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Dynamical Systems of Class C: Searching for the Most Robust Practical Implementation of the Chaotic Oscillator

JIŘÍ PETRŽELA, STANISLAV HANUS
Institute of Radio Electronics
Brno University of Technology
Purkyňova 118, 61200 Brno
CZECH REPUBLIC
http://www.urel.feec.vutbr.cz

Abstract: This contribution is about a design of universal chaotic oscillator based on the integrator synthesis, which represents a class of dynamical systems with piecewise linear vector fields. Final circuit consists of commercially available devices and provide us with good agreement between theoretical expectations and practical measurements. This is documented by a gallery of the oscilloscope screenshots. Moreover, one possible configuration of proposed circuit also exhibits the lowest sensitivity of the state space attractors with respect to the changes or uncertaines of the system parameters, which is prooved by a Monte-Carlo analysis together with the presence of positive one-dimensional Lyapunov exponent.

Key-Words: autonomous oscillator, chaos, Lyapunov exponents, eigenvalues

1 Introduction

It is well known that the most real physical system can be described by a set of nonlinear differential equations. To get a complete overview on its behavior, we further need a list of system parameters and initial conditions. In the case of deterministic systems, we should be able to make a long-time prediction of the system evolution. But, there is a small number of dynamical systems which can exhibit irregular behavior, chaos. Its typical property is an extreme sensitivity to a deviations of the initial conditions or changes in the system parameters and dense attractors with fractional topological dimension. The phenomenon of chaos is universal, it is reported from many completely different scientific fields such as chemistry, economics, mechanics, electronics, etc.

Recently, one can notice an increasing interest in practical implementation of the chaotic oscillators. There are many applications which are based on the generators of the chaotic waveforms, like a modulation techniques, masking procedures or coders. Here, we take advantage of the wide range of continuous frequency spectrum. From this point of view, the searching for the most robust circuit with well reproducible dynamic motion seems to be essential task.

In this paper, starting with the basic definitions for the systems of class C, we use the concept of qualitative equivalence [1], [2], [3], for the purpose of eigenvalue sensitivity optimization. In the next section, we verify derived results by means of numerical analysis, i.e. by the state space projections of the selected attractors, Monte-Carlo analysis of the eigenvalue sensitivities and by the corresponding estimation of the largest Lyapunov exponent (LE). In the end, the circuitry of the universal chaotic oscillator will be introduced, as well as several plane projections of chaotic attractors of some interest.

2 Mathematical Models

The dynamical system of class C is a terminology introduced by a mathematicians and covers the extensive group of autonomous deterministic dynamical systems with symmetrical vector fields, which can be expressed in compact matrix form as

$$\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \mathbf{b}\,h\left(\mathbf{w}^T\mathbf{x}\right), \tag{1}$$

where $\mathbf{x} \in \mathcal{R}^3$ are state variables, $\mathbf{A}$ is 3×3 matrix and $\mathbf{b}$, $\mathbf{w}$ are column vectors. The scalar saturation-type function

$$h\left(\mathbf{w}^T\mathbf{x}\right) = 0.5\left(\left|\mathbf{w}^T\mathbf{x}+1\right| - \left|\mathbf{w}^T\mathbf{x}-1\right|\right), \tag{2}$$

brings a nonlinearity, which separates entire state space by two parallel surfaces into into three regions. Let denote the inner region as D_0 and two outer regions as $D_{\pm1}$. For the brief analysis, we should note that the time-evolution equations are linear in each region, in detail

$$D_0: \quad \dot{\mathbf{x}} = \left(\mathbf{A} + \mathbf{b}\,\mathbf{w}^T\right)\mathbf{x}, \qquad D_{\pm1}: \quad \dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} \pm \mathbf{b}. \tag{3}$$

Qualitative aspects of the dynamical motion are determined by fixed points, i.e. real solutions of system $\dot{\mathbf{x}} = \mathbf{0}$, and corresponding eigenvalues, which are roots of the characteristic polynomial. One equilibria is allways origin. Two others exist only if det($\mathbf{A}$)≠0, then there is a solution of non-homogenous problem $\mathbf{x}_{outer}=\pm\mathbf{A}^{-1}\mathbf{b}$. For D_0 we have

$$\det\left[\lambda\,\mathbf{E} - \left(\mathbf{A} + \mathbf{b}\,\mathbf{w}^T\right)\right] = \prod_{i=1}^{3}\left(s - \lambda_{inner}^{i}\right) =$$

$$= s^3 - p_1 s^2 + p_2 s - p_3 = 0\,, \tag{4a}$$

where $\mathbf{E}$ is an unity matrix. Similarly for $D_{\pm 1}$ we get

$$\det\left[\lambda\,\mathbf{E}-\mathbf{A}\right]=\prod_{i=1}^{3}\left(s-\lambda_{outer}^{i}\right)=s^3-q_1\,s^2+q_2\,s-q_3=0\,.$$

