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Split Step Wavelet Galerkin Method Based on Parabolic Equation Model for Solving Underwater Wave Propagation

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Abstract: In this paper, split step wavelet Galerkin method (SSWGM)is proposed for solving underwater wave propagation in range-independent fluid/solid media. Parabolic equation model is applied for transforming elliptic wave to parabolic wave equation that enable us to use marching approaches in numerical algorithms. Wavelet Galerkin method is used to discretize the depth operators by using 1-periodic Daubechies scaling basis as shape functions. This discretization leads to circulant and bandlimit system which can be solved by fast Fourier transformations and this improves the accuracy and cost of computation. The numerical solution of SSWGM is applied for deep and shallow water environment involving water column over bottom. To evaluate the efficiency of the proposed method, some simulations are demonstrated and the usefulness of SSWGM is highlighted through them.

Key–Words: Wavelet Galerkin, Parabolic equation; Multiresolution analysis; Underwater wave propagation.

1 Introduction

For many years, parabolic equation techniques have been used widely in computational acoustics. The main reason of this attention is that a parabolic equation (PE) can be solved by marching method and therefore requires less computational efforts than full elliptic wave equation model [1], meanwhile, the PE methods have fast convergence within a reasonable cpu-time and memory [2]. PE model is based on factoring elliptic wave operator into a product of incoming and outgoing operators and assuming the outgoing energy dominates [3].

Several numerical techniques are applied in computational underwater acoustics, that have progressed from narrow angle methods to wide angle based methods. Among them only the finite difference, finite element and split step techniques have gained widespread use.

The methods based on finite difference and finite element approaches are used for wide angle, bottom-interacting situation environments, while, many practical ocean-acoustics of naval interest are long range, narrow angle propagation with negligible bottom interaction, Thus they can be modeled efficiently by split step based techniques especially split step Fourier method (SSFM) [4, 5] that employs fast Fourier transform (FFT) at each propagation step [6]. The strong speed and density contrasts encountered at the water-bottom interface adversely affect the efficiency of the above mentioned computational techniques [7, 4]. For maintaining the efficiency of algorithms, it requires to use excessively fine grid for range and depth segments. For example Finite element formulation requires the step size be comparable with the wavelengths which is very small in high frequency.

In this paper, we develop the use of split step wavelet Galerkin method (SSWGM) as a projection method for numerically solving PE model of underwater wave propagation with free surface and absorbing layer bottom boundary condition. Wavelet expansions may be viewed as a localized Fourier analysis with multiresolution structure that automatically distinguishes between smooth and singular region of discontinuity at water-bottom interface [8]. Expansion of such environment by Fourier expansion requires a large number of terms with complexity $\mathrm{O}(N \log_2 N)$, while, In such regions, wavelet coefficients are employed by the complexity $O(N)$ with a few number of coefficients in comparison with Fourier based methods, here, N is the number of discretization points. For more information about wavelets and their properties readers are referred to [9, 10, 11, 12, 13].

Incorporating wavelet Galerkin depth discretization with respect to 1-periodic compactly supported Daubechies scaling functions [14] and split step Pade approximation provide an fast and accurate algorithm so-called SSWGM. In matrix representation, applying this scaling basis functions for depth Galerkin discretization leads to differential matrix that is bandlimit and circulant [15]. Using of circulant matrix representation, makes it possible to execute Fast Fourier Transform (FFT) in matrix multiplications so that this substitution, efficiently increase the calculation speed and decrease complexity of computations to $\mathrm{O}(N)$.

This paper is organized as follows: In two first sections, we give brief overview about PE algorithm and Daubechies wavelet familiy respectively. Section 3, derives the SSWGM and finally in section 4, numerical implementation of proposed method for two test problems corresponding shallow and deep water, are presented.

