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Transpose of the Weighted Mean Matrix on Weighted Sequence Spaces

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Abstract: *In this paper, we concern with transpose of the weighted mean matrix (This is upper triangular matrix.) on weighted sequence spaces $\ell_p(w)$ and $L_p(w)$ which is considered by the author in [8] and [9] for special case of these operator, such as Copson on $\ell_1(w)$ and $d(w,1)$. Also, in a recent paper[7], the author has discovered the upper bound for the Copson operator on the weighted sequence spaces $d(w,p)$. Also, we establish analogous upper bound for the continuous case. The weighted mean matrices are considered by the author in [10].*

Key Words: *Transpose of Weighted Mean Matrix, Weighted Sequence Space.*
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1. Introduction and Notations:

In this note, we consider the problem of finding the norm of transpose of the weighted mean matrix $A_d = (a_{n,k})$, denoted by A_d^t, where

$$a_{n,k} = \begin{cases} \frac{d_k}{D_n} & \text{for} \quad 1 \le k \le n \\ 0 & \text{for} \quad k > n. \end{cases}$$

where the d_ns are non-negative numbers with partial sum $D_n = d_1 + \ldots + d_n$(We insist that $d_1 > 0$, so that each D_n is positive.).

These results are extension of some results which is considered by the author in [8] and [9] and Bennett[2] and [4]. If $r_n = \sum_{k=1}^{n} \frac{w_k d_k}{D_n}$, and also R_n and W_n are defined as usual, then the norm of A_d^t on $\ell_1(w)$ is the *supremum* of $\frac{R_n}{W_n}$.

Let $w = (w_n)$ be a decreasing, non-negative sequence with $\lim_{n\to\infty} w_n = 0$ and $\sum_{n=1}^{\infty} w_n$ divergent. Write $W_n = w_1 + \ldots + w_n$. Then $\ell_p(w)$ (and the Lorentz sequence space $d(w,p)$), where $p \ge 1$, is the space of sequences $x = (x_n)$ with

$$\|x\|_{\ell_p(w)} = (\sum_{n=1}^{\infty} w_n |x_n|^p)^{1/p},$$

$$\|x\|_{w,p} = (\sum_{n=1}^{\infty} w_n x_n^{*p})^{1/p}$$

convergent, where (x_n^*) is the decreasing rearrangement of $|x_n|$.

We now consider the operator A defined by $Ax = y$, where $y_n = \sum_{k=1}^{\infty} a_{n,k} x_k$. We shall write $\|A\|_{\ell_p(w)}$ for the norm of A when regarded as an operator from $\ell_p(w)$ to $\ell_p(w)$, where

$$\|A\|_{\ell_p(w)} = \sup\{\|Ax\|_{\ell_p(w)} : \|x\|_{\ell_p(w)} \le 1\},$$

$$\|A\|_{w,p} = \sup\{\|Ax\|_{w,p} : \|x\|_{w,p} \le 1\}.$$

Also, we define

$$M_{w,p}(A) = \sup\{\|Ax\|_{\ell_p(w)} : \|x\|_{\ell_p(w)} = 1\},$$

where $x = (x_n)$ is regarded as a decreasing, non-negative sequences in $\ell_p(w)$.

We assume that

1) $a_{n,k} \ge 0$ for all n, k. This implies that $|Ax| \le A(|x|)$ for all x, and hence the non-negative sequences x are sufficient to determine $\|A\|_{\ell_p(w)}$.

We assume further that each $A(e_k)$ is in $\ell_1(w)$, that is:

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2) $\sum_{n=1}^{\infty} w_n a_{n,k}$ is convergent for each k, that garantte each $A(e_k)$ is in $\ell_1(w)$.

For two finite sequence $x = (x_n)$ and $y = (y_n)$, write $y \ll x$ if

$$Y_k \leq X_k \quad (\forall k),$$

where $X_k = \sum_{i=1}^{k} x_i$ and $Y_k = \sum_{i=1}^{k} y_i$.

