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Korovkin Type Approximation Properties of Bivariate Bleimann, Butzer and Hahn Operators

ABDULLAH ALTIN[1], OGÜN DOĞRU[1] and M. ALİ ÖZARSLAN[2]

[1]Department of Mathematics
Ankara University, Faculty of Science
06100 Tandoğan, Ankara
TURKEY

[2]Eastern Mediterranean University
Faculty of Arts and Sciences
Department of Applied Mathematics and Computer Science
Gazimagosa, T.R.N.C.

http://science.ankara.edu.tr/~altin
http://science.ankara.edu.tr/~dogru

Abstract: - In the present paper, we introduce a bivariate generalization of Bleimann, Butzer and Hahn (BBH) operators. Korovkin type approximation properties and rate of convergence of these bivariate operators are established. In the last part, we obtain bounded variation properties of this generalization.

Key-Words: - Positive linear operators, bivariate Korovkin theorem, Bleimann, Butzer and Hahn operators, bivariate modulus of smoothness, bounded variation.

1 Introduction

There are many approximating linear positive operators in literature that their approximation properties are investigated.

In [6], Bleimann, Butzer and Hahn introduced an approximating operator:

$$L_n(f;x)=\frac{1}{(1+x)^n}\sum_{k=0}^{n} f\left(\frac{k}{n-k+1}\right)\binom{n}{k}x^k,\ x\geq 0\,. \quad (1.1)$$

The operators (1.1) are also called as Bleimann, Butzer and Hahn (BBH) operators.

In [6], pointwise convergence properties and rate of convergence of $L_n(f;x)$ to $f(x)$ for operators (1.1) are investigated in a compact sub-interval $[0,b]$ of $[0,\infty)$. Then many studies are given about BBH operators in the literature. Some of them, in [12], R.A. Khan obtained a rate of convergence for BBH operators using probabilistic methods. Also in [10], T. Herman investigated the behavior of the operators (1.1) when the growth condition for f is weaker than polynomial one. In [11], C. Jayasri and Y. Sitaraman, using the test functions $\left(\frac{t}{1+t}\right)^{\nu}, \nu=0,1,2,$ obtained direct and inverse result for the operators (1.1).

Some generalization of BBH operators are investigated by O. Agratini in [1], [2] and O. Doğru et. al in [7], [8].

Recently A.D. Gadjiev and Ö. Çakar [9] obtained a Korovkin type theorem and investigated the approximation properties of the BBH operators with the help of the test functions $\left(\frac{t}{1+t}\right)^{\nu}, \nu=0,1,2.$

First, D.D. Stancu [18] introduced the bivariate Bernstein polynomials and estimated order of approximation for these operators.

The main purpose of this study is to extend the operators (1.1) to the case of Stancu type operators in two variables and obtain Korovkin type approximation properties and rate of convergence of this extension.

In the last part, we shall give bounded variation properties of this bivariate operators.

2 Construction of Operators

The first aim of this part is to construct a bivariate extension of BBH operators by similar way given by D.D. Stancu in [18].

For $R_+^2=[0,\infty)\times[0,\infty)$ and $f:R_+^2\to R$, let us introduce bivariate extension of BBH operators as follows:

$$L_{n,m}(f;x,y)=\frac{1}{(1+x)^n}\frac{1}{(1+y)^m}\sum_{k=0}^{n}\sum_{s=0}^{m}f\left(\frac{k}{n-k+1},\frac{s}{m-s+1}\right)\times\binom{n}{k}\binom{m}{s}x^k y^s. \qquad (2.1)$$

Now, let us define spaces and norm using in this study.

Let $C_B(R_+^2)$ be the space of functions f which is continuous and bounded on R_+^2. Then $C_B(R_+^2)$ is a linear normed space with

$$\|f\|_{C_B(R_+^2)}=\sup_{x,y\ge 0}|f(x,y)|.$$

If we have

$$\lim_{n,m\to\infty}\|f_{n,m}-f\|_{C_B(R_+^2)}=0$$

then we say that the sequence $\{f_{n,m}\}$ converges uniformly to f.

As similarly in [9], let us introduce a space denoted by $H_\omega(R_+^2)$.