2 Parabolic Approximation

Consider a range-independent acoustic medium, bounded above by a free surface at $z = 0$ with a sound profile that supports large range propagation (for $r \longrightarrow \infty$). The acoustic pressure $p(z,r)$ due to harmonic point source with frequency f located at $(z_s, 0)$ with time dependence $\exp(-i\omega t)$ can be obtained for $r > 0$ as

$$\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial p}{\partial r}) + \rho\frac{\partial}{\partial z}(\rho^{-1}\frac{\partial p}{\partial z}) + k_0^2 N^2 p = 0, \qquad (1)$$

with complex refraction with the complex refraction index

$$N(z) = n(z) + i\alpha(z)/k_0,$$

where, $k_0 = 2\pi f/c_0$ is the reference wave number, the notations $c(z)$, $\rho(z)$ and $\alpha(z)$ denote local sound speed, density and attenuation respectively [16, 17].

By assuming the acoustic sound wave propagates along principle direction, the sound field can be separated as a slowly varying envelope and a fast oscillating phase term, i.e.,

$$\psi(z,r) = \sqrt{k_0 r}\, p(z,r) e^{-ik_0 r} \qquad (2)$$

where, this envelope function varies ψ slowly with r. Substituting the above form into equation (1), the wave equation for $k_0 r \gg 1$, is split into two terms that governs the evolution of the forward and the backward sound wave of ψ [7]. By neglecting backward sound waves, the one-way equation can be obtained as follows

$$\frac{\partial \psi}{\partial r} = ik_0(\mathcal{Q} - 1)\psi, \qquad (3)$$

where, $\mathcal{Q} = \sqrt{1+X}$ and

$$X = k_0^{-2}(\rho\partial_z(\rho^{-1}\partial_z)) + V(z),$$

with the complex valued $V(z) = N^2(z) - 1$. To solve equation (3), the square root operator $\mathcal{Q}$ need to be approximated. The standard approximation based on Taylor expansion is as

$$\sqrt{1+X} \cong 1 + \frac{1}{2}X, \qquad (4)$$

This yield the standard parabolic equation (SPE) which has been shown to be valid only for propagation angle $15 - 25^\circ$ of horizontal [4].

The formal solution of (3) at the marching step $r + \Delta r$ can be written as

$$\psi(z, r+\Delta r) = e^{ik_0\Delta r(-1+\mathcal{Q})}\psi(z,r) \qquad (5)$$

In order to implement the exponential operator of equation (5), we apply the Pade-series
Pade-product approximation given first by Collins [7]

$$e^{ik_0\Delta r(-1+\mathcal{Q})} \cong 1 + \sum_{j=1}^{m}\frac{a_{j,m}X}{1+b_{j,m}X} \cong \prod_{j=1}^{m}\frac{1+c_{j,m}X}{1+b_{j,m}X}, \qquad (6)$$

The $a_{j,m},\ b_{j,m}$ and $c_{j,m}$ are known as Pade primes. The values of Pade primes can be obtained by matching the first two m derivatives of the square root function with those of the Pade expression at $X = 1$. These coefficients are real valued. In this paper, by assuming that the square root approximation maps the real axis into itself, it dose not need to be applied complex valued Pade approximation for square root approximation [18].

Substituting Pade series (6) into (5) for all $j = 1,\ 2, \ldots,\ m$ leads to the split step Pade (SSP) solution,

$$\psi(r+\Delta r) = \psi(r) + \sum_{j=1}^{m} a_{j,m}(1 + b_{j,m}X)^{-1}X\psi(r), \qquad (7)$$

By defining, m functions q_j as

$$q_j(r+\Delta r) = a_{j,m}(1+b_{j,m}X)^{-1}X\psi(r).$$

Thus, equation (7) can be written as

$$\psi(r+\Delta r) = \psi(r) + \sum_{j=1}^{m} q_j(r+\Delta r), \qquad (8)$$

For simplicity in (7) and (8), we substitute $\psi(z,r)$ by $\psi(r)$. For one term only from the Pade series, the square root operator can be written as wide angle parabolic equation (WAPE) of Claerbout,

$$\sqrt{1+X} \cong \frac{1+p_1X}{1+q_1X}, \qquad (9)$$

where, $p_1 = \frac{3}{4}$ and $p_2 = \frac{1}{4}$.

