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A Simple Derivation of Lower Triangular Interactor Matrix

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Abstract

An interactor matrix plays several important roles in the control system theory. Recently, we presented a simple method to derive a special interactor matrix using Moore-Penrose pseudoinverse. But, the structure of the proposed interactor was not specified. A triangular structure of interactor is useful for multivariable adaptive control. In this note, it will be shown a derivation of interactor with lower triangular structure. For this, a property of the interactor which we reported will play an important role.

Key Words: interactor matrix, polynomial matrix, lower triangular structure, discrete-time systems, pseudoinverse.

1 Introduction

An interactor matrix [1] plays an important role in the design of several control problems, e.g., the inverse system [2], the decoupling problem [3], the disturbance decoupling [3] the maximum unobservable control [5], the adaptive control [6], [7], LQ regulator for singular weightings [8], etc. The derivation method of the interactor was first given in [1], which can be interpreted as another explanation of the structure algorithm [2]. Furthermore, it is known to be equivalent to the derivation method of Hermite normal form for proper and stable rational function matrix [9]. In any way, these methods are much complicated, since they need some iteration. Relatively simple methods for the derivation of interactor are proposed in [10] and [7]. In [10], the nilpotent interactor is presented by repeating QR factorization. In [7], it is shown that the interactor can be obtained by solving some matrix equation. But, for those methods, some calculation algorithms are still necessary.

Recently, we presented a “one shot” derivation of the interactor matrix [11]. As shown in [7], the coefficient matrices of the interactor can be obtained by solving a certain type of matrix equation. But, since this equation does not have a unique solution and, among all solutions, we have to find some particular solution that satisfies the condition for the interactor, we need some calculating algorithm to solve it. On the other hand, it is natural to solve this matrix equation using Moore-Penrose pseudoinverse, if such a solution qualifies as the coefficient of the interactor matrix, by which the complicated calculation algorithm can be avoided. Therefore, we focused our attention on the interactor constructed by such a solution of matrix equation. It was shown that the proposed interactor has the all-pass property in the discrete-time, so that all of its zeros lie at the origin. Moreover, by using inverted interactorizing feedback gain, nonzero Markov parameters of the closed-loop system are given by pseudoinverse of the coefficient matrix of the proposed interactor.

Unfortunately, any structural restriction was posed for our previous result. In multivariable adaptive control, a triangular structure of interactor is important to prove the stability. Thus, we consider a derivation of the interactor with lower triangular structure. It is well known that the triangularization method by elementary operations, and its calculation in state space was shown in [12]. For the calculation in [12], the Markov parameter of inverted interactor is necessary. Therefore, our previous derivation is useful for [12].

2 A Derivation of Interactor with No Structural Restriction

For a given $m \times m$ strictly proper and nonsingular transfer function matrix, $G(z)$, there exists an $m \times m$ polynomial matrix, $L(z)$, which satisfies the following equation.

$$\lim_{z\to\infty} L(z)G(z) = K \quad (nonsingular). \tag{1}$$

Such an $L(z)$ is called an interactor matrix of $G(z)$ [1] . If $K = I_m$, $L(z)$ is called an identity interactor [6]. In the following, a derivation of an identity interactor $\xi(z) := K^{-1}L(z)$ is considered.

Let $(A,\ B,\ C)$ denote a minimal realization of $G(z)$ and

$$\begin{aligned} \boldsymbol{T}_{k-1} &= \begin{bmatrix} CB & 0 & \cdots & 0 \\ CAB & CB & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ CA^{k-1}B & CA^{k-2}B & \cdots & CB \end{bmatrix}, \\ \boldsymbol{J}_{k-1} &= [I_m \ 0_{m\times m(k-1)}]. \end{aligned} \tag{2}$$

Define w as the least integer k which satisfies the following equation.

$$\text{rank} \begin{bmatrix} \boldsymbol{T}_{k-1} \\ \boldsymbol{J}_{k-1} \end{bmatrix} = \text{rank}\ \boldsymbol{T}_{k-1}. \tag{3}$$

