition 8. Let A be any subset of G. An element of Q^A, that is, a map $A \to Q$, is called a *pattern over A*.

We sometimes assume the existence of the *quiescent state* 0, that is, $\sigma(0,\ldots,0) = 0$.

Definition 9. The *support* of $x \in C$ is the set of all $g \in G$ with $x(g) \neq 0$, and denoted by $\mathrm{supp}(x)$. If $|\mathrm{supp}(x)| < \infty$, then x is called a *finite configuration*, where $|A|$ denotes the number of elements of a set A. The set of all finite configurations is denoted by C_F. We denote by $\hat{T}_\sigma$ the restriction $T_\sigma|_{C_F} : C_F \to C_F$.

We define group theoretical properties.

Definition 10. A group G is said to be *residually finite* if for any $g \in G$, there is a normal subgroup of finite index which does not contain g. A group G is said to be a *unique product* group if, for any non-empty finite subsets A and B of G, there exists an element $g \in G$ that has a unique representation in the form $g = ab$ with $a \in A$ and $b \in B$.

To define the GOE property, we must introduce the notions of mutually erasable patterns and GOE patterns. From now on we assume that the support N of a local map σ always contains e, and consequently that $N^2 = NN \supset N$. This assumption does not affect the generality of the argument since the support of a local map can always be extended to a larger set in a trivial way.

Definition 11. Let A and B are subsets of G such that $AN \subseteq B$. We define $T_{\sigma,B,A} : Q^B \to Q^A$ as follows. Let $x \in Q^B$ be any pattern over B. We can find $x_\infty \in Q^G$ such that $x_\infty|_B = x$. We put $T_{\sigma,B,A}(x) = T_\sigma(x_\infty)|_A$.

Clearly, this is well-defined.

Definition 12. Let σ be a local map and N be its support. Let A be a finite subset of G. Two patterns x and y in Q^{AN^2} are said to be *mutually erasable* (over AN^2) if

$$x|_{AN^2-A} = y|_{AN^2-A}, \quad x|_A \neq y|_A, \quad \text{and} \\ T_{\sigma,AN^2,AN}(x) = T_{\sigma,AN^2,AN}(y).$$

T_σ is called *erasing* if there exist mutually erasable patterns.

Notice that the existence of mutually erasable patterns implies that T_σ is not injective. Notice also that, if x and y are mutually erasable over AN^2, so are their translations $s_g(x)$ and $s_g(y)$ over gAN^2 for any $g \in G$.

Definition 13. A group G is said to have the *GOE property* if the condition "for any parallel map T_σ, it is surjective if and only if it is not erasing" is satisfied.

Machì and Mignosi's result is stated as follows:

Theorem (Machì and Mignosi [4]). *If G is a group of non-exponential growth, then it has the GOE property.*

The proof was conducted by counting the number of patterns over a finite set and finding inequalities that lead to contradictions when the finite set is taken large enough. The arguments in the proof are essentially the same as in Moore and Myhill's proof. However, in the non-Euclidean case the counting procedure is not straightforward.

If we assume the existence of the quiescent state, we obtain an alternative form of the GOE theorem:

Theorem (With a quiescent state[4]). *Let G be a group of non-exponential growth.* $\hat{T}_\sigma$ *is injective if and only if* T_σ *is surjective.*

3 Periods, Poisson Stability, Injectivity, and Surjectivity

This section describes a non-Euclidean extension of Sato and Honda's dynamical theory. To describe the problem, we must to add some more definitions.

The notion of period is introduced as follows.

Definition 14. Let x be a configuration. The *period* of x, denoted by $\omega(x)$, is defined as the stabilizer of x, that is,

$$\omega(x) = \{h \in G \mid s_h(x) = x\}.$$

Let $A \subset G$ be a complete set of right coset representatives of $\omega(x)\backslash G$. We sometimes call it a *fundamental transversal* of $\omega(x)$. Any element $g \in G$ is uniquely expressed as ha with $h \in \omega(x)$, $a \in A$. With this decomposition, we have $x(g) = x(ha) = x(a)$.