3 Daubechies Basis functions

In this section, a brief introduction is given about the constructions and basic properties of wavelets. More detailed discussion can be found in [14].

Given a positive integer[1] N, consider a set of constants a_k that satisfy the following four properties

$$a_k = 0 \text{ for } k \in \{0, 1, \ldots, 2N-1\}, \qquad (10)$$

$$\sum_{k=0}^{2N-1} a_k = 2, \qquad (11)$$

[1]The number N is called genus of wavelet.

$$\sum_{k=0}^{2N-1} (-1)^k k^m a_k = 0, \quad \text{for } 0 \le m \le N-1, \tag{12}$$

and for $1-N \le m \le N-1$, we have

$$\sum_{k=0}^{2N-1} a_k a_{k-2m} = 2\delta_{0m}, \tag{13}$$

In particular, for $N = 2$

$$a_0 = \frac{1+\sqrt{3}}{4}, \quad a_1 = \frac{3+\sqrt{3}}{4},$$
$$a_2 = \frac{3-\sqrt{3}}{4}, \quad a_3 = \frac{1-\sqrt{3}}{4}.$$

For each $N \ge 1$ and the corresponding constants a_k are defined in (10-13), the so called *dilation* and *wavelet equation* are defined for every $z \in \mathbb{R}$ as

$$\phi(z) = \sum_{k=0}^{2N-1} a_k \phi(2z-k) \ \text{ and}$$
$$\psi(z) = \sum_{k=0}^{2N-1} b_k \phi(2z-k), \tag{14}$$

where, $b_k = (-1)^k a_{2N-k-1}$. The function ϕ is called scaling function that can be determined by the following iteration method

$$\phi^{n+1}(z) = \sum_{k=0}^{2n-1} a_k \phi^n(2z-1), \quad n = 0, 1, \ldots, \tag{15}$$

where ϕ^0 is the unit indicator function. The scaling function ϕ and wavelet ψ in equation (14) are compactly supported

$$\text{supp } (\phi) = \text{supp } (\psi) = [0, 2N-1]. \tag{16}$$

For $j, k \in \mathbb{Z}$, the translates of scaling function ϕ define an orthogonal subspace

$$V_j = \text{closure (span}\{\phi_{j,k} : \ k \in \mathbb{Z}\} \text{)}, \tag{17}$$

and the orthogonal complement of V_j in V_{j+1} is defined as

$$W_j = \text{closure (span}\{\psi_{j,k} : \ k \in \mathbb{Z}\} \text{)}, \tag{18}$$

where,

$$\phi_{j,k} = 2^{j/2}\phi(2^j z - k) \ \text{ and } \ \psi_{j,k} = 2^{j/2}\psi(2^j z - k),$$

with the following support

$$\text{supp } (\phi_{j,k}) = \text{supp } (\psi_{j,k}) = [\frac{k}{2^j}, \frac{k+2N-1}{2^j}].$$

One can easily verify the following properties of spaces V_j and W_j for $j \in \mathbb{Z}$

$$V_{j-1} \subset V_j, \ \ W_{j-1} \subset W_j \text{ and } V_{j+1} = V_j \oplus W_j,$$

where $X \oplus Y$ denotes the orthogonal direct sum of Hilbert spaces X and Y. Note that the closures of $\bigcup_{-\infty}^{\infty} V_j$ and $\bigoplus_{-\infty}^{\infty} W_j$ are dense in $L^2(\Omega)$, which is the space of square integrable functions on domain Ω. The space $\{V_j\}_{j\in\mathbb{Z}}$ is referred to as *multiresolution analysis* for $L^2(\Omega)$.

