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# Extremes of quasi-independent random fields and clustering of high values

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*Abstract*: - Random fields on $\mathbb{Z}_+^d$, with long range weak dependence for each coordinate at a time and local restrictions on clustering of high values, behave like an i.i.d. random field. Then, for these random fields, the probability of no exceedances of high values can be approximated by $exp(-\tau)$, where $\tau$ is the limiting mean number of exceedances. An example is a nonstationary Gaussian field under a Berman's type condition on their correlations.
Random fields usually present clustering of high values. Under smooth oscillation conditions, we compute the clustering measure extremal index, from the limiting mean number of crossings of high levels.

*Key words:* random field, dependence, non-stationarity, Gaussian random field, extremal index

## 1 Introduction

Let $\mathbf{X} = \{X_\mathbf{n}\}_{\mathbf{n}\geq\mathbf{1}}$ be a random field on $\mathbb{Z}_+^d$, where $\mathbb{Z}_+$ is the set of all positive integers and $d \geq 2$. We shall consider the conditions and results for $d = 2$ since it is notationally simplest and the proofs for higher dimensions follow analogous arguments.

For a family of real levels $\{u_{\mathbf{n},\mathbf{i}} : \mathbf{i} \leq \mathbf{n}\}_{\mathbf{n}\geq\mathbf{1}}$ and a subset $\mathbf{I}$ of the rectangle of points $\mathbf{R_n} = \{1,\ldots,n_1\} \times \{1,\ldots,n_2\}$, we will denote the event $\{\bigcap_{\mathbf{i}\in\mathbf{I}} X_\mathbf{i} \leq u_{\mathbf{n},\mathbf{i}}\}$ by $\{M_\mathbf{n}(\mathbf{I}) \leq u\}$ or simply by $\{M_\mathbf{n} \leq u\}$ when $\mathbf{I} = \mathbf{R_n}$.

For each $i = 1, 2$, we say the pair $\mathbf{I}$ and $\mathbf{J}$ is in $\mathcal{S}_i(l)$ if the distance between $\Pi_i(\mathbf{I})$ and $\Pi_i(\mathbf{J})$ is great or equal to $l$, where $\Pi_i$, $i = 1, 2$ denote the cartesian projections. The distance $d(\mathbf{I},\mathbf{J})$ between sets $\mathbf{I}$ and $\mathbf{J}$ of $\mathbb{Z}_+^d$, $d \geq 1$, is the minimum of distances $d(\mathbf{i},\mathbf{j}) = max\{|i_s - j_s|, s = 1,\ldots,d\}$, $\mathbf{i} \in \mathbf{I}$ and $\mathbf{j} \in \mathbf{J}$.

Suppose that $\mathbf{X}$ satisfies a coordinatewise-mixing type condition as the $\Delta(u_\mathbf{n})$-condition introduced in [9], which exploits the past and future separation one coordinate at a time. Let $\mathcal{F}$ be a family of indexes sets in $\mathbf{R_n}$. We shall assume that there exist sequences of integer valued constants $\{k_{n_i}\}$, $\{l_{n_i}\}$, $i = 1, 2$ such that, as $\mathbf{n} = (n_1, n_2) \to \infty$, we have

$$(k_{n_1}, k_{n_2}) \to \infty,\ (\frac{k_{n_1} l_{n_1}}{n_1}, \frac{k_{n_2} l_{n_2}}{n_2}) \to \mathbf{0} \qquad (1)$$

and $(k_{n_1}\Delta_1, k_{n_1}k_{n_2}\Delta_2) \to \mathbf{0}$, where $\Delta_i$ are the components of the mixing coefficient defined as follows:

$$\Delta_1 = sup|P(M_\mathbf{n}(\mathbf{I}_1) \leq u, M_\mathbf{n}(\mathbf{I}_2) \leq u) -$$

$$P(M_{\mathbf{n}}(\mathbf{I}_1) \leq u)P(M_{\mathbf{n}}(\mathbf{I}_2) \leq u)|,$$

where the supremum is taken over pairs $\mathbf{I}_1$ and $\mathbf{I}_2$ in $\mathcal{S}_1(l_{n_1}) \cap \mathcal{F}$,

