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The Order of Approximation by Singular Integral

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Abstract: - Function approximation by convolution type sigular integrals has important applications in differential and integral equations. Thus, we develop the test conditions for the convergence of convolution type singular integral operators to approximated function in the exponential weighted space. An application for the Gauss-Weierstrass integral is given.

Key-Words: - Convolution operators, The order of approximation, the weighted modulus of continuity.

1 Introduction

Let $q>0$ be a fixed number and $1\le p<\infty$. We consider the exponential weighted space $L_{p,q}(\mathbb{R})$ of all real-valued functions f defined by

$$L_{p,q}(\mathbb{R})=\left\{f\ :\ \|f\|_{p,q}<\infty\right\}$$

where

$$\|f\|_{p,q}=\left(\int_{\mathbb{R}}|f(x)|^{p}e^{-q|x|}dx\right)^{\frac{1}{p}}.$$

In $L_{p,q}(\mathbb{R})$ space we consider the positive singular integral operators $A_\alpha(f)$ defined by

$$A_\alpha(f)(x)=\int_{\mathbb{R}}f(x+t)K_\alpha(t)\,dt \qquad (1)$$

where $\int_{\mathbb{R}}K_\alpha(t)\,dt=1$. We sometimes use the notation $A(f;x,\alpha)$ to denote this operator.

An estimate to $|f(x)-A_\alpha(f)(x)|$ in different norms for the positive singular integral operators has been established by various scientists (see [3, 4, 5] and references therein). Another approach to obtain such an estimate is by the method of test functions. In this study we derive theory of approximation of function f in $L_{p,q}$ space by positive singular operators and investigate the order of approximation using the method of test functions with the aid of the weighted modulus of continuity of f and its derivatives.

The remaining part of this paper is organised as follows. In Section 2 the weighted modulus of continuity is defined and then the rate of convergence for A_α is calculated in the weighted space. In Section 3 the global smoothness preservation property is discussed after that an application for the Gauss-Weierstrass integral is given.

2 Approximation of Functions in $L_{p,q}$ Space

We define the test set analogous to the test set defined in [3, page 70]. Our test set consists of the functions 1, $te^{\frac{q}{p}|t+x_0|}$, $t^2e^{\frac{q}{p}|t+x_0|}$ where $x_0\in\mathbb{R}$. We recall the weighted modulus of continuity of function $f\in L_{p,q}(\mathbb{R})$ denoted by $\omega_{p,q}(f;\delta)$ and defined as

$$\omega_{p,q}(f;\delta)=\sup_{|t|\le\delta}\|f(x+t)-f(x)\|_{p,q}.$$

In [5] it has been shown that

$$\omega_{p,q}(f;\lambda\delta)\le(\lambda+1)e^{\lambda\delta\frac{q}{p}}\omega_{p,q}(f;\delta) \qquad (2)$$

where λ is a positive real constant.

Now, we let $A_\alpha(f)$ be any positive linear operator that satisfies the conditions

$$A\left(te^{\frac{q}{p}|t-x_0|},x_0,\alpha\right)=C_\alpha x_0+\beta(x_0,\alpha) \qquad (3)$$

$$A\left(t^2e^{\frac{q}{p}|t-x_0|},x_0,\alpha\right)=C_\alpha x_0^2+\gamma(x_0,\alpha) \qquad (4)$$

together with $\lim_{\alpha\to0}\beta(x_0,\alpha)=0$, $\lim_{\alpha\to0}\gamma(x_0,\alpha)=0$ and $C_\alpha=\int_{\mathbb{R}}K_\alpha(t)e^{\frac{q}{p}|t|}dt<\infty$. Then we come up with the following theorems.

Theorem 1. *Let f be a function in $L_{p,q}(\mathbb{R})$ space. Then, for any positive singular integral operator $A_\alpha(f)$ with the properties* (3) *and* (4)*, the equality*

$$\|A(f;x,\alpha)-f(x)\|_{p,q}=\mathrm{O}\left(\omega_{p,q}\left(f;\sqrt{\gamma(x_0,\alpha)-2x_0\beta(x_0,\alpha)}\right)\right)$$

holds as $\alpha\to0^+$.