Let an identity interactor, $\xi(z)$, be described by

$$\begin{aligned} &\xi(z) = z\xi_1 + z^2\xi_2 + \cdots + z^w\xi_w = \boldsymbol{\xi} z S_{I_m}^{w-1}(z), \\ &\boldsymbol{\xi} := [\xi_1 \ \xi_2 \ \cdots \ \xi_w], \quad \xi_i \in R^{m\times m} \\ &S_{I_m}^{w-1}(z) := [I_m \ zI_m \ \cdots \ z^{w-1}I_m]^T, \end{aligned} \tag{4}$$

then, from [7], the following equation holds:

$$\boldsymbol{\xi}\boldsymbol{T}_{w-1} = \boldsymbol{J}_{w-1}. \tag{5}$$

Conversely, the identity interactor $\xi(z)$ can be obtained by solving this equation and the solvability of this is asserted from eq.(3). Thus, using Moore-Penrose pseudoinverse $\boldsymbol{T}_{w-1}^{\dagger}$ of $\boldsymbol{T}_{w-1}$, $\boldsymbol{\xi}$ can be calculated by

$$\boldsymbol{\xi} = \boldsymbol{J}_{w-1}\boldsymbol{T}_{w-1}^{\dagger}. \tag{6}$$

Theorem 1 Let

$$\begin{aligned} &\xi^{\sim}(z) = \xi^T(z^{-1}) = z^{-1}\xi_1^T + z^{-2}\xi_2^T + \cdots + z^{-w}\xi_w^T, \\ &F = \boldsymbol{\xi} \begin{bmatrix} CA \\ CA^2 \\ \vdots \\ CA^w \end{bmatrix}, \quad \mathcal{O}_{w-1}(C,A) = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{w-1} \end{bmatrix}, \\ &A_F := A - BF. \end{aligned} \tag{7}$$

If $\boldsymbol{\xi}$ is given by

$$\boldsymbol{\xi} = \boldsymbol{J}_{w-1}\boldsymbol{T}_{w-1}^{\dagger}, \tag{8}$$

then the following properties hold:

$$\textbf{P1}\ \ \xi(z)\xi^{\sim}(z) = \boldsymbol{\xi}\boldsymbol{\xi}^T, \tag{9}$$

$$\textbf{P2}\ \ \mathcal{O}_{w-1}(C, A_F)B = \boldsymbol{\xi}^{\dagger}, \tag{10}$$

$$\textbf{P3}\ \ CA_F^w = 0. \tag{11}$$

Example 1 Consider the following transfer function matrix [1].

$$G(z) = \begin{bmatrix} \frac{1}{z+1} & \frac{1}{z+2} \\ \frac{1}{z+3} & \frac{1}{z+4} \end{bmatrix}$$

[1] Although the definition in [1] is restricted the structure of $L(z)$ (lower triangular), we do not consider such a restriction since it is not essential in this paper.

In this case, (A, B, C) can be given by

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -3 & -4 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -8 & -6 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$
$$C = \begin{bmatrix} 3 & 1 & 4 & 1 \\ 1 & 1 & 2 & 1 \end{bmatrix}.$$

Then, using the pseudoinverse of

$$\boldsymbol{T}_2 = \begin{bmatrix} CB & 0 & 0 \\ CAB & CB & 0 \\ CA^2B & CAB & CB \end{bmatrix},$$

we have

$$\boldsymbol{\xi} = \begin{bmatrix} 0.75 & 0.75 & 0.25 & -1.25 & 0.5 & -0.5 \\ -0.5 & -0.5 & 0 & 1 & -0.5 & 0.5 \end{bmatrix},$$
$$\xi(z) = \frac{z}{2} \begin{bmatrix} z^2 + 0.5z + 1.5 & -z^2 - 2.5z + 1.5 \\ -z^2 - 1 & z^2 + 2z - 1 \end{bmatrix}.$$

3 A Derivation of Lower Triangular Interactor

The following Theorem and Corollary are important to triangularize a polynomial matrix.