$$\Delta_2 = sup|P(M_{\mathbf{n}}(\mathbf{I}_1) \leq u, M_{\mathbf{n}}(\mathbf{I}_2) \leq u)-$$

$$P(M_{\mathbf{n}}(\mathbf{I}_1) \leq u)P(M_{\mathbf{n}}(\mathbf{I}_2) \leq u)|,$$

where the supremum is taken over pairs $\mathbf{I}_1$ and $\mathbf{I}_2$ in $\mathcal{S}_2(l_{n_2}) \cap \mathcal{F}$. We say then $\mathbf{X}$ satisfies the $D(u_{\mathbf{n},\mathbf{i}})$ condition over $\mathcal{F}$.

In fact, we could consider a slightly weaker condition, as in [9], if we where concerned only with stationary random fields.

We prove, in the next section, that the maxima over disjoint rectangles behave asymptotically as independent maxima. In section 3, we introduce a local dependence condition that avoids clustering of exceedances of $u_{\mathbf{n},\mathbf{i}}$. That condition and the coordinatewise long range dependence lead to a Poisson approximation for the probability of no exceedances over $\mathbf{R_n}$. The results are applied to a nonstationary Gaussian random field. In the last section we discuss the behaviour of the maxima when clustering of high values of $\mathbf{X}$ is allowed but we restrict the local occurence of two or more crossings of the high levels $u_{\mathbf{n},\mathbf{i}}$. The smooth oscillation condition considered enables to compute a clustering measure, called extremal index, from the limiting mean number of crossings. We illustrate these results with a 1-dependent random field.

# 2 Asymptotic independence of maxima

Under the coordinatewise-mixing $D(u_{\mathbf{n},\mathbf{i}})$-condition we have the asymptotic independence for maxima over disjoint rectangles of indexes. In the following $\overline{F}_{max}$ denotes $max\{P(X_{\mathbf{i}} > u_{\mathbf{n},\mathbf{i}}) : \mathbf{i} \leq \mathbf{n}\}$.

**Proposition 2.1** *Suppose that the random field $\mathbf{X}$ satisfies the condition $D(u_{\mathbf{n},\mathbf{i}})$ over $\mathcal{F}$ such that $(\mathbf{I} \subset \mathbf{J} \wedge \mathbf{J} \in \mathcal{F}) \Rightarrow \mathbf{J} \in \mathcal{F}$ and for $\{u_{\mathbf{n},\mathbf{i}} : \mathbf{i} \leq \mathbf{n}\}_{\mathbf{n}\geq \mathbf{1}}$ such that*

$$\{n_1 n_2 \overline{F}_{max}\}_{\mathbf{n}\geq\mathbf{1}} \text{ is bounded.} \qquad (2)$$

*If $\mathbf{V}_{r,p} = I_r \times J_{r,p}$, $r = 1, \ldots, k_{n_1}$, $p = 1, \ldots, k_{n_2}$, are disjoint rectangles in $\mathcal{F}$, then, as $\mathbf{n} \to \infty$,*

$$P\Big(\bigcap_{r,p} M(\mathbf{V}_{r,p}) \leq u\Big) - \prod_{r,p} P(M(\mathbf{V}_{r,p}) \leq u) \to 0.$$

**Proof:** From (1) and (2), for the purpose of the above convergence we can assume that $\Pi_1(\mathbf{V}_{r,p}) > l_{n_1}$ or $\Pi_2(\mathbf{V}_{r,p}) > l_{n_2}$. If all the pairs of rectangles $\mathbf{V}_{r,p}$ are in $\mathcal{S}_1(l_{n_1}) \cup \mathcal{S}_2(l_{n_2})$ then the result follows inductively from the condition $D(u_{\mathbf{n},\mathbf{i}})$. On the contrary, we can eliminate $l_{n_1}$ columns or $l_{n_2}$ rows of indexes in $\mathbf{V}_{r,p}$ in order to obtain $\mathbf{V}^*_{r,p} \subset \mathbf{V}_{r,p}$, $r = 1, \ldots, k_{n_1}, p = 1, \ldots, k_{n_2}$, to which we can apply inductively the condition $D(u_{\mathbf{n},\mathbf{i}})$. □

# 3 Restriction on clustering of high values

Restrictions on clustering of high values for stationary and non-stationary time series have been considered in the form of $D'$ condition introduced in [6] ( see also [4]). We shall here introduce a $D'$ condition tailored for random fields not necessarily stationary.

