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Bounding the distance from structurally stable quadruples to non-structurally stable ones

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Abstract:- Given a quadruple of matrices (E, A, B, C) defining a generalized linear system $E\dot{x}(t) = Ax(t) + Bu(t)$, $y(t) = Cx(t)$ with $E, A \in M_n(\mathbb{C})$, $B \in M_{n\times m}(\mathbb{C})$ and $C \in M_{p\times n}(\mathbb{C})$, we present a lower bound for the distance between a structurally stable quadruple of matrices and the nearest non-structurally one, in terms of the singular values of a certain matrix associated to the quadruple.

Key-Words:- Generalized linear systems, feedback and derivative feedback.

1 Introduction

We consider generalized time-invariant linear systems given by the matrix equations $E\dot{x}(t) = Ax(t) + Bu(t)$, $y(t) = Cx(t)$ where $E, A \in M_n(\mathbb{C})$, $B \in M_{n\times m}(\mathbb{C})$, $C \in M_{p\times n}(\mathbb{C})$. We represent this systems by quadruples of matrices (E, A, B, C). These equations arise in theoretical areas as differential equations on manifolds as well as in applied areas as systems theory and control.

We are interested in obtaining lower bounds for the distance between a quadruple of matrices structurally stable under a equivalence relation defined in the set of quadruples and the nearest quadruple non structurally stable.

The structure of this paper is as follows

In Section 2 a equivalence relation is defined and a geometric study of the equivalence classes (orbits) and the tangent spaces to the orbits is presented.

Section 3 is devoted to recall the matrix norm considered and to obtain a lower bound.

2 Equivalence relation

Let us consider the set $M = \{(E, A, B, C) \mid E, A \in M_n(\mathbb{C}), B \in M_{n\times m}(\mathbb{C}), C \in M_{p\times n}(\mathbb{C})\}$ of quadruples of matrices defining a generalized time-invariant linear system. We consider the standard transformations

1) basis change in the state space $x(t) = Px_1(t)$,

2) basis change in the input space $u(t) = Ru_1(t)$

3) basis change in the output space $y_1(t) = Sy(t)$

4) feedback $u(t) = u_1(t) - Ux(t)$,

5) derivative feedback $u(t) = u_1(t) - V\dot{x}(t)$,

6) output injection $x(t) = x_1(t) + Wy(t)$

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7) Pre-multiply the state equation by an invertible matrix $QE\dot{x}(t) = QAx(t) + QBu(t)$.

This leads to the definition of the following equivalence relation in the space M

Definition 1 Two quadruples of matrices (E', A', B', C') and (E, A, B, C) in M are called equivalent if, and only if, there exist matrices $P \in Gl(n;\mathbb{C})$, $Q \in Gl(n;\mathbb{C})$, $R \in Gl(m;\mathbb{C})$, $S \in Gl(p;\mathbb{C})$, $U, V \in M_{m\times n}(\mathbb{C})$, $W \in M_{n\times p}(\mathbb{C})$ such that $(E', A', B', C') = (QEP + QBU, QAP + QBV + WCP, QBR, SCP)$.

This equivalence relation defined in M can be viewed as those induce by a Lie group action. Concretely, we consider $G = Gl(n;\mathbb{C}) \times Gl(n;\mathbb{C}) \times Gl(m;\mathbb{C}) \times Gl(p;\mathbb{C}) \times M_{m\times n}(\mathbb{C}) \times M_{m\times n}(\mathbb{C}) \times M_{n\times p}(\mathbb{C})$ where the product is defined by $(P_1, Q_1, R_1, S_1, U_1, V_1, W_1) \star (P_2, Q_2, R_2, S_2, U_2, V_2, W_2) =$
$(P_1P_2, Q_2Q_1, R_1R_2, S_2S_1, U_1P_2 + R_1V_2, V_1P_2 + R_1V_2, P_2W_1 + P_1W_2)$
being $e = (I_n, I_n, I_m, I_p, 0, 0, 0)$ its unit element.