Let $H_\omega(R_+^2)$ be a subspace of real valued functions satisfying the following condition:

$$|f(t,s)-f(x,y)|\le\omega\left(\left|(\frac{t}{1+t},\frac{s}{1+s})-(\frac{x}{1+x},\frac{y}{1+y})\right|\right) \qquad (2.2)$$

where

$$\left|(\frac{t}{1+t},\frac{s}{1+s})-(\frac{x}{1+x},\frac{y}{1+y})\right|=\sqrt{\left(\frac{t}{1+t}-\frac{x}{1+x}\right)^2-\left(\frac{s}{1+s}-\frac{y}{1+y}\right)^2}$$

and $\omega(\delta)$ is a modulus of continuity type functions so that the following conditions are satisfied:

(i) $\omega(\delta)$ is non-negative and increasing for δ,

(ii) $\lim_{\delta\to 0}\omega(\delta)=0$.

Due to (ii), we can say that $H_\omega(R_+^2)\subset C_B(R_+^2)$.

The space $H_\omega(R_+)$ was introduced by A.D. Gadjiev and Ö. Çakar [9] for the functions with one variable.

Also in [9], the following Korovkin type theorem is proved:

Theorem 2.1. [9] Let A_n be the sequence of linear positive operators, acting from $H_\omega(R_+)$ to $C_B[0,\infty)$ satisfying three conditions

$$\lim_{n\to\infty}\left\|A_n\left(\left(\frac{t}{1+t}\right)^\nu;x\right)-\left(\frac{x}{1+x}\right)^\nu\right\|_{C_B}=0,\ \nu=0,1,2\ . \qquad (2.3)$$

Then for any function $f\in H_\omega(R_+)$, we have

$$\lim_{n\to\infty}\|A_nf-f\|_{C_B}=0.$$

Gadjiev and Çakar, in [9], applied this theorem to classical BBH operator. In their application they have obtained the following equality for BBH operator:

$$L_n(1;x)=1,$$
$$L_n(\frac{t}{1+t};x)=\frac{n}{n+1}\frac{x}{1+x} \qquad (2.4)$$
$$L_n(\left(\frac{t}{1+t}\right)^2;x)=\frac{n(n-1)}{(n+1)^2}\left(\frac{x}{1+x}\right)^2+\frac{n}{(n+1)^2}\frac{x}{1+x}$$

Notice that, details about Korovkin type theorems can be found in [13] (see also [3]).

Now, we recall the following Korovkin type theorem given by Volkov [20] for the functions with two variables.

Let n and m be two positive integers, $0\le\alpha_{k,n},\beta_{j,m}\le 1$, $P_{k,j}^{(n,m)}(x,y)\ge 0$, then

$$T_{n,m}(f;x,y)=\sum_{k=0}^{n}\sum_{j=0}^{m}f(\alpha_{k,n},\beta_{j,m})P_{k,j}^{(n,m)}(x,y) \qquad (2.5)$$

is a sequence of positive linear operators.

Theorem 2.2. [20] Suppose that the operators $T_{n,m}$ defined in (2.5) satisfy the following four conditions:

(i) $\lim_{n,m\to\infty}\|T_{n,m}(e_{00})-e_{00}\|_{C([0,1]^2)}=0,$

(ii) $\lim_{n,m\to\infty}\|T_{n,m}(e_{10})-e_{10}\|_{C([0,1]^2)}=0,$

(iii) $\lim_{n,m\to\infty}\|T_{n,m}(e_{01})-e_{01}\|_{C([0,1]^2)}=0,$

(iv) $\lim_{n,m\to\infty}\|T_{n,m}(e_{20}+e_{02})-(e_{20}+e_{02})\|_{C([0,1]^2)}=0.$

Then the sequence of operators (2.5) converges uniformly to f in $[0,1]^2$ for $f\in C([0,1]^2)$. Where e_{ij} are defined as monomials $e_{ij}:(x,y)\to x^iy^j$.

Similarly, let us give the following Korovkin type theorem:

Theorem 2.3. Suppose that $A_{n,m}$ is the positive linear operators acting from $H_\omega(R_+^2)$ to $C_B(R_+^2)$ satisfying the conditions:

(i) $\lim_{n,m\to\infty}\left\|A_{n,m}(\tilde{e}_{00})-\tilde{e}_{00}\right\|_{C(R_+^2)}=0,$

(ii) $\lim_{n,m\to\infty}\left\|A_{n,m}(\tilde{e}_{10})-\tilde{e}_{10}\right\|_{C_B(R_+^2)}=0,$

(iii) $\lim_{n,m\to\infty}\left\|A_{n,m}(\tilde{e}_{01})-\tilde{e}_{01}\right\|_{C_B(R_+^2)}=0,$