Theorem 2. *Let* f *be a absolutely continuous function whose first derivative belongs to* $L_{p,q}(\mathbb{R})$. *Then, for any positive and even singular integral operator* $A_\alpha(f)$ *with the properties*(3) *and* (4), *the equality*

$$\left\|A(f;x,\alpha)-f(x)\right\|_{p,q} = \mathrm{O}\left(\omega_{p,q}\left(f';\sqrt{\gamma(x_0,\alpha)-2x_0\beta(x_0,\alpha)}\right)\right)$$

holds as $\alpha\to 0^+$.

Proofs to the these theorems will be given after the following two lemmas.

Lemma 1. *If* $f\in L_{p,q}(\mathbb{R})$ *then we have*

$$\left\|A(f;x,\alpha)\right\|_{p,q}\le C_\alpha\left\|f(x)\right\|_{p,q}. \qquad \textbf{(5)}$$

The formula (5) shows that $A_\alpha(f)$ is a positive linear operator from $L_{p,q}$ space into itself.

Recall that in the usual L_p space the modulus of continuity $\omega_p(f;\delta)$ tends to zero as δ approaches zero. Similar property holds in the weighted space [1]. The following lemma states that the weighted modulus of continuity also has the similar property.

Lemma 2. *If* $L_{p,q}(\mathbb{R})$ *then* $\lim\limits_{\delta\to 0}\omega_{p,q}(f;\delta)=0.$

Proof. Since $f\in L_{p,q}(\mathbb{R})$, then for each $\varepsilon>0$ there exists $a\in\mathbb{R}$ such that

$$\left(\int_{-\infty}^{-a}\left|f(x)\right|^p e^{-q|x|}dx\right)^{1/p}<\frac{\varepsilon}{8}$$

and

$$\left(\int_{a}^{\infty}\left|f(x)\right|^p e^{-q|x|}dx\right)^{1/p}<\frac{\varepsilon}{8}.$$

On the other hand, given $\delta>0$, we can write

$$\left(\int_{-\infty}^{-a-\delta}\left|f(x)\right|^p e^{-q|x|}dx\right)^{\frac{1}{p}}+\left(\int_{a+\delta}^{\infty}\left|f(x)\right|^p e^{-q|x|}dx\right)^{\frac{1}{p}}<\frac{\varepsilon}{4e^{\frac{q}{p}\delta}}.$$

Thus, by taking $|t|\le\delta$, we obtain

$$\left(\int_{-\infty}^{-a-\delta}\left|f(x+t)\right|^p e^{-q|x|}dx\right)^{\frac{1}{p}}+\left(\int_{a+\delta}^{\infty}\left|f(x+t)\right|^p e^{-q|x|}dx\right)^{\frac{1}{p}}<\frac{\varepsilon}{4}.$$

By using the definition of $\omega_{p,q}(f;\delta)$ and above inequalities, we get

$$\omega_{p,q}(f;\delta)\le\sup_{|t|<\delta}\left(\int_{-a-\delta}^{a+\delta}\left|\begin{matrix}f(x+t)\\ -f(x)\end{matrix}\right|^p e^{-q|x|}dx\right)^{\frac{1}{p}} +\frac{\varepsilon}{2}. \qquad \textbf{(6)}$$

It is known that by Weierstrass theorem there exists a sequence of functions $\varphi_n(x)$ with continuous derivatives in the interval $[-a-2\delta,a+2\delta]$ such that

$$\lim_{n\to\infty}\left(\int_{-a-2\delta}^{a+2\delta}\left|f(x)-\varphi_n(x)\right|^p e^{-q|x|}dx\right)^{\frac{1}{p}}=0,$$

in other words, given $\varepsilon>0$ there exists $n_0\in\mathbb{N}$ such that

$$\left(\int_{-a-2\delta}^{a+2\delta}\left|f(x)-\varphi_n(x)\right|^p e^{-q|x|}dx\right)^{\frac{1}{p}}<\frac{\varepsilon}{8e^{\frac{q}{p}\delta}}$$

whenever $n\ge n_0$ and $\delta>0$. Thus, we have

$$\begin{aligned}&\sup_{|t|\le\delta}\left(\int_{-a-\delta}^{a+\delta}\left|f(x+t)-\varphi_n(x+t)\right|^p e^{-q|x|}dx\right)^{\frac{1}{p}}\\ &\le e^{\frac{q}{p}\delta}\left(\int_{-a-2\delta}^{a+2\delta}\left|f(x)-\varphi_n(x)\right|^p e^{-q|x|}dx\right)^{\frac{1}{p}} \qquad \textbf{(7)}\\ &\le\frac{\varepsilon}{8}\end{aligned}$$

for $n\ge n_0$.