If A is a subgroup of B, we write $B \geq A$. If A is a proper subgroup of B, that is, $A \neq B$ and $B \geq A$, we write $B > A$. The following simple lemma is a key to the subsequent discussions.

Lemma 15. *Let T_σ be a parallel map. Then, $\omega(T_\sigma(x)) \geq \omega(x)$.*

Definition 16. If the period $\omega(x)$ of a configuration $x \in C$ is of finite index, that is, if $|\omega(x)\backslash G| < \infty$, then x is called a *cofinite configuration*.

Let C_P denote the set of all cofinite configurations. From Lemma 15, we know that the space C_P is invariant under parallel maps.

Definition 17. Let M be an T_σ-invariant subspace of C. A parallel map T_σ is said to be *period preserving* on M if $\omega(T_\sigma(x)) = \omega(x)$ for all $x \in M$. In particular, if $M = C$, a parallel map T_σ is simply said to be *period preserving*.

The lemma concerning the density of cofinite configurations is repeatedly used at crucial steps in the proofs of main results.

Lemma 18 (The density lemma). *If G is residually finite, then C_P is dense in C.*

Next, we introduce the notion of Poisson stability and its variants.

Definition 19. Let T_σ be a parallel map. A configuration $x \in C$ is said to be *Poisson stable* with respect to T_σ if there exists a sequence of integers $n_1 < n_2 < \cdots$ such that

$$\lim_{i\to\infty} (T_\sigma)^{n_i}(x) = x.$$

Let M be a subset of C. A parallel map T_σ is said to be *M Poisson stable* if every $x \in M$ is Poisson stable with respect to T_σ.

Definition 20. A configuration $x \in C$ is said to be *strongly Poisson stable* with respect to T_σ if there exists a nonnegative integer n_x such that $(T_\sigma)^{n_x}(x) = x$, where n_x depends on x. A parallel map T_σ is said to be *M strongly Poisson stable* if every $x \in M$ is strongly Poisson stable with respect to T_σ.

Definition 21. Let $M \subset C$ be invariant under a parallel map T_σ. The parallel map T_σ is said to be *injective on M* if the restriction of $T_\sigma : M \to M$ is injective. The parallel map T_σ is said to be *surjective on M* if the restriction of $T_\sigma : M \to M$ is surjective. For more details on Poisson stability, see [9].

The notion of order is defined as follows.

Definition 22. A parallel map T_σ is said to have *finite order* if there exists a positive integer n such that $(T_\sigma)^n = I$, where I denotes the identity map. The *order* of T_σ is defined as the minimum of such positive integers. Let $M \subseteq C$ be an invariant subspace of T_σ. A parallel map T_σ is said to have *finite order on M* if $(T_\sigma)^n|_M = I|_M$ for some positive integer n.

Now we can state Sato and Honda's result [10].

Theorem (Sato and Honda). *Let $G = \boldsymbol{Z}^d$. The following five conditions are arranged in the order of strength, that is,* (i) *implies* (ii), (ii) *implies* (iii), *and so on. In* (iii), *all the subitems are equivalent conditions.*

- (i) *injective on C.*
- (ii) *period preserving on C.*
- (iii)
  - *(a) strongly C_P-Poisson stable*
  - *(b) C_P-Poisson stable*
  - *(c) injective on C*
  - *(d) injective on C_P*
  - *(e) finite order on C_F*
  - *(f) finite order on C_P*
- (iv) *surjective and period preserving on C_P*
- (v) *surjective on C*

A non-Euclidean extension of this theorem is not straightforward since Euclidean theory uses the GOE theorem at crucial steps which does not hold in general for non-Euclidean cellular automata. Moreover, various periodic constructions in Sato and Honda's work turned out to be valid only when the underlying group has residual finiteness. The author showed that these two conditions on groups make non-Euclidean extensions of Sato and Honda's theorem possible [6]. Periodic constructions for non-Euclidean cellular automata are based on the notion of period that has been emerged from [11].