(iv) $\lim_{n,m\to\infty}\left\|A_{n,m}(\tilde{e}_{10}+\tilde{e}_{01})-(\tilde{e}_{10}+\tilde{e}_{01})\right\|_{C_B(R_+^2)}=0.$

Then the sequence of positive linear operators $A_{n,m}$ converges uniformly to f in R_+^2 for $f\in H_\omega(R_+^2)$. Where $\tilde{e}_{ij}$ are two dimensional test functions defined as

$$\tilde{e}_{ij}:R_+^2\to R_+,\ \tilde{e}_{ij}=\left(\frac{x}{1+x}\right)^i\left(\frac{y}{1+y}\right)^j.$$

Proof. If $f\in H_\omega(R_+^2)$, then, from (2.2), we have for any $\varepsilon>0$ there exists a number δ such that $|f(t,s)-f(x,y)|<\varepsilon$ if $\sqrt{\left(\frac{t}{1+t}-\frac{x}{1+x}\right)^2+\left(\frac{s}{1+s}-\frac{y}{1+y}\right)^2}<\delta$

Also, since f is bounded, there exists a positive constant M such that

$$|f(t,s)-f(x,y)|<\frac{2M}{\delta^2}\left[\left(\frac{t}{1+t}-\frac{x}{1+x}\right)^2+\left(\frac{s}{1+s}-\frac{y}{1+y}\right)^2\right]$$

if $\sqrt{\left(\frac{t}{1+t}-\frac{x}{1+x}\right)^2+\left(\frac{s}{1+s}-\frac{y}{1+y}\right)^2}\geq\delta.$ Therefore for all $(t,s),(x,y)\in R_+^2$ one can write

$$|f(t,s)-f(x,y)|<\varepsilon+\frac{2M}{\delta^2}\left[\left(\frac{t-x}{(1+t)(1+x)}\right)^2+\left(\frac{s-y}{(1+s)(1+y)}\right)^2\right] \tag{2.5}$$

By applying the positive linear operator A_n to the inequality (2.5), we have

$$\begin{aligned}\|A_n(f)-f\|_{C_B(R_+^2)}&\leq(\varepsilon+M)\|A_n(\tilde{e}_{00})-\tilde{e}_{00}\|_{C_B(R_+^2)}+\varepsilon\\&+\frac{2M}{\delta^2}\Big[\|A_n(\tilde{e}_{20}+\tilde{e}_{02})-(\tilde{e}_{20}+\tilde{e}_{02})\|_{C_B(R_+^2)} \qquad (2.6)\\&+2\|A_n(\tilde{e}_{10})-\tilde{e}_{10}\|_{C_B(R_+^2)}+2\|A_n(\tilde{e}_{01})-\tilde{e}_{0,1}\|_{C_B(R_+^2)}\Big]\end{aligned}$$

By using the equalities (2.4) in (2.6), the proof is completed.

To obtain Korovkin type approximation properties of our operators in (2.1), we need the following lemmas.

Lemma 2.4. Let $L_{n,m}$ be the operators defined in (2.1), then we have

$$L_{n,m}(f;x,y)=L_n^x(L_m^y(f;x,y))$$

and

$$L_{n,m}(f;x,y)=L_m^y(L_n^x(f;x,y))$$

where

$$L_n^x(f;x,y)=\frac{1}{(1+x)^n}\sum_{k=0}^{n}f\left(\frac{k}{n-k+1},y\right)\binom{n}{k}x^k,$$

$$L_m^y(f;x,y)=\frac{1}{(1+x)^m}\sum_{s=0}^{m}f\left(x,\frac{s}{m-s+1}\right)\binom{m}{s}y^s.$$

Proof. This proof can be easily proven using the similar technique given by D. Barbosu [5], so we will omit it.

Lemma 2.5. We have the following for operators (2.1):

(i) $L_{n,m}(\tilde{e}_{00};x,y)=1,$

(ii) $L_{n,m}(\tilde{e}_{10};x,y)=\frac{n}{n+1}\frac{x}{1+x},$

(iii) $L_{n,m}(\tilde{e}_{01};x,y)=\frac{m}{m+1}\frac{y}{1+y},$

(iv) $$L_{n,m}(\tilde{e}_{20}+\tilde{e}_{02};x,y)=\frac{n(n-1)}{(n+1)^2}\left(\frac{x}{1+x}\right)^2+\frac{n}{(n+1)^2}\frac{x}{1+x}+\frac{m(m-1)}{(m+1)^2}\left(\frac{y}{1+y}\right)^2+\frac{m}{(m+1)^2}\frac{y}{1+y}.$$

Proof. By using the Lemma 2.4 and equalities (2.4), (i-iv) can be easily shown. We will omit this proof too.