Let $\mathcal{E}(u_{\mathbf{n},\mathbf{i}})$ denote the family of indexes sets $\mathbf{I}$ such that

$$\sum_{\mathbf{i}\in\mathbf{I}} P(X_{\mathbf{i}} > u_{\mathbf{n},\mathbf{i}}) \leq \frac{1}{k_{n_1} k_{n_2}} \sum_{\mathbf{i}\leq\mathbf{n}} P(X_{\mathbf{i}} > u_{\mathbf{n},\mathbf{i}}).$$

**Definition 3.1.** *The condition $D'(u_{\mathbf{n},\mathbf{i}})$ holds for $\mathbf{X}$ if for each $\mathbf{I} \in \mathcal{E}(u_{\mathbf{n},\mathbf{i}})$ we have, as $\mathbf{n} \to \infty$,*

$$k_{n_1} k_{n_2} \sum_{\mathbf{i},\mathbf{j}\in\mathbf{I}} P(X_{\mathbf{i}} > u_{\mathbf{n},\mathbf{i}}, X_{\mathbf{j}} > u_{\mathbf{n},\mathbf{j}}) \to 0.$$

As for i.i.d. random fields, under $D'(u_{\mathbf{n},\mathbf{i}})$ and $D(u_{\mathbf{n},\mathbf{i}})$ over $\mathcal{E}(u_{\mathbf{n},\mathbf{i}})$, for $\mathbf{n}$ large, $P(\bigcap_{\mathbf{i}\leq\mathbf{n}} X_{\mathbf{i}} \leq u_{\mathbf{ni}})$ can be approximated by the limiting mean number of exceedances over $\mathbf{R_n}$.

**Proposition 3.1** *Suppose* $\mathbf{X}$ *satisfies (2),* $D'(u_{\mathbf{n},\mathbf{i}})$ *and* $D(u_{\mathbf{n},\mathbf{i}})$ *over* $\mathcal{E}(u_{\mathbf{n},\mathbf{i}})$. *Then, as* $\mathbf{n} \to \infty$, *it holds*

$$P(\bigcap_{\mathbf{i}\leq\mathbf{n}} X_{\mathbf{i}} \leq u_{\mathbf{ni}}) \to e^{-\tau},\ \tau > 0,$$

*if and only if*

$$\sum_{\mathbf{i}\leq\mathbf{n}} P(X_{\mathbf{i}} > u_{\mathbf{n},\mathbf{i}}) \to \tau > 0.$$

**Proof:** We will built $k_{n_1}k_{n_2}$ rectangles in $\mathcal{E}(u_{\mathbf{n},\mathbf{i}})$ as follows. First split $\mathbf{R_n}$ in $k_{n_1}$ quasi-rectangles $\mathbf{I}'_r = \{s_{r-1}+1\} \times \{t^*_{r-1}+1,\dots,n_2\} \cup \{s_{r-1}+2,\dots,s_r\} \times \{1,\dots,n_2\} \cup \{s_r+1\} \times \{1,\dots,t^*_r \leq n_2\}$, $r = 0,\dots,k_{n_1}$, $s_0 = 0 = t^*_0$, with $t^*_r$ maximally choosen such that

$$\sum_{\mathbf{i}\in\mathbf{I}'_r} P(X_{\mathbf{i}} > u_{\mathbf{n},\mathbf{i}}) \leq \frac{1}{k_{n_1}} \sum_{\mathbf{i}\leq\mathbf{n}} P(X_{\mathbf{i}} > u_{\mathbf{n},\mathbf{i}}).$$

