Proceedings of the 8th WSEAS International Conference on APPLIED MATHEMATICS, Tenerife, Spain, December 16-18, 2005 (pp165-169)

Differentiability of family of generalized-inverses of a differentiable family of matrices

M. ISABEL GARCÍA-PLANAS
Departament de Matemática Aplicada I
Universitat Politècnica de Catalunya,
C. Minería 1, Esc C, 1º-3ª
08038 Barcelona, Spain

Abstract:- Given a differentiable family of matrices $A \in \mathcal{D}if(M, M_{m\times n}(\mathbb{C}))$, defined over a manifold of parameters M, we study the existence of a differentiable family of matrices $X \in \mathcal{D}if(M, M_{n\times m}(\mathbb{C}))$ such that for all $t \in M$, $X(t)$ is a generalized inverse of $A(t)$.

Key-Words:- Generalized inverse, differentiable families of matrices.

1 Preliminaries

Given a complex matrix A we denote by A^* the conjugate transpose of A.

It is well know that the generalized inverse of a matrix $A \in M_{m\times n}(\mathbb{C})$ is a matrix $X \in M_{n\times m}(\mathbb{C})$ verifying

$$\begin{aligned} AXA &= A \\ XAX &= X \end{aligned}$$

The unique matrix X verifying in addition,

$$\begin{aligned} (XA)^* &= XA \\ (AX)^* &= AX \end{aligned}$$

is called the Moore-Penrose inverse, in this case we will note it by A^+.

Now we are going to give characterizations of differentiable families of subspaces.

Let M be a differentiable manifold, a differentiable family of subspaces parametrized on M is a differentiable map $\mathcal{L} : M \longrightarrow Gr_{k,n}$, from M in the Grassmann manifold $Gr_{k,n}$ of all k-dimensional subspaces of $\mathbb{K}^n$. We will write by $\mathcal{L} \in \mathcal{D}if(M, Gr_{k,n})$, the set of all differentiable families.

Analogously, we will write $\mathcal{D}if(M, Gl(n;\mathbb{C}))$, $\mathcal{D}if(M, M_{n\times m}(\mathbb{C}))$, $\mathcal{D}if(M, \mathbb{C}^n)$, etc., for the set of differentiable families of invertible matrices, of matrices, of vectors, etc., parametrized on M.

Theorem 1 ([5]) *Let M be a differentiable manifold and $\mathcal{L} : M \longrightarrow Gr_{k,n}$ a family of subspaces parametrized on M. Then, the following conditions are equivalent*

i) $\mathcal{L} \in \mathcal{D}if(M, Gr_{k,n})$.

ii) For every $t_0 \in M$ there is an open neighborhood W_{t_0} of t_0 in M, with k maps $v_i \in \mathcal{D}if(W_{t_0}, K^n)$, $1 \le i \le k$, such that $\{v_1(t), \dots, v_k(t)\}$ is a basis of $\mathcal{L}(t)$ for all $t \in W_{t_0}$.

iii) For every $t_0 \in M$ there is an open neighborhood W_{t_0} of t_0 in M, together with a map $\mathcal{S}_{t_0} \in \mathcal{D}if(W_{t_0}, Gl(n; K))$, such that $\mathcal{L}(t) = \mathcal{S}_{t_0}(t)\mathcal{L}(t_0)$, for all $t \in W_{t_0}$

In the above conditions, we say that

Proceedings of the 8th WSEAS International Conference on APPLIED MATHEMATICS, Tenerife, Spain, December 16-18, 2005 (pp165-169)

$v_1, \ldots, v_k \in \mathcal{D}(M, K^n)$ is a differentiable local basis of the family $\mathcal{L}$.

As an application:

Proposition 1 *Let $\mathcal{L}_i \in \mathcal{D}if(M, Gr_{k_i,n})$, $1 \leq i \leq 2$, be two families of subspaces such that $\mathcal{L}_1(t) \subset \mathcal{L}_2(t)$ for all $t \in M$. Then, there exists a family $\mathcal{L} \in \mathcal{D}if(M, Gr_{k_2-k_1,n})$ such that*

$$\mathcal{L}_2 = \mathcal{L}_1(t) \perp \mathcal{L}(t) \qquad \textit{for all} \quad t \in M.$$

Example 1

1) If $\mathcal{L} \in \mathcal{D}if(M, Gr_{k,n})$, then $\mathcal{L}^\perp \in \mathcal{D}if(M, Gr_{n-k,n})$.