Lemma 1: Suppose $x, y \in \mathbb{R}^n$ with $x \ll y$ and (a_i) is decreasing. Suppose also

either $a_n \geq 0$,

or $X_n = Y_n$. Then

$$\sum_{k=1}^{n} a_k x_k \leq \sum_{k=1}^{n} a_k y_k.$$

Proof: By Abel summation, it follows that

$$\begin{aligned} \sum_{k=1}^{n} a_k y_k &= \sum_{k=1}^{n} a_k (Y_k - Y_{k-1}) \quad (Y_0 = 0) \\ &= \sum_{k=1}^{n-1} Y_k (a_k - a_{k+1}) + a_n Y_n. \end{aligned}$$

Now, applying the hypothesis in both cases, we deduce that

$$\begin{aligned} \sum_{k=1}^{n-1} Y_k (a_k - a_{k+1}) + a_n Y_n &\geq \\ \sum_{k=1}^{n-1} X_k (a_k - a_{k+1}) + a_n X_n &= \sum_{k=1}^{n} a_k x_k. \end{aligned}$$

Therefore

$$\sum_{k=1}^{n} a_k x_k \leq \sum_{k=1}^{n} a_k y_k.$$

Corollary: Let x, y be decreasing, non-negative elements of $\mathbb{R}^n$ (or $\ell_1(w)$) with $x \ll y$. Then

$$\|x\|_{\ell_1(w)} \leq \|y\|_{\ell_1(w)}.$$

Proposition 1([8], Proposition 1.4.1): Suppose that (1) holds, and that

(3) for all subsets M, N of $\mathbb{N}$ having m, n elements respectively, we have

$$\sum_{i \in M} \sum_{j \in N} a_{i,j} \leq \sum_{i=1}^{m} \sum_{j=1}^{n} a_{i,j}.$$

Then

$\|Ax\|_{\ell_1(w)} \leq \|Ax^*\|_{\ell_1(w)} (\|Ax\|_{w,1} \leq \|Ax^*\|_{w,1})$ for all non-negative elements x of $\ell_1(w)(d(w,1))$, where x^* is the decreasing rearrangement of $|x_n|$. Hence decreasing, non-negative sequences are sufficient to determine $\|A\|_{\ell_1(w)} (\|A\|_{w,1})$.

Proposition 2([3], Lemma 9): Let $A = (a_{i,j})_{i,j=1}^{\infty}$ be a matrix operator with non-negative entrie s, and consider the associated transformation, $x \to y$, given by $y_i = \sum_{j=1}^{\infty} a_{i,j} x_j$. Then the following conditions are equivalent:

(i) $y_1 \geq y_2 \geq \ldots \geq 0$ whenever $x_1 \geq x_2 \geq \ldots \geq 0$.

(ii) $r_{i,n} \geq r_{i+1,n} \quad (i, n = 1, 2, ...)$,

where $r_{i,n} = \sum_{j=1}^{n} a_{i,j}$.

Proof: $(i) \Longrightarrow (ii)$ follows by taking x to be the sequence $(1, ..., 1, 0, ...)$ of n ones followed by zeros.

$(ii) \Longrightarrow (i)$: By Abel summation, it follows that

$$y_i = \sum_{j=1}^{\infty} a_{i,j} x_j = \sum_{n=1}^{\infty} r_{i,n} (x_n - x_{n+1}).$$

Since $r_{i,n} \geq r_{i+1,n}$ $(\forall i, n)$, and also (x_n) is decreasing, non-negative sequence, then

$$\begin{aligned} \sum_{n=1}^{\infty} r_{i,n} (x_n - x_{n+1}) &\geq \sum_{n=1}^{\infty} r_{i+1,n} (x_n - x_{n+1}) \\ &= \sum_{j=1}^{\infty} a_{i+1,j} x_j = y_{i+1}. \end{aligned}$$

This completes the proof of the statement.

Lemma 2: Suppose $u = (u_n)$ and $w = (w_n)$ are sequences of positive numbers.

(i) If $m \leq \frac{u_n}{w_n} \leq M$ for all n, then $m \leq \frac{U_n}{W_n} \leq M$ for all n.

(ii) If $\left(\frac{u_n}{w_n}\right)$ is increasing (or decreasing), then so is $\left(\frac{U_n}{W_n}\right)$.