In the theory of chaos, the essential task is to determine the proper ranges of system parameters. For the double-scroll attractor we adopt the following set of parameters

$$\lambda_{inner}^{1,2}=-0.319\pm j\,0.892\,,\quad \lambda_{inner}^{3}=0.728\,,\quad (5a)$$

for the inner region and

$$\lambda_{outer}^{1,2}=0.061\pm j\,,\quad \lambda_{outer}^{3}=-1.29\,,\quad (5b)$$

for both outer regions. These values directly correspond to the equivalent eigenvalues, i.e. coefficients of (4),

$$p_1=0.09\,,\quad p_2=0.433\,,\quad p_3=0.653\,,\quad (6a)$$

and

$$q_1=-1.168\,,\quad q_2=0.846\,,\quad q_3=-1.295\,,\quad (6b)$$

which are real numbers.

For the purpose of generating qualitatively equivalent dynamical systems, we can use the concept of linear topological conjugacy. Assuming the nonsingular transformation of coordinates $\mathbf{x}$=$\mathbf{T}\mathbf{y}$, where $\mathbf{T}\in\mathcal{R}^{3\times 3}$, equations

$$\dot{\mathbf{y}}=\left[\mathbf{T}^{-1}\left(\mathbf{A}+\mathbf{b}\,\mathbf{w}^T\right)\mathbf{T}\right]\mathbf{y}\,,\ \ \dot{\mathbf{y}}=\left(\mathbf{T}^{-1}\mathbf{A}\,\mathbf{T}\right)\mathbf{y}+\mathbf{T}^{-1}\mathbf{b}\,,\ (7)$$

describe the equivalent linear systems in each region of the state space. Thus, the whole piecewise-linear (PWL) system exhibits similar behavior to the original, since the eigenvalues remain unchanged. Transformation provide us with a certain degree of freedom, which can be used for constructing canonical systems with respect to the actual circuit structure or we can improve some properties of the system. Here, we choose the second possibility. The optimization criterion can be chosen in the standard form

$$\sum_{ijk}S_r^2\left(\lambda_k\,,\,a_{ij}\right)\rightarrow \min\,,\quad (8a)$$

where the relative sensitivities are given as

$$S_r\left(\lambda\,,\,a_{ij}\right)=\frac{a_{ij}}{\lambda}\frac{\partial\lambda}{\partial a_{ij}}\,.\quad (8b)$$

The analytical form of optimization criterion was found by standard mathematical method

$$\left(a_{11}-a_{22}\right)\left(t_{11}\,t_{22}+t_{21}\,t_{12}\right)=2\left(t_{11}\,t_{21}\,a_{12}-t_{12}\,t_{22}\,a_{21}\right).\ (9)$$

The symbolical form of this condition can be applied to different models of dynamical systems. Taking into account that there are different state matrices for distinct state space regions, we must make the optimization in each of these regions. Starting with a second-order state matrix $\mathbf{A}$ in Jordan form, optimized state matrices are

$$\mathbf{A}=\begin{pmatrix}\upsilon' & -K\,\upsilon''\\ K^{-1}\upsilon'' & \upsilon'\end{pmatrix},\quad \mathbf{A}+\mathbf{b}\,\mathbf{w}^T=\begin{pmatrix}\mu' & -\widetilde{K}\,\mu''\\ \widetilde{K}^{-1}\mu'' & \mu'\end{pmatrix},$$

where K, $\widetilde{K}$ are some coefficients and

$$\upsilon'=\mathrm{Re}\left(\lambda_{outer}^{1,2}\right),\quad \upsilon''=\mathrm{Im}\left(\lambda_{outer}^{1,2}\right),\quad (11)$$

belongs to the outer segments of the vector field and

$$\mu'=\mathrm{Re}\left(\lambda_{inner}^{1,2}\right),\quad \mu''=\mathrm{Im}\left(\lambda_{inner}^{1,2}\right),\quad (12)$$

determines the behavior in the inner region. Unknown coefficients K, $\widetilde{K}$ must satisfy

$$K^2\upsilon''\mu''+\widetilde{K}^2\upsilon''\mu''-K\,\widetilde{K}\left[\upsilon''^2+\mu''^2+\left(\mu'-\upsilon'\right)^2\right]=0\,,$$

which greatly simplifies if we choose K=1 into quadratic equation

$$\widetilde{K}-\left[\frac{\left(\mu''-\upsilon''\right)^2+\left(\mu'-\upsilon'\right)^2}{\upsilon''\mu''}+2\right]\widetilde{K}+1=0\,.\quad (14)$$