Let $\mathbf{I}_r = \{s_{r-1}+2,\dots,s_r\} \times \{1,\dots,n_2\}$ and now we split each rectangle $\mathbf{I}_r$ in $k_{n_1}k_{n_2}$ quasi-rectangles $\mathbf{V}'_{r,p} = \{s^*_{r,p-1}+1,\dots,s_r\} \times \{t_{p-1}+1\} \cup \{s_{r-1}+1,\dots,s_r\} \times \{t_{p-1}+2,\dots,t_p\} \cup \{s_{r-1}+1,\dots,s^*_{r,p} \leq s_r\} \times \{t_p+1\}$, $p = 1,\dots,k_{n_2}$, $t_0 = 0$, $s^*_{r,0} = s_{r-1}$, with $s^*_{r,p}$ maximally choosen such that

$$\sum_{\mathbf{i}\in\mathbf{V}'_{r,p}} P(X_{\mathbf{i}} > u_{\mathbf{n},\mathbf{i}}) \leq \frac{1}{k_{n_1}k_{n_2}} \sum_{\mathbf{i}\leq\mathbf{n}} P(X_{\mathbf{i}} > u_{\mathbf{n},\mathbf{i}}).$$

Let $\mathbf{V}_{r,p} = \{s_{r-1}+1,\dots,s_r\} \times \{t_{p-1}+2,\dots,t_p\}$. Then, by (1) and (2), it is sufficient to prove that

$$P(\bigcap_{r,p} M(\mathbf{V}_{r,p}) \leq u) \to e^{-\tau},\ \tau > 0,$$

if and only if

$$\sum_{r,p}\sum_{\mathbf{i}\in\mathbf{V}_{r,p}} P(X_{\mathbf{i}} > u_{\mathbf{n},\mathbf{i}}) \to \tau > 0.$$

This follows from Proposition 2.1, condition $D'(u_{\mathbf{n},\mathbf{i}})$ and the following relations:

$$\prod_{r,p} P(M(\mathbf{V}_{r,p}) \leq u) =$$

$$exp\Big(-(1+o(1))\sum_{r,p}(1 - P(M(\mathbf{V}_{r,p}) \leq u))\Big) =$$

$$exp\Big(-(1+o(1))\sum_{r,p}\sum_{\mathbf{i}\in\mathbf{V}_{r,p}} P(X_{\mathbf{i}} > u_{\mathbf{n},\mathbf{i}}) + o(1)\Big).$$

□

If $\mathbf{X}$ is stationary, in the above proof we can consider $k_{n_1}k_{n_2}$ rectangles of $[\frac{n_1}{k_{n_1}}][\frac{n_2}{k_{n_2}}]$ indexes and the result follows by assuming $u_{\mathbf{n},\mathbf{i}} = u_{\mathbf{n}}$, $\mathbf{i} \leq \mathbf{n}$, and condition $D'(u_{\mathbf{n},\mathbf{i}})$ as

$$n_1 n_2 \sum_{\mathbf{j}\leq([\frac{n_1}{k_{n_1}}],[\frac{n_2}{k_{n_2}}])} P(X_{\mathbf{1}} > u_{\mathbf{n}}, X_{\mathbf{j}} > u_{\mathbf{n}}) \to 0,$$

as $\mathbf{n} \to \infty$.

Let $\mathbf{X}$ be a Gaussian random field with zero means, unit variances and correlations $r_{\mathbf{i},\mathbf{j}}$, $\mathbf{i} \geq \mathbf{1}$, $\mathbf{j} \geq \mathbf{1}$. We will assume that

$$|r_{\mathbf{i},\mathbf{j}}| \leq \rho_{(|i_1-j_1|,|i_2-j_2|)} \tag{3}$$

for some $\{\rho_{\mathbf{n}} < 1\}_{\mathbf{n}\geq\mathbf{1}}$ satisfying

$$\rho_{\mathbf{n}} log(n_1 n_2) \to 0,\ as\ \mathbf{n} \to \infty. \tag{4}$$

This is a generalization of the condition $r_{\mathbf{n}} log(n_1 n_2) \to 0$ considered in [1] for stationary Gaussian random fields.

We present a class of Gaussian random fields for which the above proposition can be applied and then, from $\sum_{\mathbf{i}\leq\mathbf{n}}(1 - \Phi(u_{\mathbf{n},\mathbf{i}})) \to \tau > 0$, where $\Phi$ denotes the distribution function of standard Normal distribution, we get the convergence of $P(M_{\mathbf{n}} \leq u)$. The result is a generalization of Theorem 6.1.3 in [7] and for details of the proof see [10].