(iii) If $\frac{u_n}{w_n} \longrightarrow U$ as $n \longrightarrow \infty$, then $\frac{U_n}{W_n} \longrightarrow U$ as $n \longrightarrow \infty$ (also with $U = \infty$).

Proof. Elementary.

3. Transpose of the Weighted Mean operator

Let now A_d be the weighted mean matrix with properties (1), (2) and (3), and A_d^t be its transpose which is defined as follows:

$$(A_d^t x)(n) = \sum_{k=n}^{\infty} \frac{d_n x_k}{D_k}.$$

This is an upper triangular matrix.

Recall that (w_n) is said to be 1-regular if

$$r_1(w) = \sup_{n\geq 1} \frac{W_n}{nw_n}$$

is finite[11]. A pleasently simple statement can also be made about the norm of the weighted mean matrix operator for general $w = (w_n)$. With the previous notation,

$$r_n = \frac{1}{n}(w_1 + \ldots + w_n) = \frac{W_n}{n}.$$

Theorem 1: Suppose A_d^t is a weighted mean operator defined as before and also $d = (d_n)$ is such that $nd_n \leq D_k \quad (\forall k \geq n)$. If $w = (w_n)$ is 1-regular, then for $p > 1$, we have:

$$\|A_d^t\|_{w,p} \leq pr_1(w) \quad \& \quad \left(\|A_d^t\|_{\ell_p(w)} \leq pr_1(w)\right).$$

Proof: As mentioned before, it is sufficient to consider decreasing, non-negative sequences. Let x be in $\ell_p(w)$ or $d(w,p)$ such that $x_1 \geq x_2 \geq \ldots \geq 0$, and so $\|x\|_{w,p} = \|x\|_{\ell_p(w)}$ and also the same is true for the norm of A_d^t. Then applying [11], Theorem 4.1.6, we deduce that:

$$\begin{aligned}
\|A_d^t\|_{w,p}^p &= \sum_{n=1}^{\infty} w_n \left(\sum_{k=n}^{\infty} \frac{d_n x_k}{D_k}\right)^p \\
&\leq \sum_{n=1}^{\infty} w_n \left(\sum_{k=n}^{\infty} \frac{x_k}{k}\right)^p \\
&\leq (pr_1(w))^p \sum_{n=1}^{\infty} w_n x_n^p. \\
&= (pr_1(w))^p \|x\|_{w,p}^p.
\end{aligned}$$

Hence $\|A_d^t\|_{w,p} \leq pr_1(w)$. This completes the proof.

Theorem 2: Suppose A_d^t is an operator on $\ell_1(w)$. If

$$R = \sup \frac{R_n}{W_n} < \infty,$$

where $r_n = \sum_{k=1}^n \frac{w_k d_k}{D_n}$, and $R_n = r_1 + \ldots + r_n$ and W_n as usual. Then A_d^t is a bounded operator from $\ell_1(w)$ into itself, and we have $M_{w,1}(A_d^t) = R = \|A_d^t\|_{w,1}$.

Proof: Since $(A_d^t x)(n) \leq (A_d^t x^*)(n)$ for all n, it is sufficient to consider decreasing, non-negative sequences. Let x be in $\ell_1(w)$ such that $x_1 \geq x_2 \geq \ldots \geq 0$. Then

$$\begin{aligned}
\|A_d^t x\|_{w,1} = \|A_d^t x\|_{\ell_1(w)} &= \sum_{n=1}^{\infty} \left(\sum_{k=n}^{\infty} \frac{d_n x_k}{D_k}\right) \\
&= \sum_{n=1}^{\infty} r_n x_n \\
&= \sum_{n=1}^{\infty} R_n (x_n - x_{n+1}) \\
&\leq R \sum_{n=1}^{\infty} W_n (x_n - x_{n+1}) \\
&= R \sum_{n=1}^{\infty} w_n x_n.
\end{aligned}$$

Hence $\|A_d^t\|_{w,1} = M_{w,1}(A_d^t) \leq R$.

We have to show that this constant is the best possible. We take $x_1 = x_1 = \ldots = x_n = 1$ and $x_k = 0$ for all $k \geq n+1$. Then

$$\|x\|_{\ell_1(w)} = W_n \quad \& \quad \|A_d^t x\|_{\ell_1(w)} = R_n.$$

Therefore $M_{w,1}(A_d^t) = R = \|A_d^t\|_{w,1}$.