To finish it, for lifting these subsystems into third dimension, we use the concept of block-triangular decomposition

$$\mathbf{A}=\begin{pmatrix}\mathbf{A}_{outer} & \mathbf{A}_1\\ \mathbf{0} & \lambda_{outer}^3\end{pmatrix},\quad \mathbf{A}+\mathbf{b}\,\mathbf{w}^T=\begin{pmatrix}\mathbf{A}_{inner} & \mathbf{0}\\ \mathbf{A}_2 & \lambda_{inner}^3\end{pmatrix},$$

where matrices $\mathbf{A}_{outer}$ and $\mathbf{A}_{inner}$ are given by (10) while $\mathbf{A}_1$ and $\mathbf{A}_2$ are vectors to be computed. Following the steps published in [4], [5], we can write down the final complex decomposed system (CDSM)

$$\dot{\mathbf{x}}=\begin{pmatrix}\upsilon' & -\upsilon'' & \upsilon'-\mu'\\ \upsilon'' & \upsilon' & -b_2\\ 0 & 0 & \lambda_{outer}^3\end{pmatrix}\mathbf{x}+\begin{pmatrix}\mu'-\upsilon'\\ b_2\\ \lambda_{inner}^3-\lambda_{outer}^3\end{pmatrix}h\left(\mathbf{w}^T\mathbf{x}\right),$$

where

$$b_2=\frac{\left(\mu'-\upsilon'\right)^2}{\upsilon''-\mu''\,\widetilde{K}}\,,\quad \mathbf{w}^T=\begin{pmatrix}1 & \dfrac{\upsilon''-\mu''\,\widetilde{K}}{\mu'-\upsilon'} & 1\end{pmatrix}.\quad (17)$$

Note that there is a connection between a_{ij} of any 2×2 matrix $\mathbf{A}$ and the complex conjugated eigenvalues

$$\mathrm{tr}\left(\mathbf{A}\right)=2\,\upsilon'\,,\quad \det\left(\mathbf{A}\right)=\upsilon'^2+\upsilon''^2\,,\quad (18)$$

which can be fulfilled also by the subsystem

$$a_{11}=2\,\upsilon'\,,\ \ a_{12}=-1\,,\ \ a_{21}=\upsilon'^2+\upsilon''^2\,,\ \ a_{22}=0\,,\quad (19)$$

leading to the so-called dynamical system in elementary canonical form with state equations (ECSM)

$$\dot{\mathbf{x}}=\begin{pmatrix}2\,\upsilon' & -1 & -b_1\\ \upsilon'^2+\upsilon''^2 & 0 & -b_2\\ 0 & 0 & \lambda_{outer}^3\end{pmatrix}\mathbf{x}+\begin{pmatrix}b_1\\ b_2\\ b_3\end{pmatrix}h\left(\mathbf{w}^T\mathbf{x}\right),\quad (20)$$

where

$$b_1=2\left(\mu'-\upsilon'\right),\quad b_2=\mu'^2+\mu'^2-\upsilon'^2-\upsilon''^2\,,$$

$$b_3=\lambda_{inner}^3-\lambda_{outer}^3\,,\quad \mathbf{w}^T=\begin{pmatrix}1 & 0 & 1\end{pmatrix}.\quad (21)$$

Both systems (16) and (20) were succesfully modelled by synthesized universal oscillator.

A numerical integration of the Chua´s equations [6], its 1st canonical equivalent as well as (16) and (20) with the parameter family for the double-scroll attractor is shown in Fig.1. In the case of chaos, the global behavior is strongly affected by the changes of the internal system parameters, although the local geometry configuration of the vector field remains unchanged. So we are interested in the rate of eigenvalue migration.

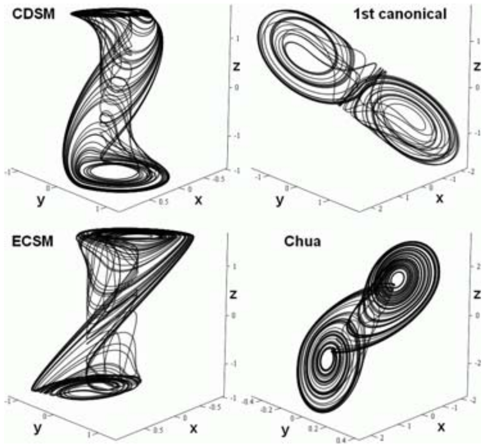

Fig.1: 3D projections of the double-scroll attractor.