On the other hand, by adding and subtracting $\varphi_n(x+t)+\varphi_n(x)$ and then applying the Minkowsky inequality we obtain

$$\begin{aligned}&\left(\int_{-a-\delta}^{a+\delta}\left|f(x+t)-f(x)\right|^p e^{-q|x|}dx\right)^{\frac{1}{p}}\\ &\le\left(\int_{-a-\delta}^{a+\delta}\left|f(x+t)-\varphi_n(x+t)\right|^p e^{-q|x|}dx\right)^{\frac{1}{p}}\\ &+\left(\int_{-a-\delta}^{a+\delta}\left|\varphi_n(x+t)-\varphi_n(x)\right|^p e^{-q|x|}dx\right)^{\frac{1}{p}}\\ &+\left(\int_{-a-\delta}^{a+\delta}\left|\varphi_n(x)-f(x)\right|^p e^{-q|x|}dx\right)^{\frac{1}{p}}.\end{aligned}$$

From**(7)** it follows that

$$\begin{aligned}&\sup_{|t|\le\delta}\left(\int_{-a-\delta}^{a+\delta}\left|f(x+t)-f(x)\right|^p e^{-q|x|}dx\right)^{\frac{1}{p}}\\ &\le\frac{\varepsilon}{4}+\sup_{|t|\le\delta}\left(\int_{-a-\delta}^{a+\delta}\left|\varphi_n(x+t)-\varphi_n(x)\right|^p e^{-q|x|}dx\right)^{\frac{1}{p}}.\end{aligned} \qquad \textbf{(8)}$$

By the properties of $\varphi_n(x)$, we can write for $|t|<\delta$ and $n\ge n_0$

$$\left|\varphi_n(x+t)-\varphi_n(x)\right|\le\frac{\varepsilon}{4C\left[2(a+\delta)\right]^{\frac{1}{p}}},$$

where the constant $C=\left[\frac{1}{q}\left(1-e^{-a-\delta}\right)\right]^{\frac{1}{p}}$. Thus, we obtain

$$\sup_{|t|\le\delta}\left(\int_{-a-\delta}^{a+\delta}\left|\varphi_n(x+t)-\varphi_n(x)\right|^p e^{-q|x|}dx\right)^{\frac{1}{p}}<\frac{\varepsilon}{4}. \quad \textbf{(9)}$$

By (8) and (9) we get

$$\sup_{|t|\le\delta}\left(\int_{-a-\delta}^{a+\delta}\left|f(x+t)-f(x)\right|^p e^{-q|x|}dx\right)<\frac{\varepsilon}{2}. \quad \textbf{(10)}$$

By the inequalities (6) and (10) we have $\omega_{p,q}(f;\delta)<\varepsilon$ for all $\varepsilon>0$. This proves the lemma.

We are now ready to prove the theorems above.

*Proof of Theorem 1***.** Since $\int_{\mathbb{R}} K_\alpha(t)dt=1$, using generalized Minkowsky inequality we get from (2) that

$$\left\|A(f;x,\alpha)-f(x)\right\|_{p,q}$$

$$=\left(\int_{\mathbb{R}}\left|\int_{\mathbb{R}}K_\alpha(t)\left(f(x+t)-f(x)\right)dt\right|^p e^{-q|x|}dx\right)^{\frac{1}{p}}$$

$$\le\int_{\mathbb{R}}\left(\int_{\mathbb{R}}\left|f(x+t)-f(x)\right|^p e^{-q|x|}dx\right)^{1/p}K_\alpha(t)dt$$

$$\le\int_{\mathbb{R}}\omega_{p,q}(f;|t|)K_\alpha(t)dt$$

$$\le\omega_{p,q}(f;\lambda^{-1})\left[\int_{\mathbb{R}}K_\alpha(t)e^{\frac{q}{p}|t|}dt+\lambda\int_{\mathbb{R}}|t|K_\alpha(t)e^{\frac{q}{p}|t|}dt\right].$$

