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Maximal inequalities for partial sums and strong law of large numbers

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Abstract- In this paper, we extend some famous maximal inequalities, and obtain strong laws of large numbers for arbitrary random variables by use of these inequalities and martingale techniques.

Key Words- Kolmogorove's inequality, Hajek-Renyi inequality, Strong Law of Large Numbers, Martingale.

1.INTRODUCTION

Let $\{X_n, n \geq 1\}$ be a sequence of random variables defined on the probability space $(\Omega,\ \mathcal{F}, \mathcal{P})$. The convergence problem with probability one for the sequence $\{\frac{1}{n}(\sum_{i=1}^{n} X_i - E(\sum_{i=1}^{n} X_i))\}$ has been studied by many authors the sequence of independent random variables . The strong laws of large numbers and complete convergence for ND random variables were studied by Amini and Bozorgnia [1],[2]. In [3] we obtained some maximal inequalities under condition $E[X_n|\mathcal{F}_{n-1}] = 0$ where $\mathcal{F}_n = \sigma(X_1, \cdots, X_n)$ for all $n \geq 1$. In this paper, we extend some famous maximal inequalities by martingale techniques , and then by using of these inequalities, we obtain strong laws of large numbers and some strong limit theorems for arbitrary random variables . To prove our main results we need the following lemma.

Lemma 1([8]) Let $\{X_n, \mathcal{F}_n,\ n \geq 1\}$ be a submartingale and $\{b_n,\ n \geq 1\}$ be a sequence of positive nondecreasing real numbers,then for every $\varepsilon > 0$,

$$P[\max_{1\leq k\leq n} \frac{|X_k|}{b_k} > \varepsilon] \leq \frac{1}{\varepsilon}(b_1^{-1}EX_1^{+} + \sum_{k=2}^{n} b_k^{-1}(EX_k^{+} - EX_{k-1}^{+})),$$

Remark 1 Under the assumption of lemma 1 for every $1 \leq m \leq n$, we have,

$$\begin{aligned} P[\max_{m\leq k\leq n} \frac{|X_k|}{b_k} > \varepsilon] = P[\max_{1\leq j\leq n-m+1} \frac{|X_{j+m-1}|}{b_{j+m-1}} > \varepsilon] &\leq \frac{1}{\varepsilon}(b_m^{-1}EX_m^{+} + \sum_{j=2}^{n-m+1} b_{j+m-1}^{-1}(EX_{j+m-1}^{+} - EX_{j+m-2}^{+})) \\ &= \frac{1}{\varepsilon}(b_m^{-1}EX_m^{+} + \sum_{k=m+1}^{n} b_k^{-1}(EX_k^{+} - EX_{k-1}^{+})). \end{aligned}$$

2. MAXIMAL INEQUALITIES

Hajek and Renyi (1955) proved the following important inequality. Let $\{X_n, n \geq 1\}$ be a sequence of independent random variables with $E(X_n) = 0,\ \ E(X_n^2) < \infty,\ n \geq 1$ and $\{b_n,\ n \geq 1\}$ is a positive nondecreasing sequence of real numbers, then for every $m < n$ and $\varepsilon > 0$,

$$P[\max_{m\leq j\leq n} \frac{|S_j|}{b_j} \geq \varepsilon] \leq \frac{1}{\varepsilon^2}(\sum_{j=m+1}^{n} \frac{\sigma_j^2}{b_j^2} + \sum_{j=1}^{m} \frac{\sigma_j^2}{b_m^2}),$$

where $\sigma_j^2 = Var(X_j)$.

This inequality has been studied by many authors. The latest literatures are given by J.Liu, S.Gan and P.Chen [8] for negative association random variables and T.C.Christofides [4] and [5]. We extend this inequality for arbitrary random variables.

Theorem 1. Let $\{X_n,\ n \geq 1\}$ be a sequence of random variables with $E(X_n) = 0, \quad EX_n^2 < \infty, n \geq 1$ and $\{b_n,\ n \geq 1\}$ be a sequence of positive nondecreasing real numbers, then for every $\varepsilon > 0$,
i)

$$P[\max_{1\leq k\leq n} \frac{|S_k|}{b_k} \geq \varepsilon] \leq \frac{8}{\varepsilon^2}(\sum_{j=1}^{n} \frac{\sigma_j^2}{b_j^2} + 2\sum_{j=2}^{n} \frac{\sigma_j \sum_{i=1}^{j-1} \sigma_i}{b_j^2}).$$

ii)

$$P[\max_{m\le k\le n}\frac{|S_k|}{b_k}\ge\varepsilon]\le\frac{8}{\varepsilon^2}(\sum_{k=1}^{m}\frac{\sigma_k^2}{b_m^2}+2\sum_{j=2}^{m}\frac{\sigma_j\sum_{i=1}^{j-1}\sigma_i}{b_m^2}+\sum_{k=m+1}^{n}\frac{\sigma_k^2}{b_k^2}+2\sum_{j=m+1}^{n}\frac{\sigma_j\sum_{i=1}^{j-1}\sigma_i}{b_j^2}).$$