**Proposition 3.2** *Let* $\mathbf{X}$ *be a Gaussian random field such that (3) and (4) hold and* $\{u_{\mathbf{n},\mathbf{i}} : \mathbf{i} \leq \mathbf{n}\}_{\mathbf{n}\geq\mathbf{1}}$ *satisfying (2) and* $\lambda_{\mathbf{n}} = min\{u_{\mathbf{n},\mathbf{i}} : \mathbf{i} \leq \mathbf{n}\} \geq c(log(n_1 n_2))^{1/2}$ *for some constant* $c > 0$. *If for each* $\mathbf{I} \in \mathcal{E}(u_{\mathbf{n},\mathbf{i}})$,

$$\{\frac{(k_{n_1}k_{n_2})^{1/2}}{n_1 n_2} |\{\mathbf{i} \in \mathbf{I} :\ u_{\mathbf{n},\mathbf{i}} > u^{(\nu)}_{\mathbf{n}}\}|\}_{\mathbf{n}\geq\mathbf{1}}$$

*is bounded, for some* $\{u_{\mathbf{n}}^{(\nu)}\}\}_{\mathbf{n}\geq\mathbf{1}}$ *satisfying* $n_1n_2(1-\Phi(u_{\mathbf{n}}^{(\nu)}))\to\nu>0$*, then* $\mathbf{X}$ *satisfies the conditions* $D(u_{\mathbf{n},\mathbf{i}})$ *over* $\mathcal{E}(u_{\mathbf{n},\mathbf{i}})$ *and* $D'(u_{\mathbf{n},\mathbf{i}})$.

# 4 Restriction on crossings of high levels

We discuss now the limiting distribution of maximum when, in addition to coordinatewise-mixing condition, we restrict the local path behaviour with respect to the number of crossings of the high levels $u_{\mathbf{n},\mathbf{i}}$.

Since the natural notion of crossing at $\mathbf{i}=(i_1,i_2)$ would get in consideration the values of the random field over the eight neighbours of $\mathbf{i}$, *id est*, over the points $\mathbf{j}$ such that $d(\mathbf{i},\mathbf{j})=1$, then by taking $\beta(\{\mathbf{i}\})=\{\mathbf{j}:d(\mathbf{i},\mathbf{j})=1\}$, we say that $\mathbf{X}$ has a crossing at $\mathbf{i}$ if occurs

$$B^*_{\mathbf{i},\mathbf{n}}=\{X_{\mathbf{i}}\leq u_{\mathbf{n},\mathbf{i}},\bigcup_{\mathbf{j}\in\beta(\{\mathbf{i}\})}X_{\mathbf{j}}>u_{\mathbf{n},\mathbf{j}}\}.$$

Using the ideas of [8], in combination with [5] and [2], to avoid clustering of crossings by a nonstationary random fields, we would assume, for each rectangle $\mathbf{I}$ satisfying

$$\sum_{\mathbf{i}\in\mathbf{I}}P(B^*_{\mathbf{i},\mathbf{n}})\leq\frac{1}{k_{n_1}k_{n_2}}\sum_{\mathbf{i}\leq\mathbf{n}}P(B^*_{\mathbf{i},\mathbf{n}}),$$

that it holds

$$k_{n_1}k_{n_2}\sum_{\mathbf{i},\mathbf{j}\in\mathbf{I}}P(B^*_{\mathbf{i},\mathbf{n}},B^*_{\mathbf{j},\mathbf{n}})\xrightarrow[\mathbf{n}\to\infty]{}0.$$

However, we verified that an i.i.d. random field doesn't satisfy the previous condition for normalized levels. In fact, for $\{u_{\mathbf{n}}\}_{\mathbf{n}\geq\mathbf{1}}$ such that $n_1n_2P(X_{\mathbf{1}}>u_{\mathbf{n}})\longrightarrow\tau$, as $\mathbf{n}\longrightarrow\infty$, and $\mathbf{I}=\{1,\ldots,\left[\frac{n_1}{k_{n_1}}\right]\}\times\{1,\ldots,\left[\frac{n_2}{k_{n_2}}\right]\}$ we have