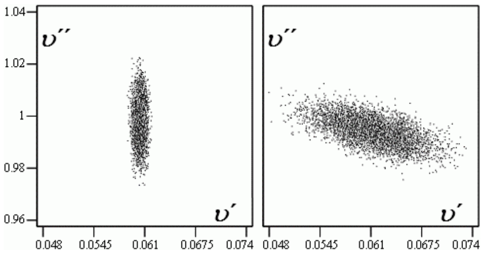

Fig.2: Eigenvalue patterns for Monte-Carlo analysis.

Fig. 2 is an illustrative example of Monte-Carlo analysis of this eigenvalues migration for 5000 generations and 1% standard deviation. The optimized subsystem (10a) is compared with the state matrix with entries $a_{11}=q_1$, $a_{12}=-1$, $a_{21}=q_2$, $a_{22}=0$, which can be somehow related to the 1st canonical model. Note that left picture (10a) is better in the sense a region of values is smaller. These results seems to be attractive, but we must ask following question: is the sensitivity criterion (8) suitable for chaotic system? The answer is unclear since there is no direct relation between the eigenvalues and the existence of chaos. Moreover, we can also ask much more general question: is there any optimization possible for chaotic system? The word optimization can be considered as the searching for the system with better properties or as the searching for the most wide hyperspace of system parameters for chaotic solution (with respect to the gradient). For identifying chaos, we are to calculate the rate of convergency of two neighborhood trajectories, i.e. we must establish the spectrum of one-dimensional LE. For chaos, due to the exponential divergency, one LE is required to be positive. Taking into account that these system are dissipative, the sum (average values) of all LE must be negative as the volume element is shrinking with time progression. To make a judgement which member of class C is less sensitive to the changes of the parameters values and thus best suited for practical implementation, we merge a program for Monte-Carlo analysis together with program for calculating spectrum of LE. Collection of resulting graphs are in Fig.3, Fig.4, Fig.5 and Fig.6. Here, we made an analysis only for 200 random generations with normal distribution and for different standard deviations, because it has a great demands on the performance of the personal computer.

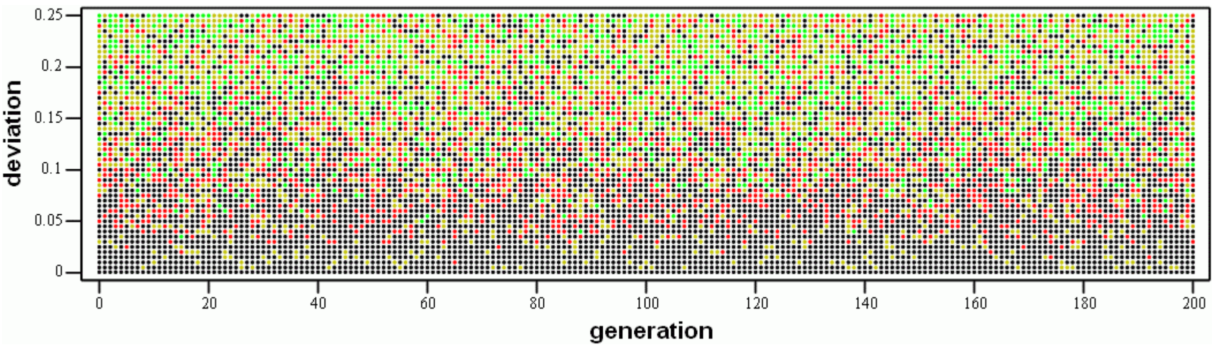

Fig.3: Monte-Carlo sensitivity analysis based on the largest LE for Chua´s equations.