Where $\sigma_i=\sqrt{Var(X_i)}$

Proof. We define $S_{1n}=\sum_{k=1}^{n}X_k^+$ and $S_{2n}=\sum_{k=1}^{n}X_k^-$. Since $E[S_{1n}|\mathcal{F}_{n-1}]\ge S_{1(n-1)},\quad w.p.1.$ and $E[S_{2n}|\mathcal{F}_{n-1}]\ge S_{2(n-1)},\quad w.p.1.$ the sequences $\{S_{1n},\ \mathcal{F}_n,\ n\ge1\}$ and $\{S_{2n},\ \mathcal{F}_n,\ n\ge1\}$ are submartingales where $\mathcal{F}_n=\sigma(X_1,\cdots,X_n)$ for all $n\ge1$. In addition if h is any real convex and nondecreasing function then $\{h(S_{1n}),\ \mathcal{F}_n,\ n\ge1\}$ and $\{h(S_{2n}),\ \mathcal{F}_n,\ n\ge1\}$ are also submartingales. Thus by lemma 1 for every $\varepsilon>0$, we have
i)

$$\begin{aligned}P[\max_{1\le k\le n}\frac{S_{1k}}{b_k}\ge\varepsilon]\le P[\max_{1\le k\le n}\frac{(S_{1k})^2}{b_k^2}\ge\varepsilon^2] &\le \varepsilon^{-2}(b_1^{-2}E(X_1^+)^2+\sum_{k=2}^{n}b_k^{-2}(E(S_{1k})^2-E(S_{1(k-1)})^2))\\ &\le \frac{1}{\varepsilon^2}(\sum_{j=1}^{n}\frac{\sigma_j^2}{b_j^2}+2\sum_{j=2}^{n}\frac{\sigma_j\sum_{i=1}^{j-1}\sigma_i}{b_j^2}),\end{aligned}$$

and similarly we have

$$P[\max_{1\le k\le n}\frac{S_{2k}}{b_k}\ge\varepsilon]\le\frac{1}{\varepsilon^2}(\sum_{j=1}^{n}\frac{\sigma_j^2}{b_j^2}+2\sum_{j=2}^{n}\frac{\sigma_j\sum_{i=1}^{j-1}\sigma_i}{b_j^2})$$

Now, by $|S_n|\le S_{1n}+S_{2n}$,we obtain

$$P[\max_{1\le k\le n}\frac{|S_k|}{b_k}\ge\varepsilon]\le P[\max_{1\le k\le n}\frac{S_{1k}}{b_k}\ge\frac{\varepsilon}{2}]+P[\max_{1\le k\le n}\frac{S_{2k}}{b_k}\ge\frac{\varepsilon}{2}]\le\frac{8}{\varepsilon^2}(\sum_{j=1}^{n}\frac{\sigma_j^2}{b_j^2}+2\sum_{j=2}^{n}\frac{\sigma_j\sum_{i=1}^{j-1}\sigma_i}{b_j^2}).$$

ii) By remark 1 for $1\le m<n$ and part i we obtain

$$\begin{aligned}P[\max_{m\le k\le n}\frac{|S_k|}{b_k}\ge\varepsilon] &\le P[\max_{m\le k\le n}\frac{S_{1k}}{b_k}\ge\frac{\varepsilon}{2}]+P[\max_{m\le k\le n}\frac{S_{2k}}{b_k}\ge\frac{\varepsilon}{2}]\\ &\le \frac{8}{\varepsilon^2}(\sum_{k=1}^{m}\frac{\sigma_k^2}{b_m^2}+2\sum_{j=2}^{m}\frac{\sigma_j\sum_{i=1}^{j-1}\sigma_i}{b_m^2}+\sum_{k=m+1}^{n}\frac{\sigma_k^2}{b_k^2}+2\sum_{j=m+1}^{n}\frac{\sigma_j\sum_{i=1}^{j-1}\sigma_i}{b_j^2}).\end{aligned}$$

Hence complete the proof. In the following we assume $\sum_{i=1}^{0}\sigma_i=0$ and $S_{1(0)}=S_{2(0)}=0$.