Daubechies wavelets that are defined on the whole of real line, can be periodized with a Poisson summation technique. For $j, k \in \mathbb{Z}$, the 1-periodic scaling function is defined as

$$\widetilde{\phi}_{j,k}(z) = \sum_{n=-\infty}^{+\infty} \phi_{j,k}(z+n).$$

and the 1-periodic wavelet as

$$\widetilde{\psi}_{j,k}(z) = \sum_{n=-\infty}^{+\infty} \psi_{j,k}(z+n).$$

The 1-periodicity can be verified as

$$\widetilde{\phi}_{j,k}(z+1) = \widetilde{\phi}_{j,k}(z) \ \text{ and } \ \widetilde{\psi}_{j,k}(z+1) = \widetilde{\psi}_{j,k}(z) \tag{19}$$

These periodic wavelets posses many of the same properties of their non periodic forms. Because of periodization, these basis functions does not commute with dilation and can not be generated by repeated translation and dilation of the mother basis functions.

Periodic functions introduced in (19) construct periodic multiresolution analysis $\{\tilde{V}_j\}_{j=-\infty}^{+\infty}$ for $L^2([0,1])$.

For $J \in \mathbb{Z}$, a function $u \in \tilde{V}_J$ can be expanded as

$$u_J(x) = \sum_{k=-\infty}^{+\infty} u_{J,k}\widetilde{\phi}_{J,k}, \tag{20}$$

where $u_{J,k} = \int_{-\infty}^{+\infty} u(z)\widetilde{\phi}_{J,k}(z)\, dz$. By starting from representation of a function in a coarse subspace at level j_0, higher resolution can be obtained by adding the spaces $\tilde{W}_j$ up to a higher level J

$$u_J(z) = \sum_{k=0}^{2^J-1} u_{J,k}\widetilde{\phi}_{J,k} + \sum_{j=j_0}^{J}\sum_{k=0}^{2^J-1} d_{J,k}\widetilde{\psi}_{J,k}. \tag{21}$$

A different kind of approximation for $u \in V_J$, well known as *sampling interpolation* is performed as

$$u_J(z) = \sum_{k=0}^{2^J-1} u(\frac{k}{2^J})\widetilde{\phi}_{J,k}, \tag{22}$$

More details about approximation properties (20-22) is referred to [14].

Finally, for any $u, \nu \in L^2(\Omega)$, the inner product is defined as

$$(u, \nu) = \int_\Omega u\nu d\Omega,$$

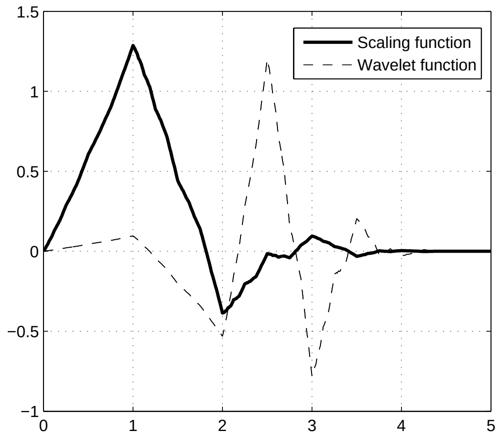

Figure 1: Daubechies scaling and wavelet functions for $N = 6$ with support on [0,5].

3.1 Split Step Wavelet Galerkin Method

Let us, briefly apply the wavelet Galerkin method for low-order PE (3) [21]. Substituting (4) in equation (3), gives the narrow angle or SPE of Tappert as

$$\frac{\partial \psi}{\partial r} = -\frac{ik_0}{2} X\psi. \tag{23}$$

Now, the depth domain $\Omega = [0, z_b]$ is covered with a set of non-coincident dyadic points $\{z_0, z_1, \dots, z_N\}$, where N is a positive integer.

Multiplying both sides of the (23) by a weighted function $\nu(z)$ and integrating over domain Ω gives

$$\int_{\Omega} [\psi_r + \frac{ik_0}{2} X\psi]\nu(z)dz = 0. \tag{24}$$

The equation (24) is called weak form of (23). In wavelet Galerkin depth discretization, the approximation space of ψ is considered at the subspace $\tilde{V_J}$ [10]. Thus, ν is substituted by 1-periodic Daubechies scaling function $\widetilde{\phi}_{J,l}$ and yields the following equation

$$(\psi_r, \widetilde{\phi}_{J,l}) + \frac{ik_0}{2}(X\psi, \widetilde{\phi}_{J,l}) = 0. \tag{25}$$