The following theorem is obtained as an extension in which the cell space $\boldsymbol{Z}^d$ is replaced by a group G.

Theorem 23 (Yukita). *Let G be a finitely generated group. For any parallel map T_σ, we have the following.*

- (i) *Injective on $C \Longrightarrow$ Period preserving on C.*
- (ii) *If G has the GOE property and is residually finite,*
  Period preserving on $C \Longrightarrow$ Injective on C_P.

(iii) *The following four conditions are equivalent.*

(a) *strongly C_P-Poisson stable*
(b) *C_P-Poisson stable*
(c) *injective on C_P*
(d) *surjective and period preserving on C_P*

(iv) *If G be residually finite,*
Surjective on $C_P \Longrightarrow$ Surjective on C.

(v) *If G has the GOE property and is residually finite.*
Period preserving on $C_P \Longrightarrow$ Surjective on C.

For the proof, see [6], where Lemmas 15 and 18 are used at crucial steps. Figure 4 summarizes Theorem 23 , where thick arrows mean implications. A thick arrow labeled with "RF", "GOE", or "GOE + RF" means an implication under the condition that the group is residually finite, that the group has the GOE property, or that the group has the GOE property and is residually finite, respectively.

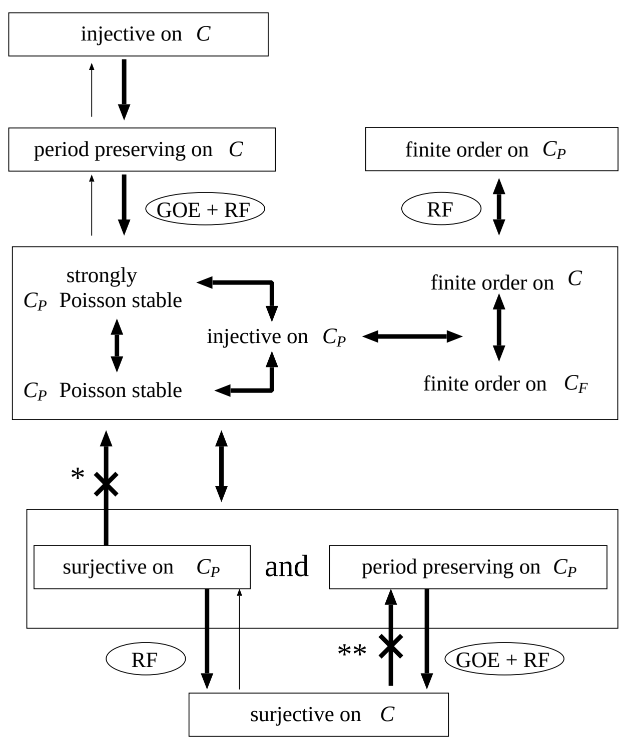


Figure 4: Properties of T_σ.

Remark 24. Let G be an infinite cyclic group. Then the diagram in Figure 4 collapses to a simpler one. It is known that injectivity on C is equivalent to injectivity on C_P and that surjectivity on C is equivalent to surjectivity on C_P [10, Figure 2 and the proof of Prop.3.1 (2)]. Therefore, thin arrows in Figure 4 also hold in this case.

Remark 25. There exists a parallel map that is surjective on C but not injective on C. This gives the non-implication in Figure 4 depicted by the crossed arrow labeled with "*."

Remark 26. D. E. Muller's example in [12, p. 131] establishes non-implications in Figure 4 depicted by the crossed arrow labeled with "**."