$$\sum_{\mathbf{i}\in\mathbf{I}}P(B^*_{\mathbf{i}})\leq\frac{1}{k_{n_1}k_{n_2}}\sum_{\mathbf{i}\leq\mathbf{n}}P(B^*_{\mathbf{i}})$$

and

$$k_{n_1}k_{n_2}\sum_{\mathbf{i},\mathbf{j}\in\mathbf{I}}P\left(B^*_{\mathbf{i}},B^*_{\mathbf{j}}\right)\geq k_{n_1}k_{n_2}$$

$$([\tfrac{n_1}{k_{n_1}}]-1)([\tfrac{n_2}{k_{n_2}}]-1)P(X_{\mathbf{1}}>u_{\mathbf{n}})P^2(X_{\mathbf{1}}\leq u_{\mathbf{n}}),$$

which tends to $\tau$, as $\mathbf{n}\to\infty$. By an analogous reasoning with subsets of the boundary $\beta(\{\mathbf{i}\})$ with more than one element, we conclude that we can only restrict the number of crossings in one of the eight directions from each $\mathbf{i}\in\mathbf{I}$.

Since the only direction that joins the notion of past and future along both coordinate axes, simultaneously, is the diagonal direction from $\mathbf{i}$ to $\mathbf{i}+\mathbf{1}$, we will here consider a condition which restricts the local occurence of two or more of these diagonal crossings, id est, a condition that restricts the local occurence of two or more events $B_{\mathbf{i},\mathbf{n}}=(X_{\mathbf{i}}\leq u_{\mathbf{n},\mathbf{i}},X_{\mathbf{i}+\mathbf{1}}>u_{\mathbf{n},\mathbf{i}+\mathbf{1}})$.

Let $\mathcal{C}(u_{\mathbf{n},\mathbf{i}})$ denote the family of indexes sets $\mathbf{I}$ such that

$$\sum_{\mathbf{i}\in\mathbf{I}}P(B_{\mathbf{i},\mathbf{n}})\leq\frac{1}{k_{n_1}k_{n_2}}\sum_{\mathbf{i}\leq\mathbf{n}}P(B_{\mathbf{i},\mathbf{n}}).$$

**Definition 4.1.** *The condition* $D''(u_{\mathbf{n},\mathbf{i}})$ *holds for* $\mathbf{X}$ *if for each* $\mathbf{I}\in\mathcal{C}(u_{\mathbf{n},\mathbf{i}})$ *we have*

$$k_{n_1}k_{n_2}\sum_{\mathbf{i},\mathbf{j}\in\mathbf{I}}P(B_{\mathbf{i},\mathbf{n}},B_{\mathbf{j},\mathbf{n}})\xrightarrow[\mathbf{n}\to\infty]{}0.$$

By using $B_{\mathbf{i},\mathbf{n}}$ instead $\{X_{\mathbf{i}}>u_{\mathbf{n},\mathbf{i}}\}$ and $D''(u_{\mathbf{n},\mathbf{i}})$ instead of $D'(u_{\mathbf{n},\mathbf{i}})$ in the proof of the Proposition 3.1 we get the next result. Therefore we omit the similar proof and only remark that, under the conditions (1) and (2), we can suppose that, for each rectangle $\mathbf{V}_{r,p}$ in the partitions that arise for $\mathbf{R_n}$, the variables $X_{\mathbf{i}}$ with indexes in the basis and in the left side of $\mathbf{V}_{r,p}$ exhibit values below the correspondents levels $\mathbf{u}_{\mathbf{n},\mathbf{i}}$. Asymptotically, the probability of the complementar of that event is negligible. So, over each rectangle $\mathbf{V}_{r,p}$ we have an exceedance if and only if it occurs some event $B_{\mathbf{i},\mathbf{n}}$.