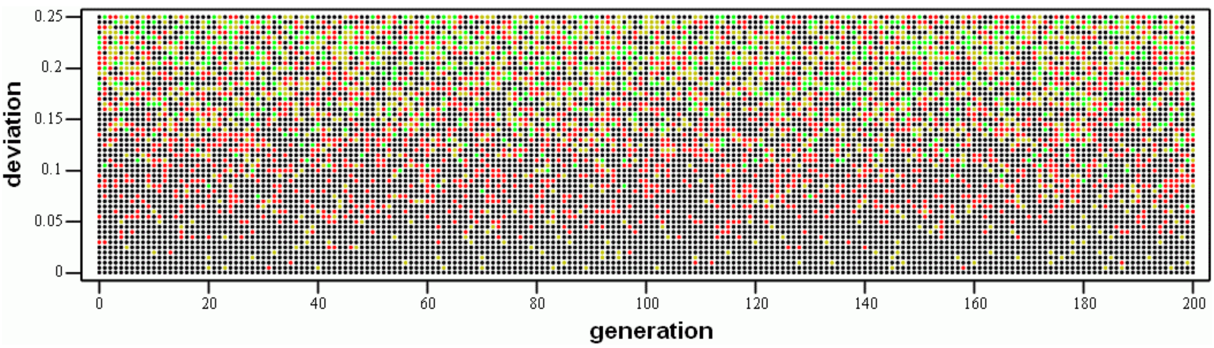

Fig.4: Monte-Carlo sensitivity analysis based on the largest LE for 1st canonical model.

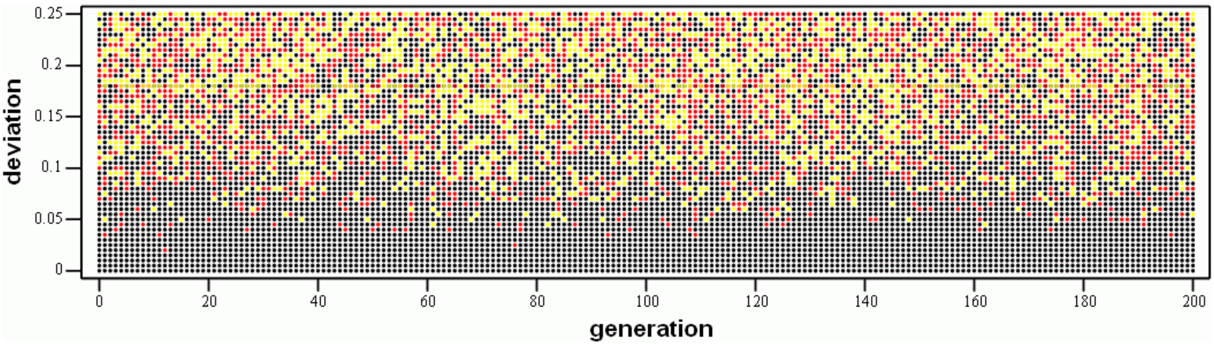

Fig.5: Monte-Carlo sensitivity analysis based on the largest LE for CDSM (optimized system).

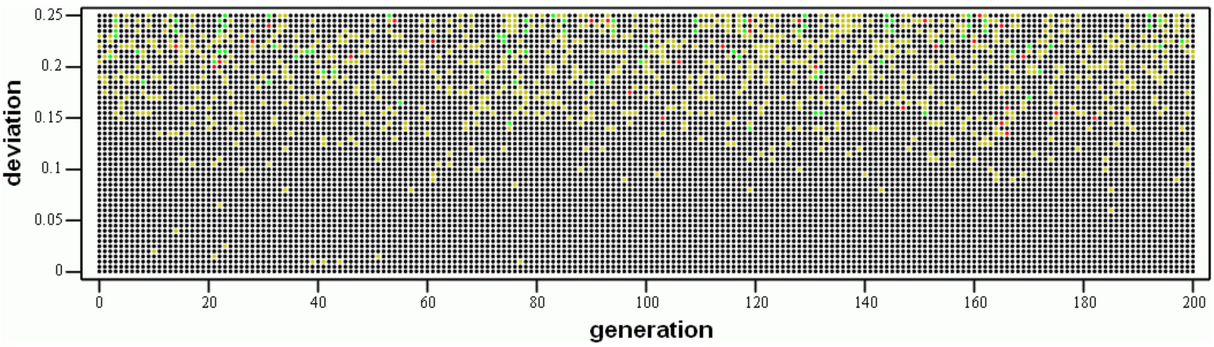

Fig.6: Monte-Carlo sensitivity analysis based on the largest LE for ECSM.

Black points are chaotic solutions, red color marks limit cycles, green color represents a fixed points and yellow denotes the suddenly unbounded solution. In each case, the initial conditions were chosen carefully to be a part of the basin of attraction. We utilize the final computation time as t_{end}=500 to ensure we are on the attractor, with a standard Gram-Smith orthogonalization procedure after κ=1 step. In both Monte-Carlo analysis, we suppose system parameters statistically independent.