4 Strong and Weak Properties

Maruoka and Kimura introduced variants of the notions of injectivity and surjectivity — the notions of weak injectivity/surjectivity and strong injectivity/surjectivity — and obtained results concerning the hierarchy among those properties [13, 14, 15], which we will call *Maruoka-Kimura's hierarchy*, or the *M-K hierarchy* for short. We will also use the same term to refer to the non-Euclidean extensions of the M-K hierarchy.

An equivalence relation $\asymp$ in C is defined as follows.

Definition 27. Two configurations x and y are said to be *asymptotically equivalent* if $x(g) = y(g)$ for all but a finite number of $g \in G$. We write $x \asymp y$ when x and y are asymptotically equivalent.

C_x denotes the equivalence class of $\asymp$ that contains x. The equivalence class C_x may be seen as the set of configurations with a given asymptotic boundary condition at "infinity." $C/\asymp$ denotes the quotient space, that is, the set of all asymptotic equivalence classes. For any $x \asymp y$, we have $T_\sigma(x) \asymp T_\sigma(y)$. This means that T_σ maps C_x into $C_{T_\sigma(x)}$ for any $x \in C$. We denote by $T_{\sigma,x}$ the map $T_\sigma|_{C_x} : C_x \to C_{T_\sigma(x)}$. Obviously we have the quotient map $T_\sigma/\asymp : C/\asymp \to C/\asymp$.

The following lemma is obvious:

Lemma 28. *For each $x \in C$, C_x is dense in C.*

Definition 29. A parallel map T_σ is said to be *weakly injective* if $T_{\sigma,x}$ is injective for some $x \in C$, and *strongly injective* if $T_{\sigma,x}$ is injective for all $x \in C$. A parallel map T_σ is said to be *weakly surjective* if $T_{\sigma,x}$ is surjective for some $x \in C$, and *strongly surjective* if $T_{\sigma,x}$ is surjective for all $x \in C$. A parallel map T_σ is said to be *residually injective* if no two asymptotically non-equivalent configurations have asymptotically equivalent successors. A parallel map T_σ is said to be *residually surjective* if any configuration x has an asymptotically equivalent configuration that has a predecessor.

The terms *totally injective/surjective* are meant for surjectivity and injectivity on C.

Maruoka and Kimura's result shows that relations among properties of injectivity, surjectivity, their strong and weak versions form a hierarchical structure:

Theorem (Maruoka and Kimura). *Let $G = \boldsymbol{Z}^d$. In each of the following* (i) *and* (iii)*, all the conditions are equivalent. Further, conditions* (i) *implies* (ii)*, and* (ii) *implies* (iii)*.*

(i) *(a) residually injective*
(b) totally injective
(c) strongly surjective

(ii) *weakly surjective*

(iii) *(a) strongly injective*
(b) weakly injective
(c) totally surjective
(d) residually surjective

Attempts at non-Euclidean extensions must face the following problem. Maruoka and Kimura's discussions depend heavily on the notions of *balancedness* and *hardness* and the following facts. Surjectivity and injectivity of parallel maps are characterized as: For any parallel map,

(i) surjective $\Longleftrightarrow$ balanced.
(ii) injective $\Longleftrightarrow$ hard.

Neither balancedness, hardness, nor these characterizations work well for non-Euclidean cellular automata. Therefore, the author had to seek other approaches for a non-Euclidean extension and eventually obtained several versions of modified hierarchies in [7], where various conditions are imposed in turn on the groups that generate the tessellation. The conditions considered were the *GOE property*, *residual finiteness*, and their combination.

The following theorem is obtained as an extension in which the cell space $\boldsymbol{Z}^d$ is replaced by a group G, where the condition on groups is taken the most general.