2) If $A \in \mathcal{D}if(M, M_{n\times m}(K))$ such that $\operatorname{rank} A(t) = k$ for all $t \in M$, then locally one can choose k columns of $A(t)$ such that form a basis of $\operatorname{Im} A(t)$. Hence, $\operatorname{Im} A \in \mathcal{D}if(M, Gr_{k,n})$.

3) In the above conditions, since $\ker A(t) = (\operatorname{Im} A(t)^*)^\perp$, we have $\ker A \in \mathcal{D}if(M, Gr_{(n-k),n})$.

4) From the above examples, it follows that if $\mathcal{L}_1 \in \mathcal{D}if(M, Gr_{k_1,n})$, $\mathcal{L}_2 \in \mathcal{D}if(M, Gr_{k_2,n})$ and $\dim(\mathcal{L}_1(t) + \mathcal{L}_2(t)) = k$ for all $t \in M$, then $\mathcal{L}_1 + \mathcal{L}_2 \in \mathcal{D}if(M, Gr_{k,n})$ y $\mathcal{L}_1 \cap \mathcal{L}_2 \in \mathcal{D}if(M, Gr_{k_1+k_2-k,n})$.

In the following we assumes that M is a contractible real manifold of class C^r. We have the following fundamental result.

Proposition 2 ([5]) *If M is a contractible real manifold of class C^r, any C^r principal bundle over M is trivial.*

Remark 1 For continuous principal bundles, Husemoller [7], shows that they are trivial over any contractible paracompact topological space. Thus the following theorem holds under these conditions.

If M is contractible real manifold, the local properties can be globalized:

Theorem 2 *Let M be a contractible real C^r-manifold, and $\mathcal{L} \in C^r(M, Gr_{k,n})$, then:*

i) There exist k maps $v_i \in C^r(M, \mathbb{C}^n)$, $1 \leq i \leq k$, such that

$\{v_1(t), \ldots, v_k(t)\}$ is a basis of $\mathcal{L}(t)$ for every $t \in M$.

ii) There exists a map $\mathcal{S} \in C^r(M, Gl(n; \mathbb{C}))$ and a k-subspace $L \in Gr_{k,n}$ such that

$$\mathcal{L}(t) = \mathcal{S}(t)L, \qquad \textit{for every} \quad t \in M$$

In particular, $\mathcal{L}(t) = \mathcal{S}(t)\mathcal{S}(t_0)^{-1}\mathcal{L}(t_0)$, for all $t, t_0 \in M$.

2 Differentiable families of Moore-Penrose inverses

Given a family $A \in \mathcal{D}if(M, M_{m\times n}(\mathbb{C}))$ matrices, we will call generalized inverse of this family to the family $X : M \longrightarrow M_{n\times m}(\mathbb{C})$ such that, for each $t \in M$ the matrix $X(t)$ is a generalized inverse of $A(t)$. If, in addition, for each $t \in M$ the matrix $X(t)$ is the Moore-Penrose inverse $A^+(t)$ de $A(t)$, we will say that the family $X : M \longrightarrow M_{n\times m}(\mathbb{C})$ is the Moore-Penrose inverse of the family A, and we will note by A^+.

We are going to present a local characterization of the differentiability of the A^+ family. Also we we announce that these local properties can be extended to the parameter manifolds if they are contractible.

Theorem 3 *Let M be a differentiable manifold and $A \in \mathcal{D}if(M, M_{m\times n}(\mathbb{C}))$ a differentiable family of matrices having constant rank, then*

i) *The Moore Penrose inverse family is differentiable*

ii) *If the manifold M is real and contractible, the family can be obtained from A by means the following differentiable matrix operations:*

$$A^+ = S \begin{pmatrix} 0 & 0 \\ \left[\begin{pmatrix} Id_k & 0 \end{pmatrix} T^{-1} A S \begin{pmatrix} 0 \\ Id_k \end{pmatrix}\right]^{-1} & 0 \end{pmatrix} T^{-1}$$

where $T \in \mathcal{D}if(M, M_m(\mathbb{C}))$, $S \in \mathcal{D}if(M, M_n(\mathbb{C}))$.

Remark 2 Other local differentiable families of generalized inverses can be obtained in the case the orthogonality of matrices S and T is not required and these inverses will be global if the variety M is real and contractible.