Theorem 2 Let (A, B, C) denote a realization of $G(z)$ and $\boldsymbol{M}$ denote a left annihilating matrix of $\mathcal{O}_\nu(C, A)$, where ν is the observability index of (C, A). Then, the polynomial matrices

$$M(z) = \boldsymbol{M} S_{I_m}^{\nu}(z), \quad U(z) = \boldsymbol{M} \begin{bmatrix} 0_{m\times m\nu} \\ \boldsymbol{T}_{\nu-1} \end{bmatrix} S_{I_m}^{\nu-1}(z) \tag{12}$$

satisfy the following relation:

$$G(z) = M^{-1}(z)U(z). \tag{13}$$

If $G(z)$ does not have any finite zeros, then $U(z)$ is a unimodular matrix.

Corollary Let c_i denote the i-th row of C and $\nu_1, \ldots, \nu_m$ be observability indices of $\mathcal{O}_\nu(C, A)$ determined by searching the *crate diagram* by rows, i.e., choose linearly independent rows $c_1, \ldots, c_1A^{\nu_1-1}, c_2, \ldots, c_2A^{\nu_2-1}, c_3, \ldots, c_mA^{\nu_m-1}$, where $c_1A^{\nu_1}$ is row span of $c_1, \ldots, c_1A^{\nu_1-1}$, $c_2A^{\nu_2}$ is row span of $c_1, \ldots, c_1A^{\nu_1-1}, c_2, \ldots, c_2A^{\nu_2-1}$ etc. Then define

$$\boldsymbol{M} = \begin{bmatrix} -\Lambda & I_m \end{bmatrix} V, \tag{14}$$

where Λ is the solution matrix of

$$\Lambda\hat{\mathcal{O}} = \tilde{\mathcal{O}}, \quad \hat{\mathcal{O}} := \begin{bmatrix} c_1 \\ \vdots \\ c_1A^{\nu_1-1} \\ c_2 \\ \vdots \\ c_mA^{\nu_m-1} \end{bmatrix}, \quad \tilde{\mathcal{O}} = \begin{bmatrix} c_1A^{\nu_1} \\ \vdots \\ c_mA^{\nu_m} \end{bmatrix}$$

and V is a row selection matrix such that

$$V\mathcal{O}_\nu(C, A) = \begin{bmatrix} \hat{\mathcal{O}} \\ \tilde{\mathcal{O}} \end{bmatrix}.$$

Using $\boldsymbol{M}$ given in eqn.(14), $M(z)$ given in eqn.(12) is in a lower triangular form.

In the above Theorem, the triangularizing unimodular matrix in z is derived. To keep the main property given in eqn.(1), it is necessary to derive the unimodular in z^{-1}. Thus, we consider the variable transformation $\lambda := z^{-1}$. An algorithm to derive a lower triangular interactor is as follows:

Algorithm

Step 1 Derive the interactor $\xi(z) = \boldsymbol{\xi} z S_{I_m}^{w-1}(z)$ using eqn.(8).

Step 2 Consider the variable transformation $\lambda = z^{-1}$. Define the polynomial matrix $\eta(\lambda)$ by

$$\eta(\lambda) = \begin{bmatrix} \xi_w & \xi_{w-1} & \dots & \xi_1 \end{bmatrix} \lambda S_{I_m}^{w-1}(\lambda) \tag{15}$$

(Note that $\eta(\lambda) = z^{-(w+1)}\xi(z)$.)

Step 3 Calculate the pseudoinverse of $\begin{bmatrix} \xi_w & \xi_{w-1} & \dots & \xi_1 \end{bmatrix}$, which gives nonzero Markov parameters of $\eta^{-1}(\lambda)$.

Step 4 Consider the Hankel matrix

$$H = \mathcal{O}_{w-1}(\bar{C}, \bar{A}) \begin{bmatrix} \bar{B} & \bar{A}\bar{B} & \dots & \bar{A}^{w-1}\bar{B} \end{bmatrix} = \begin{bmatrix} \xi_w^\dagger & \xi_{w-1}^\dagger & \cdots & \xi_1^\dagger \\ \xi_{w-1}^\dagger & \xi_{w-2}^\dagger & \cdot^{\cdot^{\cdot}} & 0 \\ \vdots & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \vdots \\ \xi_1^\dagger & 0 & \dots & 0 \end{bmatrix}, \tag{16}$$

where $(\bar{A}, \bar{B}, \bar{C})$ denote a minimal realization of $\eta^{-1}(\lambda)$. Thus, an left annihilating matrix of H also annihilates $\mathcal{O}_{w-1}(\bar{C}, \bar{A})$ and it can be used the results in Theorem 2 ad its Corollary. Calculate the left annihilating matrix $\boldsymbol{M}$ according to Theorem 2.