Theorem 30 (Yukita). *Let G be a finitely generated infinite group, and let T_σ is a parallel map. The following equivalences or implications hold for T_σ.*

(i) *Totally injective $\Longrightarrow$ Strongly(weakly) injective.*
(ii) *Strongly injective $\Longleftrightarrow$ Weakly injective.*
(iii) *Strongly surjective $\Longrightarrow$ Weakly surjective.*
(iv) *Weakly surjective $\Longrightarrow$ Totally surjective.*
(v) *Totally surjective $\Longrightarrow$ Residually surjective.*

The most strict condition on groups considered is "GOE + Residual Finiteness." Under this condition we can restore nearly all of the M-K hierarchy.

Theorem 31 (Yukita). *Assume that G has the GOE property and is residually finite. For any local map σ, relations among properties of T_σ are summarized as in Figure 5.*

For the proofs, see [7].

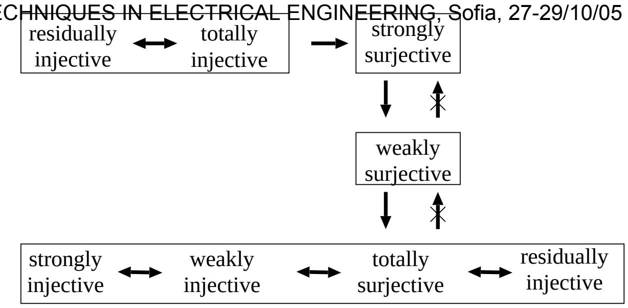


Figure 5: The M-K hierarchy with GOE and residual finiteness

5 Arithmetic Properties of Linear Cellular Automata

In this section, we focus on linear cellular automata. The existence of the quiescent state is automatically guaranteed. The *quiescent configuration*, a configuration having the quiescent state 0 at every cell, is also denoted by 0.

A concise algebraic notation for linear cellular automata is given as follows. Let G be a group. We consider a formal sum $x = \sum_{g\in G} x_g g$, where $x_g \in \boldsymbol{Z}_m$. $\boldsymbol{Z}_m[[G]]$ denotes the space of all such formal sums. The obvious addition operation is defined by $x + y = \sum_{g\in G}(x_g + y_g)g$ for $x = \sum_{g\in G} x_g g$ and $y = \sum_{g\in G} y_g g$. The *support* of $x \in \boldsymbol{Z}_m[[G]]$ is the set of all $g \in G$ with $x_g \neq 0$. $\boldsymbol{Z}_m[G]$ denotes the space of all formal sums $x \in \boldsymbol{Z}_m[[G]]$ with $|\mathrm{supp}(x)| < \infty$. Clearly, $\boldsymbol{Z}_m[G]$ is a submodule of $\boldsymbol{Z}_m[[G]]$ and has an extra ring structure where multiplication is the convolution product defined by $x * y = \sum_{g\in G}(\sum_{ab=g} x_a y_b)g$. Notice that $\sum_{ab=g} x_a y_b$ is a finite sum, and hence the convolution operation is well-defined in $\boldsymbol{Z}_m[G]$. Given any $x \in \boldsymbol{Z}_m[[G]]$ and $y \in \boldsymbol{Z}_m[G]$ the convolution $x * y$ and $y * x$ are also well-defined.

$\boldsymbol{Z}_m$, $\boldsymbol{Z}_m[[G]]$, and $\boldsymbol{Z}_m[G]$ are regarded as the set of states, the space of configurations, and the space of finite configurations, respectively. Given $\sigma \in \boldsymbol{Z}_m[G]$ with $\mathrm{supp}(\sigma) = N$, the map $T_\sigma : \boldsymbol{Z}_m[[G]] \to \boldsymbol{Z}_m[[G]]$ is defined by $T_\sigma(x) = x * \sigma$ for all $x \in \boldsymbol{Z}_m[[G]]$. We can see that this σ plays the role of a local map. The map $\hat{T}_\sigma : \boldsymbol{Z}_m[G] \to \boldsymbol{Z}_m[G]$ is defined by $\hat{T}_\sigma(x) = x * \sigma$ for all $x \in \boldsymbol{Z}_m[G]$. The dynamical system $(\boldsymbol{Z}_m[[G]], T_\sigma)$ is a *linear cellular automaton over $\boldsymbol{Z}_m$*. The shift $s_u : \boldsymbol{Z}_m[[G]] \to \boldsymbol{Z}_m[[G]]$ induced by $u \in G$ is given by $s_u(x) = u * x$.