Proof. Taking into account that $\operatorname{rank} A(t) = k$ for each $t \in M$, we have

$$\begin{aligned} \operatorname{Im} A \in & \mathcal{D}if(M, Gr_{k,m}) \\ \ker A \in & \mathcal{D}if(M, Gr_{(n-k),n}) \end{aligned}$$

so, $\forall t_0 \in M$ there exists a neighborhood $\mathcal{W}^1_{t_0}$ of t_0 in M such that

$$(x_1(t), \dots, x_{n-k}(t))$$

is a differentiable basis for $\ker A(t)$ for all $t \in \mathcal{W}^1_{t_0}$, and $\forall t_0 \in M$ there is a neighborhood $\mathcal{W}^2_{t_0}$ of t_0 in M such that

$$(y_1(t), \dots, y_k(t))$$

is a differentiable basis of $\operatorname{Im} A(t)$ for all $t \in \mathcal{W}^1_{t_0}$.

Obviously, A^* is a differentiable family of matrices verifying $\operatorname{rank} A^*(t) = k$ constant for all $t \in M$, then, as before, $\forall t_0 \in M$ there is a neighborhood $\mathcal{W}^3_{t_0}$ of t_0 in M such that

$$(y_{k+1}(t), \dots, y_m(t))$$

is a differentiable basis for $\ker A^*(t) = (\operatorname{Im} A(t))^\perp$ for all $t \in \mathcal{W}^3_{t_0}$, and $\forall t_0 \in M$ there is a neighborhood $\mathcal{W}^4_{t_0}$ of t_0 in M such that

$$(x_{n-k+1}(t), \dots, x_n(t))$$

is a differentiable basis of $\operatorname{Im} A^*(t) = (\ker A(t))^\perp$ for all $t \in \mathcal{W}^4_{t_0}$.

Let be now, the neighborhood $\mathcal{W}_{t_0} = \cap_{i=1}^4 \mathcal{W}^i_{t_0}$, of t_0, we have that

$$(y_1(t), \dots, y_k(t), y_{k+1}(t), \dots, y_m(t))$$

is a differentiable basis of $\mathbb{C}^m$.

Let $T(t)$ be the matrix whose columns are the components of the vectors of this basis with respect the canonical basis of $\mathbb{C}^m$.

In the neighborhood $\mathcal{W}_{t_0}$

$$(x_1(t), \dots, x_{n-k}(t), x_{n-k+1}(t), \dots, x_n(t))$$

is a differentiable basis of $\mathbb{C}^n$.

Let $S(t)$ be the matrix whose columns are the components of the vectors of this basis with respect the canonical basis of $\mathbb{C}^n$.

The matrix of the map $A(t)$ in these bases is

$$\begin{pmatrix} 0 & A_0(t) \\ 0 & 0 \end{pmatrix} = T^{-1}(t) A(t) S(t)$$

We observe that $A_0(t)$ is the matrix in the bases $(x_{n-k+1}(t), \dots, x_n(t))$ and $(y_1(t), \dots, y_k(t))$ of the map

$$(\ker A)^\perp \longrightarrow \operatorname{Im} A$$

restriction of $A(t)$. For each $t \in \mathcal{W}_{t_0}$ is invertible and differentiable.

We consider the matrix

$$\begin{pmatrix} 0 & 0 \\ A_0^{-1}(t) & 0 \end{pmatrix}$$

of the linear map

$$\begin{aligned} &\mathbb{C}^m \longrightarrow \mathbb{C}^n \\ &y_j(t) \longrightarrow A_0^{-1}(t) y_j(t) \quad \forall\, j = 1, \dots, k \\ &y_j(t) \longrightarrow 0 \qquad\qquad\quad\; \forall\, j = k+1, \dots, m \end{aligned}$$

in the bases

$$(y_1(t), \dots, y_k(t), y_{k+1}(t), \dots, y_m(t))$$

and

$$(x_1(t), \dots, x_{n-k}(t), x_{n-k+1}(t), \dots, x_n(t))$$

of $\mathbb{C}^m$ and $\mathbb{C}^n$ respectively. In the canonical bases of $\mathbb{C}^m$ and $\mathbb{C}^n$ is

$$S(t) \begin{pmatrix} 0 & 0 \\ A_0^{-1}(t) & 0 \end{pmatrix} T^{-1}(t) = X(t)$$

The family $X(t)$ defined in this way, is differentiable in $\mathcal{W}_{t_0}$ and for each $t \in \mathcal{W}_{t_0}$ is a generalized inverse of $A(t)$:

$$\begin{aligned} & A(t)X(t)A(t) = \\ & = T(t) \begin{pmatrix} 0 & A_0(t) \\ 0 & 0 \end{pmatrix} S^{-1}(t) S(t) \begin{pmatrix} 0 & 0 \\ A_0^{-1}(t) & 0 \end{pmatrix} T^{-1}(t) \\ & \qquad T(t) \begin{pmatrix} 0 & A_0(t) \\ 0 & 0 \end{pmatrix} S^{-1}(t) = \\ & = T(t) \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & A_0(t) \\ 0 & 0 \end{pmatrix} S^{-1}(t) = \\ & = T(t) \begin{pmatrix} 0 & A_0(t) \\ 0 & 0 \end{pmatrix} S^{-1}(t) = A(t) \end{aligned}$$