Interpolation formula for ψ in resolution J is derived from equation (20) as

$$\psi_J(z, r) = \sum_{k=0}^{2^J-1} c_{J,k}(r)\widetilde{\phi}_{J,k}(z). \tag{26}$$

By substituting (26) in equation (25) the wavelet Galerkin depth discretization is obtained as

$$\begin{aligned} \sum_k \frac{\partial}{\partial r} c_{J,k}(r)(\widetilde{\phi}_{J,k}(z), \widetilde{\phi}_{J,l}(z)) &\quad + \\ \sum_k c_{J,k}(r)[\frac{i}{2k_0}(\widetilde{\phi}''_{J,k}(z), \widetilde{\phi}_{J,l}(z)) &\quad + \\ \frac{ik_0}{2}(V(z)\widetilde{\phi}_{J,k}(z), \widetilde{\phi}_{J,l}(z))] &\quad = \quad 0, \end{aligned}$$

where, $l = 0, \dots, 2^J - 1$. In matrix notation the above formulation is changed as

$$\frac{\partial c_J}{\partial r} = [\mathcal{L} + \mathcal{N}]c_J = \mathcal{H}_{op}\, c_J, \tag{27}$$

The next step to SSWGM consists of solving (27) as an ordinary differential equation in r [20]. This gives

$$c_J(r + \Delta r) = \exp(\int_r^{r+\Delta r} \mathcal{H}_{op} dr')\, c_J(r). \tag{28}$$

In equation (27), we have

$$\begin{aligned} \mathcal{L}_{k,l} &= \frac{i}{2k_0} \int \widetilde{\phi}''_{J,k}(z)\widetilde{\phi}_{J,l}(z)\, dz, \\ \mathcal{N}_{k,l} &= \frac{ik_0}{2} \int V(z)\widetilde{\phi}_{J,k}(z)\widetilde{\phi}_{J,l}(z)\, dz, \end{aligned}$$

and

$$\int \widetilde{\phi}''_{J,k}(z)\widetilde{\phi}_{J,l}(z)\, dz = \delta(k, l),$$

where, δ is Kronecker function. These integral components are called two term *connection coefficients* with the following general form

$$\Gamma_k^d = \int_{-\infty}^{+\infty} \phi(z)\phi_k^d(z)dx, \;\; 2 - N \le k \le N - 2. \tag{29}$$

Since the support of ϕ and ϕ^d overlap only for $1-N \le k \le N-1$ There are $2N-3$ nonzero connection coefficients that can be calculated by the following properties

$$\begin{aligned} &\Gamma_k^d = (-1)^k \Gamma_{-k}^d, \;\; k \in [2 - N, N - 2], \\ &\sum_{r=0}^{N-1}\sum_{s=0}^{N-1} a_r a_s \Gamma_{2k+s-r}^d = \frac{1}{2^d}\Gamma_k^d, \;\; k \in [2 - N, N - 2], \\ &\sum_{k=2-N}^{n-2} M_k^d \Gamma_k^d = d!, \end{aligned}$$

More details can be referred in [19, 11, 15]. The matrix representation of these coefficients is sparse and circulant then solution can be handled completely by use of the FFT for decreasing computational cost. interpolating parameter c_J and its range derivative, between two range level n and $n+1$ for equation (27) are defined as

$$c_J(r + \Delta r) = \exp(\Delta r \mathcal{L}) \exp(\Delta r \mathcal{N}) c_J(r). \tag{30}$$