Corollary 1. Under the assumptions of Theorem 1 we have
i)

$$P[\max_{1\le k\le n}\frac{|S_k|}{b_n}\ge\varepsilon]\le\frac{8}{\varepsilon^2}(\sum_{j=1}^{n}\frac{\sigma_j^2}{b_j^2}+2\sum_{j=2}^{n}\frac{\sigma_j\sum_{i=1}^{j-1}\sigma_i}{b_j^2})$$

ii)

$$P[\max_{m\le k\le n}\frac{|S_k|}{b_n}\ge\varepsilon]\le\frac{8}{\varepsilon^2}(\sum_{k=m+1}^{n}\frac{\sigma_k^2}{b_k^2}+\sum_{k=1}^{m}\frac{\sigma_k^2}{b_m^2}+2\sum_{j=2}^{m}\frac{\sigma_j\sum_{i=1}^{j-1}\sigma_i}{b_m^2}+2\sum_{j=1}^{n}\frac{\sigma_j\sum_{i=1}^{j-1}\sigma_i}{b_j^2})$$

Remark 2 We have the following inequalities
i)

$$ES_{1k}^2-ES_{1(k-1)}^2=E(X_k^+)^2+2E(X_k^+\sum_{j=1}^{k-1}X_j^+)\le\sigma_k^2+2\sigma_k\sum_{j=1}^{k-1}\sigma_j,$$

the last inequality holds by Cauchy- Schwartz' inequality. Similarly $ES_{2k}^2-ES_{2(k-1)}^2\le\sigma_k^2+2\sigma_k\sum_{j=1}^{k-1}\sigma_j$.
ii)

$$\sum_{j=1}^{n}\frac{\sigma_j^2}{b_j^2}+2\sum_{j=2}^{n}\frac{\sigma_j\sum_{i=1}^{j-1}\sigma_i}{b_j^2}\le\sum_{j=1}^{n}\frac{1}{b_j^2}(\sigma_j^2+2\sigma_j\sum_{i=1}^{j-1}\sigma_i)\le2\sum_{j=1}^{n}\frac{\sigma_j\sum_{i=1}^{j}\sigma_i}{b_j^2}.$$

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iii) If $\sum_{j=1}^{\infty}\frac{\sigma_j\sum_{i=1}^{j}\sigma_i}{b_j^2}$ converges then series $\sum_{j=1}^{\infty}\frac{\sigma_j^2}{b_j^2}$ and $\sum_{j=1}^{\infty}\frac{\sigma_j\sum_{i=1}^{j-1}\sigma_i}{b_j^2}$ are converge.

The following corollary is an extension of Kolmogorov's inequality for arbitrary random variables. In fact we obtain an upper bound that is greater than Kolmogorov's bound, bout this inequality implies our results.

Corollary 2 Let $\{X_n,\ n\geq 1\}$ be a sequence of random variables with $E(X_n)=0$, $EX_n^2<\infty$, $n\geq 1$ and $b_k=1$, $k\geq 1$.Then for every $\varepsilon>0$,

$$P[\max_{1\leq k\leq n}|S_k|\geq\varepsilon]\leq\frac{8}{\varepsilon^2}(\sum_{j=1}^{n}\sigma_j^2+2\sum_{j=2}^{n}\sigma_j\sum_{i=1}^{j-1}\sigma_i)$$

3. SOME STRONG LIMIT THEOREMS

In this section, we use the results of section 2 to prove some strong limit theorems for arbitrary random variables.

Theorem 2. Let $\{X_n,\ n\geq 1\}$ be a sequence of random variables and $\{b_n,\ n\geq 1\}$ be a sequence of positive nondecreasing real numbers. If $\sum_{j=1}^{\infty}\frac{\sigma_j\sum_{i=1}^{j}\sigma_i}{b_j^2}<\infty$, then for every $0<\beta<2$,

i) $E(\sup_n(\frac{|S_n|}{b_n})^{\beta})<\infty$. and *ii)* If $b_n\longrightarrow\infty$, as $n\to\infty$, then $\frac{S_n}{b_n}\longrightarrow 0$, $w.p.1$.

Where $S_n=\sum_{k=1}^{n}(X_k-E(X_k))$, $\sigma_n^2=Var(X_n)$ and $\sigma_n=\sqrt{Var(X_n)}$, $n\geq 1$.