The action $\alpha : \mathcal{G} \times M \longrightarrow M$ is defined as follows:

$$\begin{aligned}&\alpha((P, Q, R, S, U, V, W), (E, A, B, C)) =\\ &(QEP + QBU, QAP + QBV + WCP, QBR, SCP)\end{aligned} \tag{1}$$

Any equivalence class coincides with the orbit of any quadruple in it under this action. For any quadruple (E, A, B, C) we will denote by $\mathcal{O}(E, A, B, C)$ the orbit of this quadruple under the action α.

Let us denote by $T_{(E,A,B,C)}\mathcal{O}(E, A, B, C)$ the tangent space to the orbit of the quadruple (E, A, B, C) at (E, A, B, C) under the Lie group action α. The tangent space can be characterized in the following manner

Proposition 1 ([4]) *Let $(E, A, B, C) \in M$. Then*

$$\begin{aligned}&T_{(E,A,B,C)}\mathcal{O}(E, A, B, C) =\\ &\{(EP + QE + BU, AP + QA + BV + WC,\\ &BR + QB, SC + CP) | P \in M_n(\mathbb{C}), Q \in M_n(\mathbb{C}),\\ &U \in M_{m\times n}(\mathbb{C}), V \in M_{m\times n}(\mathbb{C}),\\ &W \in M_{n\times p}(\mathbb{C}), R \in M_m(\mathbb{C}), S \in M_p(\mathbb{C})\}.\end{aligned}$$

Let us consider the following matrix $T(E, A, B, C)$

$$\begin{pmatrix} X_{11} & X_{12} & 0 & 0 & X_{15} & 0 & 0\\ X_{21} & X_{22} & 0 & 0 & 0 & X_{26} & X_{27}\\ X_{31} & 0 & X_{33} & 0 & 0 & 0 & 0\\ 0 & X_{42} & 0 & X_{44} & 0 & 0 & 0 \end{pmatrix} \tag{2}$$

where $X_{11} = E^t \otimes I_n$, $X_{12} = -I_n \otimes E$, $X_{15} = -I_n \otimes B$, $X_{21} = A^t \otimes I_n$, $X_{22} = -I_n \otimes A$, $X_{26} = -I_n \otimes B$, $X_{27} = C^t \otimes I_n$, $X_{31} = B^t \otimes I_n$, $X_{33} = -I_m \otimes B$, $X_{42} = -I_n \otimes C$, $X_{44} = C^t \otimes I_p$.

This matrix give us a characterization of the tangent space.

Proposition 2 *Given any quadruple $(E, A, B, C) \in M$, $T_{(E,A,B,C)}\mathcal{O}(E, A, B, C) =$ range$\,T(E, A, B, C)$*

Proof. The proof is based on the properties of the vec operator (see [6] for its definition and properties) and its relationship with the Kronecker product. $\square$

3 Bounding the distance from structurally stable quadruples to non-structurally stable ones

The concept of structural stability used in this paper is as appears in [7]

Definition 2 Let X be a topological space where an equivalence realtion is defined. An element $x \in X$ is said to be structurally stable if and only if there exists an open neighborhood U in X such that for all $x' \in U$, x' is equivalent to x.

Remark In the case where the topological space X is a differentiable or complex manifold and the equivalence relation is that induced by the action of a Lie group, giving rise to orbits which are (differentiable or complex) submanifolds, then it is a straightforward consequence of the definition above that the following statements are equivalent:

1. x is structurally stable,
2. the orbit of x, $\mathcal{O}(x)$, is an open manifold,
3. $\dim \mathcal{O}(x) = \dim X$,
4. $\dim T_x\mathcal{O}(x) = \dim X$.

The we have the following characterization of quadruples which are structurally stable under the equivalence relation considered.

Proposition 3 *A quadruple of matrices $(E, A, B, C) \in M$ is structurally stable if and only if the matrix $T(E, A, B, C)$ has full rank.*

Our goal is to obtain a bound for the value of the radius of a ball which is neighborhood of a structurally stable element, containing only elements which are also structurally stable.

The distance we will deal with is that deduced from the Frobenius norm. We recall that given a matrix $A = (a_{ij}) =\in M_{n\times m}(\mathbb{C})$, its Frobenius norm is defined as $\|A\| = \sqrt{\sum_{ij} a_{ij}^2}$.

This norm leads to the natural definition of the norm of quadruples in M and the corresponding definition of the distance in M.