3 Circuitry Implementation

The most straightforward design of chaotic oscillator is based on the integrator block schematic, entire circuitry is designed by using three fundamental building blocks: inverting integrator, differential amplifier and diode limiter. Final voltage-mode circuit is presented in Fig. 7 and consists of seven operational amplifiers TL084, two diodes 1N4148, several linear capacitors and resistors. It can model any dynamical system which can be recasted into the normalized expression

$$\begin{aligned} -dx/d\tau &= \varepsilon_{11}y+\varepsilon_{12}\left[h(x+\varepsilon y+z)-z\right]+ \\ &\quad +\varepsilon_{13}\left[x+z-h(x+\varepsilon y+z)\right], \qquad (22) \\ -dy/d\tau &= \varepsilon_{21}x+\varepsilon_{22}y+\varepsilon_{23}\left[h(x+\varepsilon y+z)-z\right], \\ -dz/d\tau &= \varepsilon_{31}\left[h(x+\varepsilon y+z)-z\right]+\varepsilon_{32}h(x+\varepsilon y+z). \end{aligned}$$

We choose the following listing of circuit elements: C=33nF, R=47kΩ, R_t=1.5kΩ, R_s=10kΩ, R_x=15kΩ. To model the behavior of (16), we are to set $\varepsilon_{11}=\upsilon''$, $\varepsilon_{12}=-\mu'$, $\varepsilon_{13}=-\upsilon'$, $\varepsilon_{21}=-\upsilon''$, $\varepsilon_{22}=-\upsilon'$, $\varepsilon_{23}=-b_2$, $\varepsilon_{31}=\lambda^3_{outer}$, $\varepsilon_{32}=-\lambda^3_{inner}$ and $\varepsilon=w_2$. Similarly, we can reconfigure the circuit for modelling (20) simply by setting $\varepsilon_{11}=1$, $\varepsilon_{12}=2\mu'-4\upsilon''$, $\varepsilon_{13}=-2\upsilon'$, $\varepsilon_{21}=-\upsilon'^2-\upsilon''^2$, $\varepsilon_{22}=0$, $\varepsilon_{23}=-b_2$, $\varepsilon_{31}=\lambda^3_{outer}$, $\varepsilon_{32}=-\lambda^3_{inner}$ and $\varepsilon=0$.

A nodes marked ~ serve as an input for the reference sine voltage for precise adjusting of individual system parameters. Note that we can set them continuously and independently on each other, what is of big advantage. We can change the signum of the parameter simply by overstrain the associated switch. The breakpoint are defined by an external dc sources and can handle arbitrary values. Fig. 8 shows a gallery of screenshot photos measured on digital oscilloscope HP54603B, which were carefully selected to demonstrate a chaotic nature of oscillations. These laboratory experiments uncovers that this circuit exhibits a very good agreement between numerical integration and real behavior. It is worth nothing that also PSpice simulations give us the same results. This is the reason why is this implementation well suited for educational purposes. The transfer characteristics of the double-sided diode limiter is demonstrated in Fig.9. For easier imagination about the time-dependance of a chaotic signal we also include Fig.10, which display both x and z state variable. Finally, the physical implementation (SMD) of presented oscillator is shown in Fig.11.