3. Copson Operator on Weighted Sequence Spaces:

We now consider the Copson operator C on $\ell_1(w)$ and $d(w,1)$, which is defined by $y = Cx$, where

$$y_i = \sum_{j=i}^{\infty} \frac{x_j}{j}.$$

It is given by the transpose of the matrix of the Averaging operator A:

$$a_{i,j} = \begin{cases} \frac{1}{j} & \text{for} \quad i \leq j \\ 0 & \text{for} \quad i > j \end{cases}$$

This is an upper triangular matrix. The classical inequality of Copson [5] and [6] states that $\|C\|_p = \|A^t\|_p = p(p > 1)$ as an operator on ℓ_p spaces.

Proposition 3: If $w = (w_n)$ is 1-regular, then C maps $\ell_1(w)$ into $\ell_1(w)$. Also, we have

$$\|C\|_{w,1} = M_{w,1}(C) \leq \|C\|_{\ell_1(w)} \leq r_1(w).$$

Proof: Since

$$r_n = \frac{W_n}{n} \leq r_1(w) w_n \quad (\forall n),$$

then by Lemma 2(i) and Theorem 2, it follows that

$$\|C\|_{w,1} = M_{w,1}(C) \leq \|C\|_{\ell_1(w)} \leq r_1(w).$$

Corollary 1([8], Theorem 2.3.1): If

$$\sup \frac{1}{W_n} \sum_{n=1}^{n} \frac{W_k}{k} < \infty,$$

then the Copson operator is a bounded operator from $d(w,1)$ into itself, and also we have

$$M_{w,1}(C) = \|C\|_{w,1} = \sup \frac{1}{W_n} \sum_{n=1}^{n} \frac{W_k}{k}.$$

Write

$$u_n = \frac{1}{n^r}, \quad (r > 0) \quad v_n = \int_{n-1}^{n} \frac{1}{t^r} dt$$

and(as usual) $U_n = u_1 + \ldots + u_n$, etc. For $r < 1$, the usual integral comparison gives

$$v_2 + \ldots + v_n \leq U_n \leq V_n,$$

or

$$\frac{1}{1-r}(n^{1-r} - 1) \leq U_n \leq \frac{n^{1-r}}{1-r},$$

we need to know that $\frac{U_n}{V_n}$ is increasing. The following is the key lemma.

Lemma 3: With v_n as above (for any $r > 0$), $n^r v_n$ decreases with n and $n^r v_{n+1}$ increases with n.

Proof: Write $t_n = n^r v_n$. Then

$$t_{n+1} = (n+1)^r \int_{n}^{n+1} \frac{ds}{s^p} = (n+1)^r \int_{n-1}^{n} \frac{ds}{(s+1)^r}.$$

For $n-1 \leq s \leq n$, we have $\frac{n+1}{n} \leq \frac{s+1}{s}$, hence $\frac{(n+1)^r}{(s+1)^r} \leq \frac{n^r}{s^r}$. Therefore $t_{n+1} \leq t_n \quad (\forall n)$. Similarly for the second statement.

Proposition 4: Let $0 < r < 1$ and let $U_n = \sum_{j=1}^{n} \frac{1}{j^p}$. Then $\frac{U_n}{n^{1-r}}$ increases and tends to $\frac{1}{1-r}$.

Proof: With v_n as above, by Lemma 3, $\frac{u_n}{v_n}$ increases with n, and so applying Lemma 1, we deduce that $\frac{U_n}{V_n}$ is increasing. The limit follows from the inequalities above.

We now consider the tail of the series for $\zeta(1+p)$. For the tail of a series, the analogous result to Lemma 2(ii) is the following.

Lemma 4: Suppose that $v_n > 0$, $u_n > 0$ for all n and that $\sum_{n=1}^{\infty} v_n$ and $\sum_{n=1}^{\infty} u_n$ are convergent. Let $U_{(n)} = \sum_{j=n}^{\infty} u_j$, similarly $V_{(n)}$. If $\left(\frac{u_n}{v_n}\right)$ is increasing (or decreasing), then so is $\left(\frac{U_{(n)}}{V_{(n)}}\right)$.