Definition 3 Given a quadruple $(E, A, B, C) \in M$ we define its norm as

$$\|(E, A, B, C)\| = \sqrt{\|E\|^2 + \|A\|^2 + \|B\|^2 + \|C\|^2}$$

and the distance between the quadruples (E, A, B, C), (E', A', B', C') is

$$d((E, A, B, C), (E', A', B', C')) = \|(E - E', A - A', B - B', C - C')\|$$

Finally we define the distance between a quadruple satisfying a property and the nearest quadruple not-satisfying it is considered to be

$$\inf\|(\delta E, \delta A, \delta B, \delta C)\|$$

where $(\delta E, \delta A, \delta B, \delta C)$ is a quadruple such that $(E + \delta E, A + \delta A, B + \delta B, C + \delta C)$ does not satisfies the given property.

The starting point to find a bound is the relationship between the norm of the associated matrix $T(E, A, B, C)$ to the quadruple (E, A, B, C) and the norm of this quadruple.

Proposition 4 *For all $(E, A, B, C) \in M$, $\|T(E, A, B, C)\| = \leq \sqrt{3n + m + p}\|(E, A, B, C)\|$*

Proof. By direct calculation we have
$\|T(E, A, B, C)\| =$
$2n\|E\|^2 + 2n\|A\|^2 + (3n + m)\|B\|^2 + (2n + p)\|C\|^2$

Then
$\|T(E, A, B, C)\|^2 \leq$
$(3n + m + p)(\|E\|^2 + \|A\|^2 + \|B\|^2 + \|C\|^2 =$
$(3n + m + p)\|(E, A, B, C)\|^2$. □

Let us assume (E, A, B, C) is a structurally stable quadruple of matrices with respect the equivalence relation considered. A bound for distance from this quadruple to the nearest non-structurally stable one $(E + \delta E, A + \delta A, B + \delta B, C + \delta C)$, is given in the following theorem.

Theorem 1 *Given a structurally stable quadruple $(E, A, B, C) \in M$ a lower bound for the distance to the nearest non-structurally stable quadruple is given by*

$$\|(\delta E, \delta A, \delta B, \delta C)\| \geq \sigma_{2n^2+mn+mp} T(E, A, B, C)$$

where $\sigma_{2n^2+mn+mp} T(E, A, B, C)$ denotes the smallest non-zero singular value of $T(E, A, B, C)$.

Proof. We know that $\operatorname{rank} T(E, A, B, C) = 2n^2 + nm + np$ and that if $(E + \delta E, A + \delta A, B + \delta B, C + \delta C)$ is not structurally stable, $\operatorname{rank} T(E + \delta E, A + \delta A, B + \delta B, C + \delta C) \leq 2n^2 + nm + np - 1$. □

The Eckart-Young and Minkowski theorem states that the smallest perturbation in the Frobenius norm that reduces the rank of a matrix M with $\operatorname{rank} M = r$ from r to $r - 1$ is $\sigma_r(M)$, the smallest non-zero singular value of M. Therefore, the norm of the perturbation of the matrix $T(\delta E, \delta A, \delta B, \delta C)$ must be at least $\sigma_{2n^2+nm+np}(T(E, A, B, C))$. The only fact which needs to be noted is that

$$T(E+\delta E, A+\delta A, B+\delta B, C+\delta C) = \\ T(E,A,B,C) + T(\delta E, \delta A, \delta B, \delta C)$$

which yields

$$\|T(E+\delta E, A+\delta A, B+\delta B, C+\delta C)\| \\ \leq \|T(E,A,B,C)\| + \|T(\delta E, \delta A, \delta B, \delta C)\|$$

Hence, abound for the distance from (E,A,B,C) to the nearest non-structurally stable quadruple, taking into account above proposition is

$$\|(\delta E, \delta A, \delta B, \delta C))\| \\ \geq \frac{1}{\sqrt{3n+m+p}} \|T(\delta E, \delta A, \delta B, \delta C)\| \\ \geq \frac{1}{\sqrt{3n+m+p}} \sigma_{2n^2+nm+np}(T(E,A,B,C)).$$

4 Conclusions

In this paper we obtain a lower bound for the distance between a structurally stable quadruple of matrices and the nearest non-structurally one, in terms of the singular values of a matrix defining the tangent space to the orbit of equivalent quadruples to the given one.

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