The exponential transform of linear operator $\mathcal{L}$ is defined by Taylor expansion as

$$\exp(\Delta r\mathcal{L}) = I + \Delta r\mathcal{L} + \frac{(\Delta r)^2}{2!}\mathcal{L}^2 + \cdots , \qquad (31)$$

that based on matrix multiplications with computational complexity O(N^3). Thus, computational cost of approximating (31) can be very expensive. Another useful approximation for this exponential term has been applied at follows

$$c_J^{n+1} = \exp(\Delta r\frac{\mathcal{L}}{2})\exp(\Delta r\mathcal{N})\exp(\Delta r\frac{\mathcal{L}}{2})c_J^n, \qquad (32)$$

with initial field $c_J(0)$ that we use Gaussian starter [4]. The left side of (31) is circulant. Thus, it can be discretize as

$$\begin{aligned} \exp(\Delta r\mathcal{L}) &= \mathcal{F}^{-1}\Lambda_e\mathcal{F}, \\ \Lambda_e &= diag(\hat{e}), \\ \hat{e} &= \mathcal{F}e, \end{aligned}$$

where, e is the first column of matrix $\exp(\Delta r\mathcal{L})$, and operator $\mathcal{F}$ is the FFT.

4 Numerical Result

To illustrate many features of idealized ocean for deep water wave propagation, we use Munk sound speed profile as

$$c(z) = 1500.0\,[1.0 + \varepsilon(\tilde{z} + 1 - e^{-\tilde{z}})], \qquad (33)$$

the quantity ε is as

$$\varepsilon = 0.00737,$$

while, the scaled depth $\tilde{z}$ is taken as

$$\tilde{z} = \frac{2(z-1300)}{1300}.$$

This sound speed profile is plotted in figure 2. Taking a homogeneous, halfspace bottom with velocity, density and attenuation 1700m/s,1500kg/m^3 and 0.5dB/km, respectively, the numirical results based on our proposed method is shown in figure 3 for source depth and frequency 100m and 50Hz, respectively. In the next example, we consider a shallow water environment with strong contrasts in velocity, density and attenuation. This model consists wedge-shaped ocean for 2.86° wedge with a penetrable lossy bottom. The initial water depth is 200m decreasing linearly to zero at a range of 4km for the depth of the source 100m. Figure 4 shows the results of frequency 200Hz and figures 5-8 show transmission loss for receiver depth 30m and frequencies, 20Hz, 200Hz, 1000Hz and 2000Hz respectively. As can be seen from this figures, our method is in good agreement with the physical behavior of wave propagation in deep and shallow water [4].

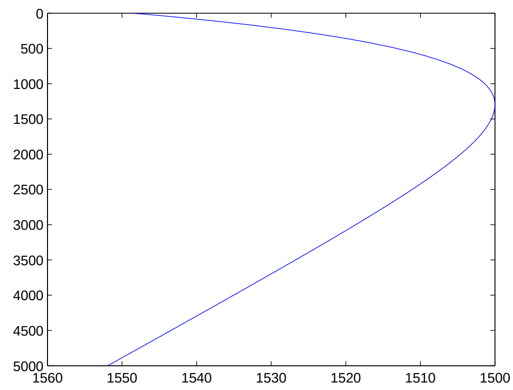

Figure 2: The Munk sound speed profile.

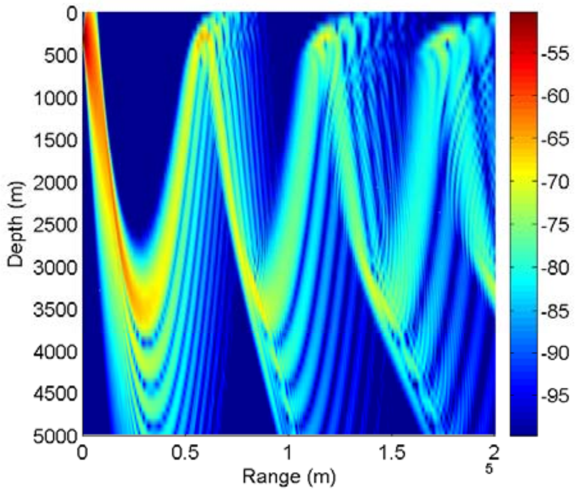

Figure 3: Transmission loss shade for the Munk profile for waterborn mode.