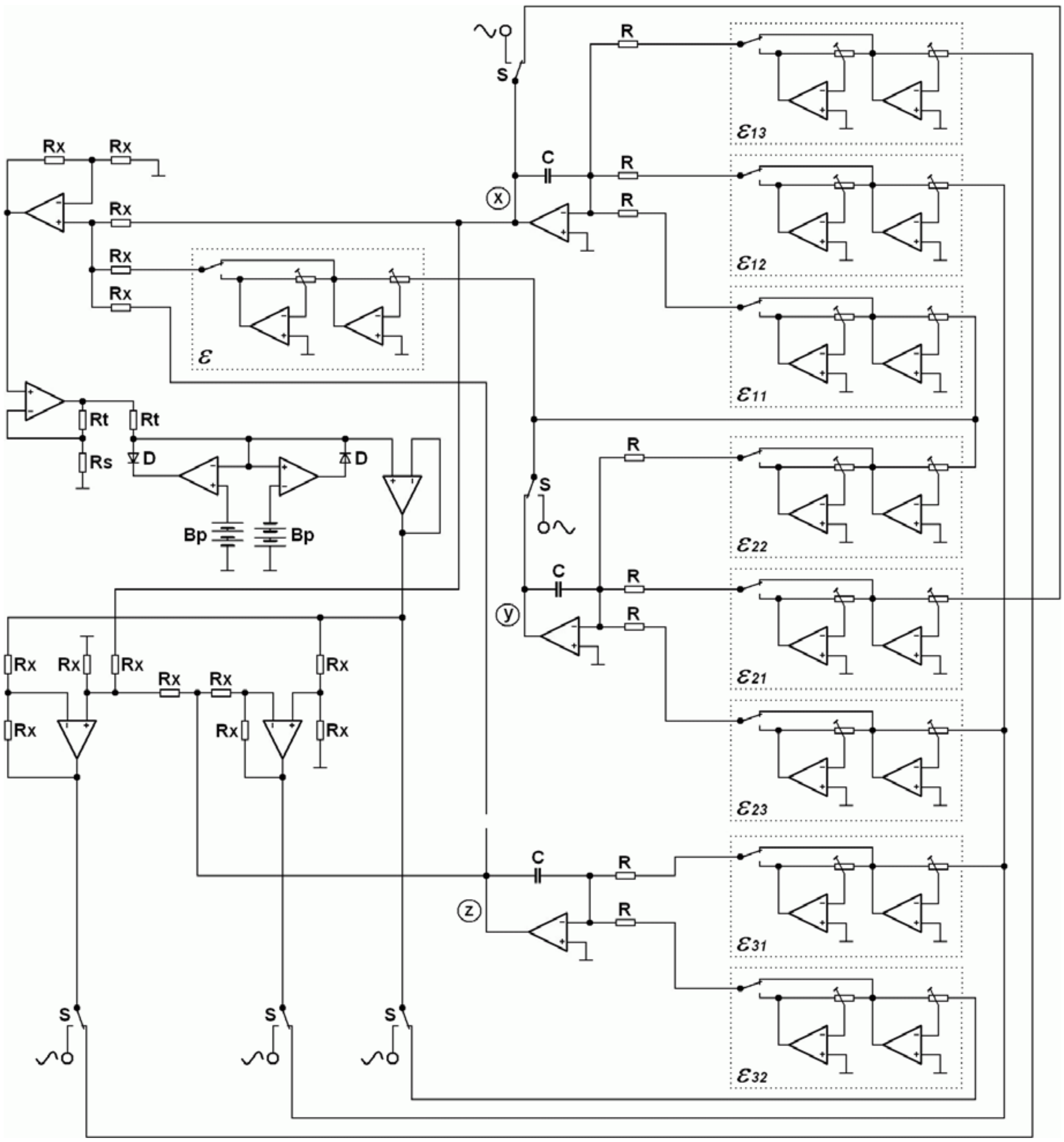

Fig.7: Circuitry implementation of the universal, fully analog chaotic oscillator.

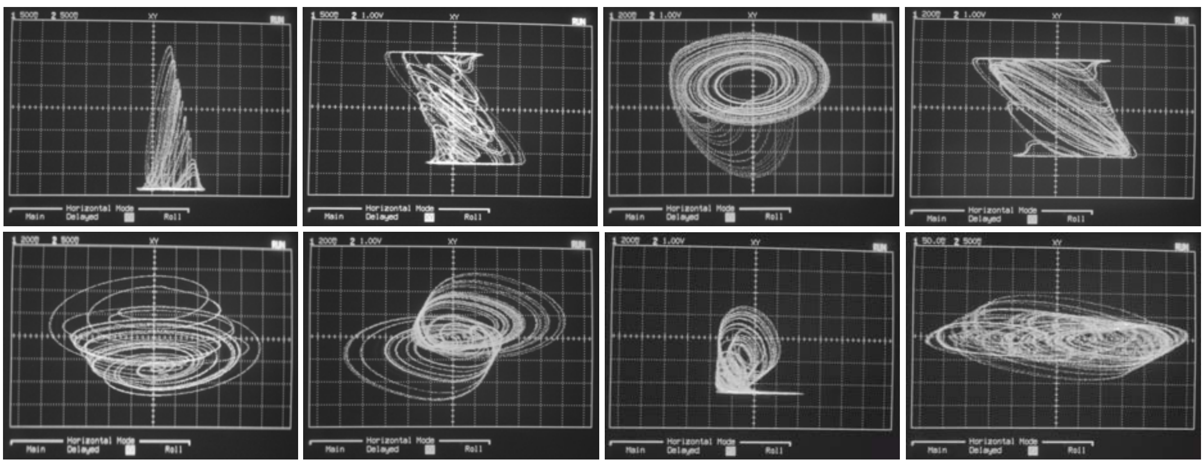

Fig.8: Laboratory experiments, selected chaotic steady states.

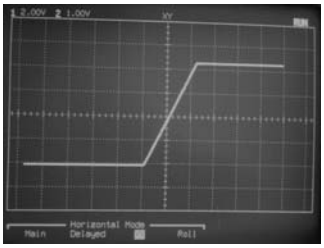

Fig.9: PWL transfer function.