In the light of Theorem 2.3 and Lemmas 2.4 and 2.5, we can give our main result as follows:

Theorem 2.6. The sequence of operators (2.1) converges uniformly to f for any $f\in H_\omega(R_+^2)$.

Proof. Due to Lemma 2.5, all hyphotesis of Theorem 2.3 are satisfied for the operators (2.1) which gives the proof.

3 Rates of Convergence

Let $f\in H_\omega(R_+^2)$, then we introduce the following modulus

$$\tilde{\omega}(f;\delta):=\sup_{t,x\geq0,\ \left|\frac{t}{1+t}-\frac{x}{1+x}\right|\leq\delta}|f(t)-f(x)|. \tag{3.1}$$

It is clear that, similarly to the classical modulus of continuity, $\tilde{\omega}(f;\delta)$ satisfies the following properties:

(i) $\lim_{\delta\to0}\tilde{\omega}(f;\delta)=0,$

(ii) $|f(t)-f(x)|\leq\tilde{\omega}(f;\delta)\left(\frac{\left|\frac{t}{1+t}-\frac{x}{1+x}\right|}{\delta}+1\right).$

Theorem 3.1. Let L_n be the sequence of operators in (1.1). Then we have

$$|L_n(f;x)-f(x)|\leq2\tilde{\omega}(f;\delta_n(x)) \tag{3.2}$$

for all $x \geq 0$ where

$$\delta_n(x) = \sqrt{\frac{1-n}{(n+1)^2}\left(\frac{x}{1+x}\right)^2 + \frac{n}{(n+1)^2}\frac{x}{1+x}}\,.$$

Proof. We will use Popoviciu's technique [17].
By linearity, monotonicity of L_n and (ii), we can write

$$\begin{aligned} \left|L_n(f;x) - f(x)\right| &\leq L_n(\tilde{\omega}(f;\left|\tfrac{t}{1+t} - \tfrac{x}{1+x}\right|);x) \\ &\leq \tilde{\omega}(f;\delta_n(x))L_n\left(1 + \frac{\left|\frac{t}{1+t} - \frac{x}{1+x}\right|}{\delta_n};x\right). \end{aligned} \quad (3.3)$$

By using the Cauchy-Schwarz inequality in (3.3), we have

$$\left|L_n(f;x) - f(x)\right| \leq \tilde{\omega}(f;\delta_n(x))\left(1 + \tfrac{1}{\delta_n(x)}\sqrt{\mu_{n,2}(x)}\right), \quad (3.4)$$

where

$$\mu_{n,2}(x) = L_n\left(\left(\frac{t}{1+t} - \frac{x}{1+x}\right)^2;x\right).$$

By taking $q = 1$ in Lemma 2.2 of [8], we get

$$\mu_{n,2}(x) = \frac{1-n}{(n+1)^2}\left(\frac{x}{1+x}\right)^2 + \frac{n}{(n+1)^2}\frac{x}{1+x}\,.$$

By choosing $\delta_n = \sqrt{\mu_{n,2}(x)}$ in (3.4), we have (3.2) immediately.

Now, let us introduce the following modulus of smoothness for bivariate case similarly in [15] (see, for details, [4, Sec. 2.3]):

$$\tilde{\omega}(f;\alpha,\beta) = \sup\left\{\left|f(t,s) - f(x,y)\right| : \begin{array}{c}(t,s) \in R_+^2, (x,y) \in R_+^2 \\ \left|\frac{t}{1+t} - \frac{x}{1+x}\right| \leq \alpha,\ \left|\frac{s}{1+s} - \frac{y}{1+y}\right| \leq \beta\end{array}\right\}$$

It is clear that if $f \in H_\omega(R_+^2)$ then we have

$$\tilde{\omega}(f;\alpha,\beta) \to 0 \text{ for } \alpha \to 0 \text{ and } \beta \to 0 \quad (3.5)$$

Also one can write

$$\left|f(t,s) - f(x,y)\right| \leq \tilde{\omega}(f;\alpha,\beta)\left(1 + \frac{\left|\frac{t}{1+t} - \frac{x}{1+x}\right|}{\alpha}\right)\left(1 + \frac{\left|\frac{s}{1+s} - \frac{y}{1+y}\right|}{\beta}\right) \quad (3.6)$$

Theorem 3.2. Let $L_{n,m}$ be the sequence of operators in (2.1).Then we have, for all $(x,y) \in R_+^2$, we obtain

$$\left|L_{n,m}(f;x,y) - f(x,y)\right| \leq 4\,\tilde{\omega}(f;\alpha_n(x),\beta_m(y)) \quad (3.7)$$

where

$$\alpha_n(x) = \delta_n(x) \text{ and } \beta_m(y) = \delta_m(y)$$

and $\delta_n(x)$ is similarly as in Theorem 3.1.