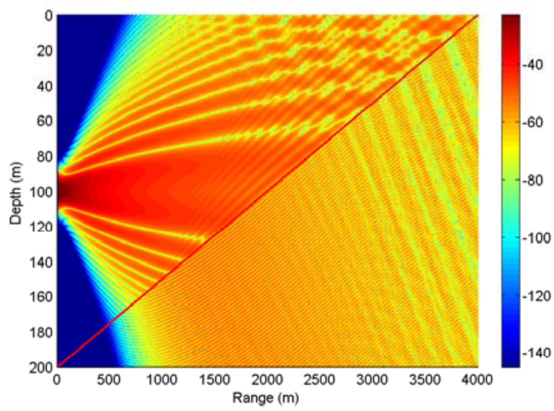

Figure 4: Transmission loss shade for source frequency 200Hz.

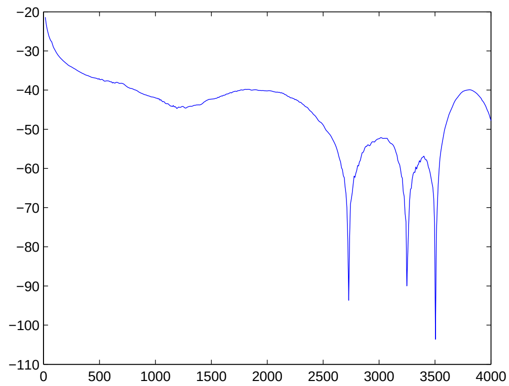

Figure 5: Transmission loss profile for source frequency 20Hz and receiver depth 30m.

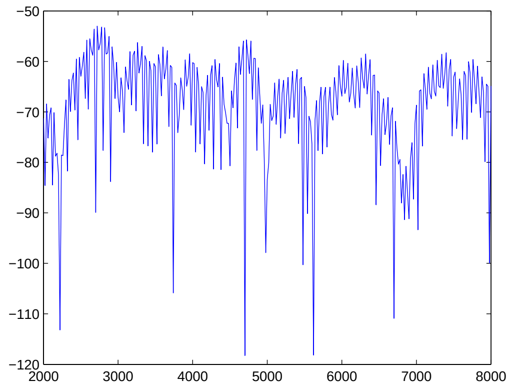

Figure 6: Transmission loss profile for source frequency 200Hz and receiver depth 30m.

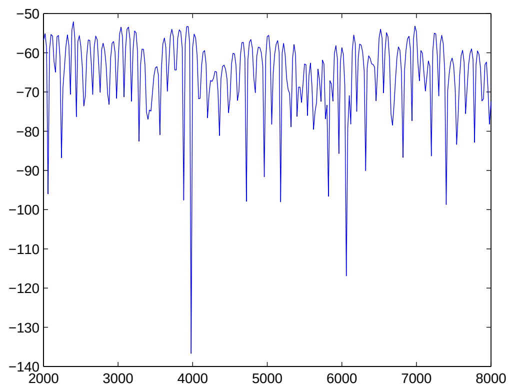

Figure 7: Transmission loss profile for source frequency 1000Hz and receiver depth 30m.

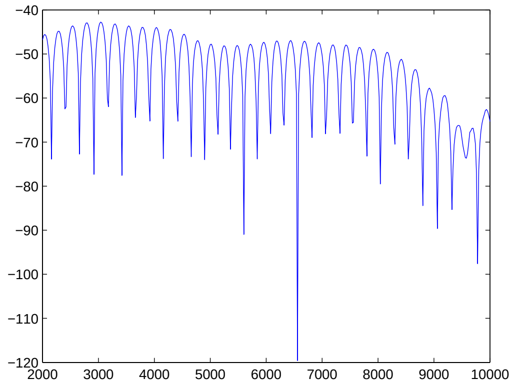

Figure 8: Transmission loss profile for source frequency 2000Hz and receiver depth 30m.

5 Conclusion

In this paper, split step wavelet Galerkin method (SS-WGM)was formulated for solving underwater wave propagation in range-independent fluid/solid media. Parabolic equation model was applied for transforming elliptic wave to parabolic wave equation that enable us to use marching approaches in numerical algorithms. The numerical solution of SSWGM was applied for deep and shallow water environment involving water column over bottom. To evaluate the efficiency of the proposed method, some simulations was demonstrated and the usefulness of SSWGM was highlighted through them.