We apply the Hölder inequality to obtain

$$\left\|A(f;x,\alpha)-f(x)\right\|_{p,q}$$

$$\le\omega_{p,q}(f;\lambda^{-1})\left[\int_{\mathbb{R}}K_\alpha(t)e^{\frac{q}{p}|t|}dt\right.$$

$$\left.+\lambda\left(\int_{\mathbb{R}}t^2K_\alpha(t)e^{\frac{q}{p}|t|}dt\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}}K_\alpha(t)e^{\frac{q}{p}|t|}dt\right)^{\frac{1}{2}}\right]$$

$$\le M_\alpha\omega_{p,q}(f;\lambda^{-1})\left[1+\lambda\left(\int_{\mathbb{R}}t^2K_\alpha(t)e^{\frac{q}{p}|t|}dt\right)^{\frac{1}{2}}\right]$$

where $M_\alpha=\max\left\{C_\alpha,\sqrt{C_\alpha}\right\}$. From (3) and (4) it follows that

$$\int_{\mathbb{R}}t^2K_\alpha(t)e^{\frac{q}{p}|t|}dt=\gamma(x_0,\alpha)-2x_0\beta(x_0,\alpha).$$

If we choose $\lambda^{-1}=\sqrt{\gamma(x_0,\alpha)-2x_0\beta(x_0,\alpha)}$ then we get

$$\left\|A(f;x,\alpha)-f(x)\right\|_{p,q}$$

$$\le 2M_\alpha\omega_{p,q}\left(f;\sqrt{\gamma(x_0,\alpha)-2x_0\beta(x_0,\alpha)}\right),$$

which completes the proof.

*Proof of Theorem 2***.** Since f is absolutely continuous function then by mean value theorem, there exists ξ between x and $A_\alpha(f)$ such that

$$f(x+t)-f(x)=tf'(\xi)$$

$$=tf'(x)+t\left(f'(\xi)-f'(x)\right).$$

Then, we have

$$A_\alpha(f)(x)-f(x)=$$

$$f'(x)\int_{\mathbb{R}}tK_\alpha(t)dt+\int_{\mathbb{R}}t\left(f'(\xi)-f'(x)\right)K_\alpha(t)dt.$$

Since the kernel $K_\alpha(t)$ is even, $\int_{\mathbb{R}}tK_\alpha(t)dt=0.$ Therefore, in view of generalized Minkowsky inequality we have

$$\left\|A(f;x,\alpha)-f(x)\right\|_{p,q}\le\int_{\mathbb{R}}\left\|f'(\xi)-f'(x)\right\|_{p,q}|t|K_\alpha(t)dt.$$

On the other hand, by $|\xi-x|<|t|$ and (2)

$$\left\|A(f;x,\alpha)-f(x)\right\|_{p,q}$$

$$\le\int_{\mathbb{R}}\omega_{p,q}(f';|\xi-x|)|t|K_\alpha(t)dt$$

$$\le\int_{\mathbb{R}}\omega_{p,q}(f';|t|)|t|K_\alpha(t)dt$$

$$\le\omega_{p,q}(f';\lambda^{-1})\int_{\mathbb{R}}(1+\lambda|t|)e^{\frac{q}{p}|t|}|t|K_\alpha(t)dt$$

$$\le\omega_{p,q}(f';\lambda^{-1})\left(\int_{\mathbb{R}}|t|e^{\frac{q}{p}|t|}K_\alpha(t)dt+\lambda\int_{\mathbb{R}}t^2e^{\frac{q}{p}|t|}K_\alpha(t)dt\right)$$

$$\le\omega_{p,q}(f';\lambda^{-1})\left[\sqrt{C_\alpha}\left(\int_{\mathbb{R}}t^2e^{\frac{q}{p}|t|}K_\alpha(t)dt\right)^{\frac{1}{2}}+\int_{\mathbb{R}}t^2e^{\frac{q}{p}|t|}K_\alpha(t)dt\right]$$

$$\le\omega_{p,q}(f';\lambda^{-1})\left(1+\sqrt{C_\alpha}\right)\int_{\mathbb{R}}t^2e^{\frac{q}{p}|t|}K_\alpha(t)dt.$$

So, the equality

$$\int_{\mathbb{R}}t^2e^{\frac{q}{p}|t|}K_\alpha(t)dt=\gamma(x_0,\alpha)-2x_0\beta(x_0,\alpha)$$

and taking $f_{\nu,n}(x)=[u_n]^{-1}\chi_u(x)$ it follows that

$$\left\|A(f;x,\alpha)-f(x)\right\|_{p,q}$$

$$=\mathrm{O}\left(\omega_{p,q}\left(f';\sqrt{\gamma(x_0,\alpha)-2x_0\beta(x_0,\alpha)}\right)\right),$$

for $\alpha\to0^+$. This proves Theorem 2.