**Proposition 4.1.** *Suppose that the random field* $\mathbf{X}$ *verifies conditions (2),* $D''(u_{\mathbf{n},\mathbf{i}})$ *and* $D(u_{\mathbf{n},\mathbf{i}})$ *over* $\mathcal{C}(u_{\mathbf{n},\mathbf{i}})$*. Then, as* $\mathbf{n}\longrightarrow\infty$,

$$P(\bigcap_{\mathbf{i}\leq\mathbf{n}}X_{\mathbf{i}}\leq u_{\mathbf{n},\mathbf{i}})\to e^{-\nu},\ \nu>0,$$

*if and only if*

$$\sum_{\mathbf{i}\leq\mathbf{n}} P\left(B_{\mathbf{i},\mathbf{n}}\right) \rightarrow \nu > 0.$$

If $\mathbf{X}$ is stationary the result follows by assuming $u_{\mathbf{n},\mathbf{i}} = u_{\mathbf{n}}$, $\mathbf{i} \leq \mathbf{n}$, and condition $D''\left(u_{\mathbf{n}}\right)$ as

$$n_1 n_2 \sum_{\mathbf{j}\leq\left(\left[\frac{n_1}{k_{n_1}}\right],\left[\frac{n_2}{k_{n_2}}\right]\right)} P\left(B_{\mathbf{1},\mathbf{n}}, B_{\mathbf{j},\mathbf{n}}\right) \xrightarrow[\mathbf{n}\rightarrow\infty]{} 0.$$

Weaker local dependence conditions can be consider as in [3].

Accordingly [1], the stationary random field $\mathbf{X}$ has extremal index $\theta \in [0,1]$ if, for each $\tau > 0$, there exists $\{u_{\mathbf{n}}^{(\tau)}\}_{\mathbf{n}\geq\mathbf{1}}$ such that, as $\mathbf{n} \longrightarrow \infty$, $n_1 n_2 P(X_{\mathbf{1}} > u_{\mathbf{n}}^{(\tau)}) \longrightarrow \tau$ and $P(M_{\mathbf{n}} \leq u_{\mathbf{n}}^{(\tau)}) \longrightarrow exp(-\theta\tau)$.

If $\mathbf{X}$ is an i.i.d random field or a stationary random field satisfying the conditions of the Proposition 3.1 then the extremal index equals to 1.

For nonstationary random fields the extremal index can be defined in a similar way:

$$\theta(\tau) = \frac{-log \lim_{\mathbf{n}} P(\bigcap_{\mathbf{i}\leq\mathbf{n}} X_{\mathbf{i}} \leq u_{\mathbf{n},\mathbf{i}}^{(\tau)})}{\tau}$$

where $\tau = \lim_{\mathbf{n}} \sum_{\mathbf{i}\leq\mathbf{n}} P(X_{\mathbf{i}} > u_{\mathbf{n},\mathbf{i}}^{(\tau)})$.
Here the extremal index may depend on $\tau$, as pointed out examples in [4].

The following result gives a convenient existence criterion of the extremal index, assuming $D''\left(u_{\mathbf{n},\mathbf{i}}\right)$, and follows immediately from Proposition 4.1.

**Corollary 2.1.** *If the random field* $\mathbf{X}$ *satisfies (2),* $D''\left(u_{\mathbf{n},\mathbf{i}}\right)$ *and* $D\left(u_{\mathbf{n},\mathbf{i}}\right)$ *on* $\mathcal{C}(u_{\mathbf{n},\mathbf{i}})$ *with* $u_{\mathbf{n},\mathbf{i}} \equiv u_{\mathbf{n},\mathbf{i}}^{(\tau)}$, *then there exists* $\theta(\tau)$ *if and only if there exists*

$$\nu = \lim_{\mathbf{n}\longrightarrow\infty} \sum_{\mathbf{i}\leq\mathbf{n}} P\left(B_{\mathbf{i},\mathbf{n}}\right)$$

*and, in this case, it holds*

$$\theta(\tau) = \frac{\nu}{\tau}.$$

The clustering measure extremal index can be considered for sub-fields of $\mathbf{X}$. Let $\{\mathbf{I}_{\mathbf{n}}\}_{\mathbf{n}\geq\mathbf{1}}$ be an increasing sequence of sub-rectangles of $\mathbf{R}_{\mathbf{n}}$. If for each $\tau > 0$ there exists a family of levels $\{v_{\mathbf{n},\mathbf{i}}^{(\tau)}, \mathbf{i} \in \mathbf{I}_{\mathbf{n}}\}_{\mathbf{n}\geq\mathbf{1}}$ such that