References:

[1] A. Z. Hyaric, Wide-angle nonlocal boundary conditions for the parabolic wave equation, IEEE transaction on antennas and propagation, Vol. 49, NO. 6, 2001.

[2] T. Tsuchiya, T. Anada and N. Endoh, An efficient method combined the Douglas operator scheme to split-step Pade approximation of higher order parabolic equation, IEEE ultrasonics symposium, 2001.

[3] M. D. Collins, W.A. Kuperman b, B. Edward McDonaldc and W. L. Siegmann, Parabolic equation modeling of azimuthally advected gravity waves, Wave Motion, Vol. 31, PP. 131138, 2000.

[4] B. Jensen, W.A. Kuperman, M.B. Porter and H. Schmidt, Computational Ocean Acoustics, AIP series in modern acoustics and signal processing, 1993.

[5] M. D. Collins, W. A. Kuperman, B. E. McDonald and W. L. Siegmann, Parabolic equation modeling of az-

imuthally advected gravity waves, Wave Motion, Vol, 31, PP. 131-138, 2000.

[6] T. Kremp, Pade approximation for improved split step methods, Optical Soc. Am., 2003.

[7] M.D. Collins, A higher-order parabolic equation for wave propagation in an ocean overlying an elastic bottom, J. Acoust. Soc. Am., 1989.

[8] G. Beylkin, J.M. Keiser,, On the addaptive numerical solution of nonlinear partial differential equations in wavelet bases, Journal of computational physics, Vol.132, PP. 233-259, 1997.

[9] G. Beylkin, J.M. Keiser and L. Vozovoi, A new class of time discretization schemes for the solution of nonlinear PDEs, Journal of computational physics, Vol. 147, No. 362, 1998.

[10] J.C. Xu, X.C. Shann , Galerkin wavelet method for two point boundary value problem , J. Num. Math. , Vol. 63, No. 1, PP. 123-142, 1992.

[11] E.B. Lin, X. Zhou, Connection coefficient on an interval and wavelet solution of Burger equation. Journal of computational and applied mathematics, Vol. 135, PP. 63-78, 2001.

[12] E. Barcy, S. Mallat and G. Papanicolau, A wavelet based space time adaptive numerical method for partial differential equations, Mathematical Modeling and Numerical Analysis, Vol. 26, No. 793, 1992.

[13] S. Qian, J. Wiess, Wavelets and the numerical solution of partial differential equations. J. Comp. Phys., Vol. 106, PP. 155-175, 1993.

[14] I. Daubechies, Orthonormal bases of compactly supported wavelets, Bell Laboratories communications on pure and applied mathematics, PP. 909-996, 1988.

[15] O.M. Nielson, Wavelet in scientific computing, Ph.D thesis, Technical university of Denmark, 1998.

[16] G.H. Brooke and D.J. Thomson, Non-local boundary conditions for high-order parabolic equation algorithms, Wave Motion, Vol. 31, PP. 117-129, 2000.

[17] M. Ehrhadt and A. Zisowsky, Discrete non-local boundary conditions for Split-Step pade approximation of the One-Way Helmholtz equation, Preprint to Journal of computational and applied mathematics, 2005.

[18] Y.Y. Lu, A complex coefficient rational approximation of, Applied numerical mathematics, Vol. 27, PP. 141-154, 1998.

[19] G. Beylkin, On the representation of operations in basis of compactly support wavelet, Siam J. Num. Anal., Vol. 6, No. 6, 1716-1740, 1992.

[20] J. N. McMullin, Explicit split step propagation beam method, IEEE photonics technology letters, Vol. 6, NO. 4, 1994.

[21] Avudainayagam, A. and Vani, C., Wavelet Galerkin method for integro differential equation. Applied Nummerical Mathematics, Vol. 32, PP. 247-254, 2000.