Proof. *i)* We have

$$E(\sup_n(\frac{|S_n|}{b_n})^{\beta})<\infty\quad\Leftrightarrow\quad\int_1^{\infty}P[\sup_n\frac{|S_n|}{b_n}>t^{\frac{1}{\beta}}]dt<\infty.$$

Now by Theorem 1 we obtain

$$\int_1^{\infty}P[\sup_n\frac{|S_n|}{b_n}>t^{\frac{1}{\beta}}]dt\leq 8\int_1^{\infty}t^{\frac{-2}{\beta}}(\sum_{n=1}^{\infty}\frac{\sigma_n^2}{b_n^2}+2\sum_{n=1}^{\infty}\frac{\sigma_n\sum_{i=1}^{n-1}\sigma_i}{b_n^2})dt=C(\sum_{n=1}^{\infty}\frac{\sigma_n^2}{b_n^2}+2\sum_{n=1}^{\infty}\frac{\sigma_n\sum_{i=1}^{n-1}\sigma_i}{b_n^2})<\infty.$$

Where $0<C<\infty$. The last inequality holds by remark 2.

ii) By Theorem 1, for every $\varepsilon>0$,

$$P[\sup_{k\geq m}\frac{|S_k|}{b_k}>\varepsilon]\leq\frac{8}{\varepsilon^2}(\sum_{j=1}^{m}\frac{\sigma_j^2}{b_m^2}+2\sum_{j=2}^{m}\frac{\sigma_j\sum_{i=1}^{j-1}\sigma_i}{b_m^2}+\sum_{j=m+1}^{\infty}\frac{\sigma_j^2}{b_j^2}+2\sum_{j=m+1}^{\infty}\frac{\sigma_j\sum_{i=1}^{j-1}\sigma_i}{b_j^2}),$$

now by the assumption of $\sum_{j=1}^{\infty}\frac{\sigma_j\sum_{i=1}^{j}\sigma_i}{b_j^2}<\infty$, and Kronecker's lemma we obtain $\lim_{m\to\infty}P[\sup_{k\geq m}\frac{|S_k|}{b_k}>\varepsilon]=0$. Hence lemma 7.1 in [6] completes the proof.

Corollary 3. Under the assumptions of Theorem 2, if $\sup_n(\sigma_n\sum_{j=1}^{n}\sigma_j)<\infty$, then for every $\alpha>\frac{1}{2}$, $\varepsilon>0$ and $m\geq 1$.

i)

$$P[\sup_{j\geq m}\frac{|S_j|}{j^{\alpha}}>\varepsilon]\leq\frac{16}{\varepsilon^2}\times\frac{2\alpha}{2\alpha-1}\times m^{1-2\alpha}\sup_n(\sigma_n\sum_{j=1}^{n}\sigma_j).$$

ii) $\frac{S_n}{n^{\alpha}}\longrightarrow 0$, $w.p.1$. as $n\to\infty$, and *iii)* For every $0<\beta<2$, $E\sup_n(\frac{|S_n|}{n^{\alpha}})^{\beta}<\infty$.

Theorem 3 Let $\{X_n\ ,\ n\geq 1\}$ be a sequence of random variables,if $\sum_{j=1}^{\infty}\sigma_j\sum_{i=1}^{j}\sigma_i<\infty$ and $E(X_n)=0$. Then $\sum_{k=1}^{\infty}X_k$ converges W.P.1.

Proof By corollary 2 for every $\varepsilon>0$, we have

$$P[\sup_{k\geq 1}|S_{n+k}-S_n|>\varepsilon]=\lim_{m\longrightarrow\infty}P[\max_{1\leq k\leq m}|S_{n+k}-S_n|>\varepsilon]\leq\frac{8}{\varepsilon^2}(\sum_{k=n+1}^{\infty}\sigma_k^2+2\sum_{j=n+1}^{\infty}\sigma_j\sum_{i=1}^{j-1}\sigma_i).$$

Now by Remark 2 the left hand side of the above inequality tends to zero when $n \longrightarrow \infty$. Hence $\sum_{k=1}^{n} X_k$ converges $w.p.1$ by Lemma 7.1 in [6].

The following corollary that is an extension of Kolmogorov's theorem provides strong law of large numbers arbitrary random variables.

Corollary 4 Let $\{X_n \ , \ n \geq 1\}$ be a sequence of random variables with $E[X_n] = \mu_n$ and $Var(X_n) = \sigma_n^2$ for all $n \geq 1$. If $\sum_{j=1}^{\infty} \frac{\sigma_j \sum_{i=1}^{j} \sigma_i}{j^2} < \infty$ then

$$\frac{1}{n} \sum_{k=1}^{n} (X_k - \mu_k) \longrightarrow 0 \qquad w.p.1.$$

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