3 Global Smoothness Preservation Property

In this section we state an estimation that satisfies the global smoothness preservation property. This property is studied for different modulus of smoothness in [2].

Theorem 3. *For every* $f \in L_{p,q}(\mathbb{R})$ *and* $\sigma > 0$

$$W_{p,q}(A_\alpha(f);\sigma) \le D_\alpha W_{p,q}(f;\sigma),$$

where $D_\alpha = \int_{\mathbb{R}} e^{q|t|} K_\alpha(t) < \infty$.

Proof. Since $\int_{\mathbb{R}} K_\alpha(t) = 1$, we have

$$A_\alpha(f;x+h) - A_\alpha(f;x) = \int_{\mathbb{R}} \left(f(x+t+h) - f(x+t) \right) K_\alpha(t)\,dt.$$

Using this equality we obtain

$$\left(\int_{\mathbb{R}} \left| A_\alpha(f;x+h) - A_\alpha(f;x) \right|^p e^{-q|x|} dx \right)^{1/p}$$

$$\le \left(\int_{\mathbb{R}} \left| \int_{\mathbb{R}} \left(f(x+t+h) - f(x+h) \right) K_\alpha(t)\,dt \right|^p e^{-q|x|} dx \right)^{1/p}$$

$$\le \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \left| \left(f(x+t+h) - f(x+h) \right) \right|^p e^{-q|x|} dx \right)^{1/p} K_\alpha(t)\,dt.$$

By the definition of the weighted modulus of smoothness we have

$$\left(\int_{\mathbb{R}} \left| A_\alpha(f;x+h) - A_\alpha(f;x) \right|^p e^{-q|x|} dx \right)^{1/p}$$

$$\le W_{p,q}(f;h) \int_{\mathbb{R}} e^{q|t|} K_\alpha(t)\,dt.$$

Thus we have

$$W_{p,q}(A_\alpha(f);h) \le D W_{p,q}(f;h).$$

This proves Theorem 3.

Example 1. *Let* S_α *be the positive Gauss-Weierstrass operators defined as*

$$S(f;x,\alpha) =: S_a(f)(x) = \frac{1}{2\alpha} \int_{-\infty}^{\infty} f(x+t) e^{-\frac{|t|}{\alpha}} dt.$$

(see [1] for more details). It is easy to calculate that

$$S\left(t e^{\frac{q}{p}|t-x_0|}, x_0, \alpha \right) = x_0 \frac{p}{(p-\alpha q)}$$

$$S\left(t^2 e^{\frac{q}{p}|t-x_0|}, x_0, \alpha \right) = x_0^2 \frac{p}{(p-\alpha q)} + \frac{2\alpha^2 p^3}{(p-\alpha q)^3},$$

where $0 < \alpha < \frac{p}{q}$. *Then, by comparing these with* (2) *and* (3) *we obtain*

$$C_\alpha = \frac{p}{(p-\alpha q)},$$

$$\beta(x_0, \alpha) = 0,$$

$$\gamma(x_0, \alpha) = \frac{\alpha^2 p^3}{(p-\alpha q)^3},$$

and

$$D_\alpha = \frac{1}{1-\alpha q} < \infty \text{ for } 0 < \alpha < \frac{1}{q}.$$

Note that $\sqrt{\gamma(x_0,\alpha) - 2x_0\beta(x_0,\alpha)} \to 0$ *and* $C_\alpha < \infty$ *for* $0 < \alpha < \frac{p}{q}$. *Applying theorems 1, 2 and 3, we obtain*

$$\left\| S_a(f)(x) - f(x) \right\|_{p,q} = O\left(\omega_{p,q}\left(f; \sqrt{\gamma(x_0,\alpha)} \right) \right),$$

$$\left\| S_a(f)(x) - f(x) \right\|_{p,q} = O\left(\omega_{p,q}\left(f'; \sqrt{\gamma(x_0,\alpha)} \right) \right),$$

where $f \in L_{p,q}$, $f' \in L_{p,q}$ *and* $0 < \alpha < \frac{p}{q}$. *Also we have ,for* $0 < \alpha < \frac{1}{q}$,

$$W_{p,q}(S_\alpha(t);\sigma) \le \frac{1}{1-\alpha q} W_{p,q}(t;\sigma)$$

with $\sigma > 0$.

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