$$
\begin{aligned}
&X(t)A(t)X(t) = \\
&S(t)\begin{pmatrix} 0 & 0 \\ A_0^{-1}(t) & 0 \end{pmatrix} T^{-1}(t)T(t)\begin{pmatrix} 0 & A_0(t) \\ 0 & 0 \end{pmatrix} S^{-1}(t) \\
&\qquad S(t)\begin{pmatrix} 0 & 0 \\ A_0^{-1}(t) & 0 \end{pmatrix} T^{-1}(t) = \\
&= S(t)\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} 0 & 0 \\ A_0^{-1}(t) & 0 \end{pmatrix} T^{-1}(t) = \\
&= S(t)\begin{pmatrix} 0 & 0 \\ A_0^{-1}(t) & 0 \end{pmatrix} T^{-1}(t) = X(t).
\end{aligned}
$$

Then, locally, there exist differentiable families of generalized inverses.

Applying Gram-schmidt to the bases $(y_1(t), \dots, y_k(t), y_{k+1}(t), \dots, y_m(t))$ and $(x_1(t), \dots, x_{n-k}(t), x_{n-k+1}(t), \dots, x_n(t))$ of $\mathbb{C}^m$ and $\mathbb{C}^n$ respectively, we obtain

$$(\overline{y}_1(t), \dots, \overline{y}_k(t), \overline{y}_{k+1}(t), \dots, \overline{y}_m(t))$$

and

$$(\overline{x}_1(t), \dots, \overline{x}_{n-k}(t), \overline{x}_{n-k+1}(t), \dots, \overline{x}_n(t))$$

where $(\overline{x}_{n-k+1}(t), \dots, \overline{x}_n(t))$ is a basis of $(\ker(t)A)^{\perp}$ and $(\overline{y}_1(t), \dots, \overline{y}_k(t))$ is a basis of $\operatorname{Im} A(t)$, then we can repeat the above construction but using these new bases, obtaining a differentiable family of matrices $X(t)$, such that for each $t \in \mathcal{W}_{t_0}$, $X(t)$ is the Moore-Penrose inverse:

$$
\begin{aligned}
&(X(t)A(t))^* = \\
&= \left(S(t)\begin{pmatrix} 0 & 0 \\ A_0^{-1}(t) & 0 \end{pmatrix} T^{-1}(t)T(t)\begin{pmatrix} 0 & A_0(t) \\ 0 & 0 \end{pmatrix} S^{-1}(t)\right)^* = \\
&= \left(S(t)\begin{pmatrix} 0 & 0 \\ A_0^{-1}(t) & 0 \end{pmatrix}\begin{pmatrix} 0 & A_0(t) \\ 0 & 0 \end{pmatrix} S^{-1}(t)\right)^* \\
&= \left(S(t)\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} S^{-1}(t)\right)^* = \\
&= (S^{-1}(t))^*\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}(S(t))^* = \\
&= S(t)\begin{pmatrix} 0 & 0 \\ A_0^{-1}(t) & 0 \end{pmatrix} T^{-1}(t)T(t)\begin{pmatrix} 0 & A_0(t) \\ 0 & 0 \end{pmatrix} S^{-1}(t) = \\
&= X(t)A(t)
\end{aligned}
$$

$$
\begin{aligned}
&(A(t)X(t))^* = \\
&= \left(T(t)\begin{pmatrix} 0 & A_0(t) \\ 0 & 0 \end{pmatrix} S^{-1}(t)S(t)\begin{pmatrix} 0 & 0 \\ A_0^{-1}(t) & 0 \end{pmatrix} T^{-1}(t)\right)^* = \\
&= \left(T(t)\begin{pmatrix} 0 & A_0(t) \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ A_0^{-1}(t) & 0 \end{pmatrix} T^{-1}(t)\right)^* \\
&= \left(T(t)\begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix} T^{-1}(t)\right)^* = \\
&= (T^{-1}(t))^*\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}(T(t))^* = \\
&= T(t)\begin{pmatrix} 0 & A_0(t) \\ 0 & 0 \end{pmatrix} S^{-1}(t)S(t)\begin{pmatrix} 0 & 0 \\ A_0^{-1}(t) & 0 \end{pmatrix} T^{-1}(t) = \\
&= A(t)X(t).
\end{aligned}
$$