$$\sum_{\mathbf{i}\in\mathbf{I}_{\mathbf{n}}} P(X_{\mathbf{i}} > v_{\mathbf{n},\mathbf{i}}^{(\tau)}) \xrightarrow[\mathbf{n}\rightarrow\infty]{} \tau$$

and

$$P(\bigcap_{\mathbf{i}\in\mathbf{I}_{\mathbf{n}}} X_{\mathbf{i}} \leq v_{\mathbf{n},\mathbf{i}}^{(\tau)}) \xrightarrow[\mathbf{n}\rightarrow\infty]{} \exp\left(-\theta\tau\right),$$

we say that $\mathbf{X}$ has extremal index $\theta$ *over* $\bigcup_{\mathbf{n}\geq\mathbf{1}} \mathbf{I}_{\mathbf{n}}$.

In general, we can't compare the extremal indexes over regions with the extremal index of the random field since the normalized levels are not, in general, coincident.

We illustrate the results with a simple 1-dependent random field defined as follows. Let $\mathbf{Y} = \{Y_{\mathbf{n}}\}_{\mathbf{n}\geq\mathbf{1}}$ be an i.i.d. random field and $\{u_{\mathbf{n}}\}_{\mathbf{n}\geq\mathbf{1}}$ such that $n_1 n_2 P(X_{\mathbf{1}} > u_{\mathbf{n}}) \longrightarrow \tau$.
Consider $\mathbf{X} = \{X_{\mathbf{n}}\}_{\mathbf{n}\geq\mathbf{1}}$ with $X_{\mathbf{i}} = X_{(i_1+1,i_2)} = X_{(i_1,i_2+1)} = Y_{\mathbf{i}}$ and $X_{\mathbf{i}+\mathbf{1}} = max\left\{Y_{\mathbf{i}}, Y_{(i_1+1,i_2)}, Y_{(i_1,i_2+1)}\right\}$, for each $\mathbf{i} = (i_1, i_2) = (2k+1, 2s+1)$, $k, s \geq 0$.
The nonstationary random field $\mathbf{X}$ satisfies $D\left(u_{\mathbf{n}}\right)$ and $D''\left(u_{\mathbf{n}}\right)$ conditions and has extremal index $\theta = \frac{5}{6}$.
For instance, for the region $\bigcup_{\mathbf{n}\geq\mathbf{1}} \mathbf{I}_{\mathbf{n}}$ with $\mathbf{I}_{\mathbf{n}} = \{\mathbf{i} : i_1 \leq n_1, i_2 \in \{1,2\}\}$, $\mathbf{n} \geq \mathbf{1}$, we find $\theta_{\mathbf{I}} = \frac{1}{6}$.

# References


[1] Choi, H. (2002) *Central limit theory and extremes of random fields.* PhD Dissertation, Univ. of North Carolina at Chapel Hill.

[2] Ferreira, H. (1994) *Condições de dependência local em teoria de valores extremos.* PhD Dissertation. University of Coimbra.

[3] Ferreira, H. e Pereira, L. (2005). How to compute the extremal index of stationary random fields. Pre-print Univ. of Lisbon.

[4] Hüsler, J. (1986) Extreme values of non-stationary random sequences. *J. Appl. Prob.* **23**, 937-950.

[5] Hüsler, J. (1993) A note on exceedances and rare events of non-stationary sequences. *J. Appl. Prob.* 30, 877-888.

[6] Leadbetter, M.R. (1983) Extremes and local dependence in stationary sequences. *Z. fur Wahrs. verw. Gebiete*, 65, 291-306.

[7] Leadbetter, M.R., Lindgren, G. and Rootzén, H. (1983) *Extremes and related properties of random sequences and processes.* Springer, New York.

[8] Leadbetter, M. R. and Nandagopalan, S. (1989) On exceedances point processes for stationary sequences under mild oscillation restrictions, em *Extremes values,* eds. Hüsler, J. e R.-D. Reiss, Springer-Verlag, 69-80.

[9] Leadbetter, M.R. and Rootzén, H. (1998) On extreme values in stationary random fields *Stochastic processes and related topics*, 275-285, *Trends Math.* Birkhauser Boston, Boston.

[10] Pereira, L. and Ferreira, H. (2005) On extreme values in non stationary random fields. *Preprint.* Univ. of Lisbon.