Proof: Elementary.

Proposition 5: Let $r > 0$ and let $U_{(n)} = \sum_{j=n}^{\infty} \frac{1}{j^{1+r}}$. Then $n^r U_{(n)}$ decreasing, $(n-1)^r U_{(n)}$ increasing. Both tend to $\frac{1}{r}$ as $n \longrightarrow \infty$.

Proof: Let $u_n = \frac{1}{n^{1+r}}$ and $v_n = \int_{n-1}^{n} \frac{1}{t^{1+r}} dt$. Then $V_{(n+1)} = \frac{1}{rn^r}$. By the usual integral comparison,

$$\frac{1}{rn^r} \leq U_{(n)} \leq \frac{1}{r(n-1)^r},$$

which implies the stated limits. By Lemma 3, $\left(\frac{u_n}{v_{n+1}}\right)$ is decreasing, so by Lemma 2(ii), $\frac{U_{(n)}}{V_{(n+1)}} = rn^r U_{(n)}$ is decreasing. Similarly, $\frac{U_{(n)}}{V_{(n)}}$ is increasing.

Remark: This is stated without proof in [1], Remark 4.10.

Theorem 3: If $w = (\frac{1}{n^p})$, $\;0 < p \leq 1$, then the Copson operator C is a bounded operator from $\ell_1(w)(d(w,1))$ into itself. Also, we have

$$M_{w,1}(C) = \|C\|_{w,1} = \|C\|_{\ell_1(w)} = \frac{1}{1-p}.$$

Proof: Since $r_n = \frac{w_n}{n}$, then

$$\frac{r_n}{w_n} = \frac{W_n}{nw_n} = \frac{W_n}{n^{1-p}}.$$

Also, since W_n is the U_n of the Proposition 4, then $\frac{W_n}{n^{1-p}}$ increases with n and tends to $\frac{1}{1-p}$. Hence applying Proposition 2, we deduce that

$$M_{w,1}(C) = \|C\|_{w,1} = \|C\|_{\ell_1(w)} = \frac{1}{1-p}.$$

Remark: When $p = 1$, so that $w_n = \frac{1}{n}$, we have

$$\frac{r_n}{w_n} = W_n \longrightarrow \infty \quad (\text{as } n \longrightarrow \infty),$$

so the Copson operator C is not a bounded operator on $d(w,1)$, although of course is satisfies condition (2).

Theorem 4: Let w_n be defined by $W_n = n^{1-p}$, where $0 < p < 1$. Then the Copson operator is a bounded operator from $d(w, 1)$ into itself. Also, we ahve

$$\|C\|_{w,1} = M_{w,1}(C) = \frac{1}{1-p}.$$

Proof: We now have

$$R_n = \sum_{k=1}^{n} \frac{W_k}{k} = \sum_{k=1}^{n} \frac{1}{k^p},$$

so the new $\frac{R_n}{W_n}$ is exactly the $\frac{r_n}{w_n}$ of Theorem 3 and Proposition 4 again gives the statement.

4. Continous Version of the Copson Opreator:

In this section, we consider the analogous problem for the continuous case concern the space $Ł_p(w)$. In the continuous case, the Copson operator C is given by:

$$(Cf)(x) = \int_x^{\infty} \frac{f(t)}{t} dt.$$

Let $w(x)$ be a decreasing, non-negative function on $(0, \infty)$. We assume that $W(x) = \int_0^x w(t)dt$ is finite for each x(Hence $\frac{1}{x^\alpha}$ is permited for $0 < \alpha < 1$, but not for $\alpha = 1$.). Then $Ł_p(w)$ is the space of functions f having

$$\int_0^{\infty} w(x)|f(x)|^p dx$$

convergent, with norm

$$\|f\|_{Ł_P(w)} = \left(\int_0^{\infty} w(x)|f(x)|^p dx\right)^{1/p}.$$

Proposition 6: Let $f \geq 0$ be in $Ł_p(w)$, $a(x) = \frac{w(x)}{W(x)} f(x)$, and also $A_\infty(x) = \int_x^\infty a(t)dt$. Then $A_\infty(x)$ is finite and also we have:

$$\|A_\infty\|_{Ł_p(w)} \leq p\|f\|_{Ł_p(w)}.$$

Proof: Fix x_0. For any $x < x_0$, let $\int_x^{x_0} a(t)dt = A_{x_0}(x)$. Then $\frac{d}{dx} A_{x_0}(x)^p = -pA_{x_0}(x)^{p-1}a(x)$, and so

$$\begin{aligned} A_{x_0}(x)^p &= A_{x_0}(x)^p - A_{x_0}(x_0)^p \\ &= p\int_x^{x_0} A_{x_0}(t)^{p-1}a(t)dt. \end{aligned}$$

Hence, applying Holder's inequality, we deduce that:

$$\begin{aligned} \int_0^{x_0} w(x)A_{x_0}(x)^p dx &= \\ p\int_0^{x_0} w(x)\int_x^{x_0} A_{x_0}(t)^{p-1}a(t)dtdx &= \\ p\int_0^{x_0} A_{x_0}(t)^{p-1}a(t)\int_0^t w(x)dxdt &= \\ p\int_0^{x_0} A_{x_0}(t)^{p-1}a(t)W(t)dt &= \\ p\int_0^{x_0} w(t)A_{x_0}(t)^{p-1}a(t)f(t)dt &\leq \end{aligned}$$

$$p\left(\int_0^{x_0} w(t)f(t)^p dt\right)^{1/p}\left(\int_0^{x_0} w(t)A_{x_0}(t)^p dt\right)^{1/p^*}.$$

Therefore

$$\left(\int_0^{x_0} w(t)A_{x_0}(t)^p dt\right)^{1/p} \leq p\|f\|_{Ł_p(w)}.$$

The above inequality is true for all $x_0 > 0$, and so true with x_0 replacing by infinity. This completes the proof.

Proposition 7: If $\frac{W(x)}{w(x)} \leq r_1(w) \quad (\forall x > 0)$, then

$$\|C\|_{Ł_p(w)} \leq pr_1(w).$$

Proof: We have

$$\frac{1}{t} \leq r_1(w)\frac{w(t)}{W(t)},$$

and so

$$(Cf)(x) \leq r_1(w)\int_x^{\infty} \frac{w(t)}{W(t)} f(t)dt = r_1(w)A_\infty(x).$$

This establishes the statement.

Theorem 5: If $w(x) = \frac{1}{x^\alpha}$, where $0 \leq \alpha < 1$, then

$$\|C\|_{Ł_p(w)} = \frac{p}{1-\alpha}.$$

Attained by action of C on decreasing positive functions.

Proof: (i) We have $W(x) = \frac{x^{1-\alpha}}{1-\alpha}$, and so

$$\frac{W(x)}{xw(x)} = \frac{1}{1-\alpha} \quad (\forall x > 0).$$

Hence

$$\|C\|_{Ł_p(w)} \leq \frac{p}{1-\alpha}.$$

(ii) Now, by taking $\varepsilon > 0$, and define r by: $\alpha + rp = 1 + \varepsilon$, we deduce that:

$$f(x) = \begin{cases} \frac{1}{x^r} & \text{for } x \geq 1 \\ 1 & \text{for } 0 \leq x < 1. \end{cases}$$

Then f is decreasing and in $Ł_p(w)$, since $\int_0^1 \frac{1}{x^\alpha} dx$ and $\int_0^\infty \frac{1}{x^{\alpha+rp}} dx$ are convergent. Also, we have:

$$(Cf)(x) = \int_x^\infty \frac{1}{t^{r+1}} dt = \frac{1}{rx^r} \quad \text{for } x \geq 1,$$

and also we have:

$$(Cf)(x) \geq (Cf)(1) = \frac{1}{r} \quad \text{for } 0 < x < 1.$$

Hence $(Cf)(x) \geq \frac{1}{r} f(x) \quad (\forall x > 0)$, and so

$$\|Cf\|_{Ł_p(w)} \geq \frac{1}{r} \|f\|_{Ł_P(w)},$$

where $\frac{1}{r} = \frac{p}{1-\alpha+\varepsilon}$. Now, applying (i) and (ii) implies the statement.

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