Taking into account that for each $t \in M$ exists a unique Moore-Penrose inverse $A^+(t)$, the family $X = A^+$ is differentiable and locally can be obtained by means matrix operations, for that it suffices to remark that

$$A_0(t)^{-1} = \left[\begin{pmatrix} Id_k & 0 \end{pmatrix} T(t)^{-1}A(t)S(t)\begin{pmatrix} 0 \\ Id_k \end{pmatrix}\right]^{-1}$$

Suppose now, that the space M is real and contractible, then the families of matrices S and T are global, consequently all family can be obtained by means matrix operations:

$$A^+ = S\begin{pmatrix} 0 & 0 \\ \begin{pmatrix} Id_k & 0 \end{pmatrix} T^{-1}AS\begin{pmatrix} 0 \\ Id_k \end{pmatrix} & 0 \end{pmatrix} T^{-1}$$

□

2 Differentiable families of Drazin inverses

Given a matrix $A \in M_{n\times n}(\mathbb{C}))$, we will call Drazin inverse of A to the matrix $A^D \in M_{n\times n}(\mathbb{C})$ such that,

$$
\begin{aligned}
A^D A A^D &= A^D \\
A A^D &= A^D A \\
A^{k+1} A^D &= A^k
\end{aligned}
$$

where $k = \operatorname{ind} A$ (index of A), is the smallest nonnegative integer k such that $\operatorname{rank} A^k = \operatorname{rank} A^{k+1}$.

Let $A : M \longrightarrow M_{n\times m}(\mathbb{C})$ a differentiable family of matrices, we will call Drazin inverse of this family to the family $A^D : M \longrightarrow M_{n\times m}(\mathbb{C})$ such that, for each $t \in M$ the matrix $A^D(t)$ is the Drazin inverse of $A(t)$.

As for the case of Moore-Penrose inverses, we are going to present a local characterization of the differentiability of the A^D family. Also we we announce that these local properties can be extended to the parameter manifolds if they are contractible.

Proposition 3 *Let M be a simple connected differentiable manifold and $A \in \mathcal{D}if(M, M_{m\times n}(\mathbb{C}))$ a differentiable family of matrices having the same Segre type (that is to say, having the same discrete invariants varying at most in the value of eigenvalues) and having constant index, then*

i) *The Drazin inverse family is differentiable*

ii) *If the manifold M is real and contractible, the family can be obtained from A by means the following differentiable matrix operations:*

$$A^D = P^{-1}(t) \begin{pmatrix} J^{-1}(t) & 0 \\ 0 & 0 \end{pmatrix} P(t)$$

where $P \in \mathcal{D}if(M, M_m(\mathbb{C}))$ *is such that* $P(t)A(t)P^{-1} = \begin{pmatrix} J(t) & 0 \\ 0 & N \end{pmatrix}$ *and* N *is a nilpotent matrix of index* k.

Remark 3 Taking into account that for all $t \in M$ $A(t)$ have the same Segre type and the same index, the Jordan part of $A(T)$ corresponding to the zero eigenvalue is the same for all t.

Proof. In these conditions there exists a differentiable Jordan basis for $A(t)$, (see [5]). □

Conclusions

In this paper locally differentiable families of generalized inverses are obtained and these families are global if the parameter space is real and contractible.

These families can be used to obtain solutions for generalized linear systems depending on parameters.

References

[1] Absil, P.-A, Sepulchre, R. Van Dooren, P., Mahony, R. *Cubically convergent iterations for invariant subspace computation.* SIAM J. Matrix Anal. Appl, 26 (1), 2004, pp. 70-96.

[2] A. Ben-Israel and T. Greville. *Generalized Inverses: Theory and Applications*, John Wiley and Sons, New York, 1974

[3] J. Ferrer, Mª I. García, F. Puerta, "Differentiable familles of subspaces" L.A.A **199** (1994), 229-252.

[4] X. Puerta, F. Puerta, J. Ferrer, *Global reduction to the Kronecker canonical form of a* C^r *family of time invariant linear systems.* LAA 346 (2002) pp 77-45.

[5] Mª I. García-Planas, *Estudio geométrico de familias diferenciables de parejas de matrices* PhD Thesis, (1994). Ed. UPC, Barcelona Spain.

[6] I. Goberg, P. Lancaster, L. Rodman, "Invariant Subspaces of matrices with applications," Wiley, 1986.

[7] D. Husemoller, "Fibre Bundles," Springer-Verlag, New York, 1975.