Proof. Using the Cauchy-Schwarz inequality in (3.6), the proof follows by the Theorem 3.1.

Remark 3.3. According to the (3.5), if $f \in H_\omega(R_+^2)$ then (3.7) gives the pointwise rate of convergence of $L_{n,m}(f;x,y)$ to $f(x,y)$.

Remark 3.4. We mention that, a similar result given in Theorem 3.2 can be obtained by means of the bivariate modulus of smoothness introduced by Martinez [16].

4 Derivative Properties

First explicit formula for derivatives of Bernstein polynomials with difference operator was given by G.G. Lorentz [14]. In [14], the author also obtained an estimate between the variation of the Bernstein polynomials $B_n(f;x)$ and total variation of the function *f*. Then in [19], D.D. Stancu obtained the monotonicity properties of Bernstein polynomials in different orders by means of divided differences. Recently, in [7], O. Doğru obtained a formula for the variation of BBH type generalization of Balász operators by means of total variation of function *f*.

In this part, for $(x,y) \in R_+^2$, let us introduce variation of any bivariate operator $A_{n.m}(f;x,y)$ as follows:

$$\operatorname{var} A_{n,m}(f;x,y) = \int_0^\infty\int_0^\infty \left|\frac{\partial^2}{\partial x \partial y}A_{n,m}(f;x,y)\right| dxdy$$

Also let us introduce total variation of bivariate function as follows:

$$\operatorname{var} f(x,y) = \sum_{k=0}^{n-1}\sum_{s=0}^{m-1}\left|\Delta f(a_{n,k}, b_{m,s})\right|$$

where

$$\Delta f(a_{n,k}, b_{m,s}) = f(a_{n,k+1}, b_{m,s+1}) - f(a_{n,k}, b_{m,s})\,.$$

Theorem 4.1. For the operators (2.1), the inequality

$$\operatorname{var} L_{n,m}(f;x,y) \leq \operatorname{var} f(x,y)\,. \quad (4.1)$$

Proof. If we use Lemma 2.4, we obtain

$$\begin{aligned} \frac{\partial^2}{\partial x \partial y}L_{n,m}(f;x,y) &= \frac{\partial}{\partial x}\left(L_n^x\left(\frac{\partial}{\partial y}L_m^y(f;x,y)\right)\right) \\ &= \frac{nm}{(1+x)^{n+1}(1+y)^{m+1}}\sum_{k=0}^{n-1}\sum_{s=0}^{m-1}\left(f\left(\tfrac{k+1}{n-k},\tfrac{s+1}{m-s}\right) - f\left(\tfrac{k}{n-k+1},\tfrac{s}{m-s+1}\right)\right) \\ &\quad \times \binom{n-1}{k}\binom{m-1}{s}x^k y^s \end{aligned}$$

Thus we have

$$\begin{aligned} \operatorname{var} L_{n,m}(f;x,y) &\leq \sum_{k=0}^{n-1}\sum_{s=0}^{m-1} nm\left(f\left(\tfrac{k+1}{n-k},\tfrac{s+1}{m-s}\right) - f\left(\tfrac{k}{n-k+1},\tfrac{s}{m-s+1}\right)\right) \\ &\quad \times \binom{n-1}{k}\binom{m-1}{s}\int_0^\infty\int_0^\infty (1+x)^{-n-1}x^k(1+y)^{-m-1}y^s dxdy\,. \end{aligned} \quad (4.2)$$

On the other hand, it is obvious that

$$\int_0^\infty (1+x)^{-n-1} x^k dx = \frac{\Gamma(k+1)\Gamma(n-k)}{\Gamma(n+1)} = \frac{k!(n-k-1)!}{n!} \tag{4.3}$$

and

$$\int_0^\infty (1+x)^{-n-1} x^k dx = \frac{\Gamma(k+1)\Gamma(n-k)}{\Gamma(n+1)} = \frac{k!(n-k-1)!}{n!} \tag{4.4}$$

are satisfied. By using (4.3) and (4.4) in (4.2), we obtain desired result.

Remark 4.2. Notice that, all results given in this paper can be extended to cases of *n*-variate functions.

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