Step 5 A lower triangular interactor $M(z)$ is given by

$$M(z) = z^{w+1}\boldsymbol{M} S_{I_m}^{w}(\lambda) \tag{17}$$

and corresponding unimodular matrix $U(\lambda)$ is given by

$$U(\lambda) = \boldsymbol{M} \begin{bmatrix} 0 & 0 & \dots & 0 \\ \xi_w^\dagger & 0 & \dots & 0 \\ \xi_{w-1}^\dagger & \xi_w^\dagger & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \xi_2^\dagger & \xi_3^\dagger & \dots & \xi_w^\dagger \end{bmatrix} S_{I_m}^{w-1}(\lambda). \tag{18}$$

Example 2 Consider the same transfer function matrix as shown in Example 1.

Step 1 : See Example 1.

Step 2 : From the previous example

$$\begin{aligned} \eta(\lambda) &= z^{-4}\xi(z) = \begin{bmatrix} 0.75\lambda^3 + 0.25\lambda^2 + 0.5\lambda & 0.75\lambda^3 - 1.25\lambda^2 - 0.5\lambda \\ -0.5\lambda^3 - 0.5\lambda & -0.5\lambda^3 + \lambda^2 + 0.5\lambda \end{bmatrix} \\ &= \underbrace{\begin{bmatrix} 0.5 & -0.5 \\ -0.5 & 0.5 \end{bmatrix}}_{\xi_3} \underbrace{\begin{matrix} 0.25 & -1.25 \\ 0 & 1 \end{matrix}}_{\xi_2} \underbrace{\begin{matrix} 0.75 & 0.75 \\ -0.5 & -0.5 \end{matrix}}_{\xi_1} \lambda S_I^2(\lambda). \end{aligned}$$

Step 3 : Pseudoinverse is given by

$$\begin{bmatrix} \xi_3 & \xi_2 & \xi_1 \end{bmatrix}^\dagger = \begin{bmatrix} \xi_3^T \\ \xi_2^T \\ \xi_1^T \end{bmatrix} (\xi_1\xi_1^T + \xi_2\xi_2^T + \xi_3\xi_3^T)^{-1} = \begin{bmatrix} -1 & -1.5 \\ 1 & 1.5 \\ 2 & 2.5 \\ 0 & 0.5 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Step 4 : The Hankel Matrix H is given by

$$H = \begin{bmatrix} -1 & -1.5 & 2 & 2.5 & 1 & 1 \\ 1 & 1.5 & 0 & 0.5 & 1 & 1 \\ 2 & 2.5 & 1 & 1 & 0 & 0 \\ 0 & 0.5 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

According to Theorem 2, we choose 1, 2, 3 and 5-th rows of H as a linearly independent vectors. Thus, the left annihilating matrix $\boldsymbol{M}$ is given by

$$\boldsymbol{M} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 2 & 0 & 0 & 0 \end{bmatrix}.$$

Step 5 : Finally we obtain

$$M(z) = z^4 \boldsymbol{M} S_I^3(\lambda) = z^4 \begin{bmatrix} \lambda^3 & 0 \\ 2\lambda^2 - \lambda & \lambda \end{bmatrix} = \begin{bmatrix} z & 0 \\ -z^3 + 2z^2 & z^3 \end{bmatrix}.$$

Corresponding unimodular is given by

$$U(\lambda) = \begin{bmatrix} 1 & 1 & 2 & 2.5 & -1 & -1.5 \\ 6 & 8 & -2 & -3 & 0 & 0 \end{bmatrix} S_I^2(\lambda) = \begin{bmatrix} -\lambda^2 + 2\lambda + 1 & -1.5\lambda^2 + 2.5\lambda + 1 \\ -2\lambda + 6 & -3\lambda + 8 \end{bmatrix}.$$

4 Conclusion

In this note, a derivation of interactor with lower triangular structure was discussed. The method is based on our previous research [11], [12]. Especially, the property of nonzero Markov parameters of the inverted interactorizing system was used effectively.

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