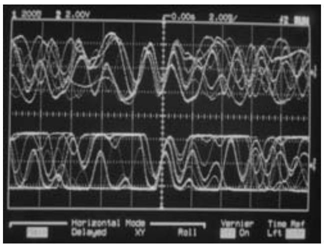

Fig.10: Chaotic waveforms in time domain.

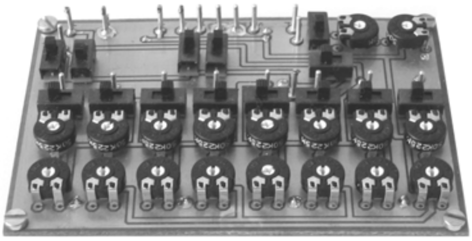

Fig.11: Universal chaos generator.

4 Conclusion

This paper shows a new method how to classify chaotic systems with respect to their sensitivities. We found the member of class C systems with the best performance, i.e. its chaotic double-scroll attractor is less sensitive to a changes of system parameters than any other existing model. If we can generalize this statement to any global attractor (chaotic or even non-chaotic) is the topic of our further study.

We describe the type of attractor by means of one-dimensional LE. The positive LE is a result not a reason why a dynamical system is chaotic. In spite of this, it suggests that a given system exhibits a sensitive dependance on initial conditions and bounded solution.

CDSM and ECSM are not the only dynamical system covered by our universal oscillator. It can also model some members of class F, which have three real distinct eigenvalues in the inner region of the state space and thus behaves like an overdamped circuit [7]. The general condition (9) was applied also on this system and the optimized system was found [8]. Oscillators based on the integrators can be easily upgraded to fourth dimension for the chance of hyperchaos [9], but at the cost of large amount of circuit elements.

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References:

[1] Ch. W. Wu, L. O. Chua, On Linear Topological Conjugacy of Lure Systems, *IEEE Trans on CAS I,* Vol.43, No.2, 1996, pp. 158-161.

[2] U. Feldmann, W. Schwarz, Linear Topological Conjugacy of n-Dimensional Piecewise-Linear Systems, *IEEE Trans. on CAS I*, Vol.41, No.2, 1994, pp. 190-192.

[3] L. M. Kocarev, T. D. Stojanovski, Linear Conjugacy of Vector Fields in Lure Form, *IEEE Trans. on CAS I*, Vol.43, No.9, 1996, pp. 782-785.

[4] Z. Kolka, Synthesis of Optimized Piecewise-Linear Systems Using Similarity Transformation, Part I: Basic Principles, *Radioengineering*, Vol.10, No.3, 2001, pp. 5-7.

[5] J. Pospíšil, Z. Kolka, J. Horská, Synthesis of Optimized Piecewise-Linear Systems Using Similarity Transformation, Part II: Second-Order Systems, *Radioengineering*, Vol.10, No.3, 2001, pp. 8-10.

[6] L. O. Chua, M. Komuro, T. Matsumoto, The Double-Scroll Family, *IEEE Trans. on CAS I*, Vol.33, No.11, 1986, pp. 1073-1117.

[7] Ch. P. Silva, L. O. Chua, The Overdamped Double-Scroll Family, Part I: Piecewise-Linear Geometry and Normal Form, *International Journal of Circuit Theory and Applications,* Vol.16, 1988, pp. 233-302.

[8] J. Pospíšil, Z. Kolka, S. Hanus, V. Michálek, J. Brzobohatý, Optimized State Model of Piecewise-Linear Dynamical Systems, *Radioengineering*, Vol.12, No.1, 2003, pp. 27-29.

[9] J. Pospíšil, Z. Kolka, J. Horská, J. Brzobohatý, Synthesis of Optimized Piecewise-Linear Systems Using Similarity Transformation, Part III: Higher-Order Systems, *Radioengineering*, Vol.11, No.2, 2002, pp. 11-13.

[10] J. Petržela, S. Hanus, Nonlinear Oscillator and Its Circuitry Implementation Using General Impedance Converter, In. *Proc. of the Electronic Devices and Systems IMAPS CS International Conference*, 2005, pp. 77-82.

[11] M. Itoh, Synthesis of Electronic Circuits for Simulating Nonlinear Dynamics, *International Journal of Bifurcation and Chaos*, Vol.11, No.3, 2001, pp. 605-655.

[12] J. Horská, Comparison of Sensitivity Properties of Selected Models of Dynamical Systems, *Radioengineering*, Vol.9, No.3, 2000, pp. 17-21.