Let S denote the set of all coefficients a_i appearing in the specification of σ. Let $\mathrm{Spec}\,(m) = \{p_1, \ldots, p_s\}$ be the set of all prime factors of m. This set is partitioned as $\mathrm{Spec}\,(m) = W_{m,S} \cup P^{(1)}_{m,S} \cup \cdots \cup P^{(n)}_{m,S} \cup Q_{m,S}$, where each set is determined as follows. $W_{m,S}$ is the set of prime factors of m that divide all of $a_1, \ldots, a_n$. $P^{(i)}_{m,S}$ is the set of prime factors of m that do not divide a_i but divide all other coefficients. $Q_{m,S}$ is the set of prime factors of m that do not divide at least two coefficients $a_i, a_j\,(i \neq j)$.

Ito, Osato, and Nasu obtained the following two theorems that claim that injectivity and surjectivity of parallel maps of linear cellular automata are completely determined by the corresponding local rules [16]. Further studies on this track were conducted by Aso and Honda [17] and recently by Manzini and Margara [18, 19].

Theorem (Ito, Osato, and Nasu). *Let $G = \boldsymbol{Z}^d$. The following three properties are equivalent.*
(i) $\hat{T}_\sigma$ *is injective.*
(ii) T_σ *is surjective.*
(iii) $W_{m,S} = \emptyset$.

Theorem (Ito, Osato, and Nasu). *Let $G = \boldsymbol{Z}^d$. The following three properties are equivalent.*
(i) T_σ *is injective.*
(ii) $\hat{T}_\sigma$ *is surjective.*
(iii) $W_{m,S} = Q_{m,S} = \emptyset$.

A non-Euclidean extension must face the difficulty caused by the absence of commutativity. Ito-Osato-Nasu and Aso-Honda's arguments heavily depend on the algebraic nature of the group $\boldsymbol{Z}^d$ or Abelian groups. The author examined how an attempt of non-Euclidean extension fails for various groups and obtained a sufficient condition on groups that allows Ito-Osato-Nasu type theorems [8]. The above result on injectivity and surjectivity is derived as a corollary of the author's result. In addition, the new proofs clarify the algebraic nature of original Ito-Osato-Nasu's theorems, which was only implicitly described in their paper. The proofs in [8] utilize properties of unique product groups and Machì and Mignosi's GOE theorem.

Theorem 32 (Yukita). *Let G be a unique product group with a finite set of generators N and have the GOE property. The following three properties are equivalent.*
(i) $\hat{T}_\sigma$ *is injective.*
(ii) T_σ *is surjective.*
(iii) $W_{m,S} = \emptyset$.

Theorem 33 (Yukita). *Let G be a unique product group with a finite set of generators N and have the GOE property. The following three properties are equivalent.*
(i) T_σ *is injective.*
(ii) $\hat{T}_\sigma$ *is surjective.*
(iii) $W_{m,S} = Q_{m,S} = \emptyset$.

For the proofs, see [8].

Remark 34. Since any torsion free nilpotent group is known to be a unique product group, in particular, so is $\boldsymbol{Z}^d$.

6 Conclusion

Problems of the GOE patterns, dynamical properties, asymptotic boundary conditions, and arithmetic properties of automata are studied in the non-Euclidean cell spaces. Difference between Euclidean and non-Euclidean is characterized by the properties of groups such as GOE, residual finiteness, and the unique product properties.

Future work includes investigating other dynamical properties or phenomena such as ergodicity, attractors, and topological classification.

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