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Some Approximation Properties of Beta Operators

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Abstract: In this paper, we study the order of convergence of the linear positive Beta operators by means of the Peetre-K functional and of the functions from Lipschitz class. Furthermore we introduce a generalization of r-th order of these operators and also investigate approximation properties of them. Finaly we give an applications to differential equations.

Key Words: Beta Operators, K-Functional of Peetre, Lipschitz Class.

1. Introduction

The operators

$$B_n(f;x)=\int_0^1 \frac{t^{nx-1}(1-t)^{n(1-x)-1}}{B(nx,n(1-x))}f(t)dt \quad (1)$$

are known as Beta operators, where $n\in N=\{1,2,...\}$, $x\in[0,1]$ and $f\in C[0,1]$ and $B(.,.)$ denotes the familiar Beta function. It is clear that Beta operators are linear and positive. From the well-known theorem of Korovkin [5], it is easily verified that

$$\|B_n(f;x)-f(x)\|_{C[0,1]}\to 0 \quad as \quad n\to\infty. \quad (2)$$

The order of convergence of Beta operators by means of modulus of continuity was investigated by Khan [3]. Some approximation properties of Beta operators may be viewed in [3], [1] and references therein.

In this paper we first investigate the order of convergence of Beta operators, defined by (1), by means of the Peetre-K functional and of the functions from Lipschitz class. Next we give a generalization of $r-th$ order of these operators, and study their approximation properties. Finally we give an application to differential equations.

2. Rates of convergence

In the following, we first give the definition of the Peetre-K functional which we shall use.

Let $C^2[0,1]$ denote the space of those functions f for which $f, f', f''\in C[0,1]$, then the Peetre-K functional is defined as follows (see[2]):

$$K(f,\delta_n)=\inf_{g\in C^2[0,1]}\left\{\|f-g\|_{C[0,1]}+\delta_n\|g\|_{C^2[0,1]}\right\},$$

where $f\in C[0,1]$, $\delta_n\geq 0$ and the norm in the space $C^2[0,1]$ is defined by

$$\|g\|_{C^2[0,1]}:=\|g\|_{C[0,1]}+\|g'\|_{C[0,1]}+\|g''\|_{C[0,1]},$$

for $g\in C^2[0,1]$.

In this section we give the rates of convergence of (2) by means of the Peetre-K functional and elements of Lipschitz class.

We note that Khan proved the rate of convergence of (2) using modulus of continuity as follows (see[3]): For $f\in C[0,1]$,

$$|B_n(f;x)-f(x)|\leq(1+x(1-x))w(f,\delta_n),$$

where $\delta_n=\frac{1}{\sqrt{n+1}}$, $n\geq 1$, and $w(f,\delta_n)$ is the modulus of continuity of f.

Taking maximum over $[0,1]$ on each side of the last inequality we obtain that

$$\|B_n(f;x)-f(x)\|_{C[0,1]} \le \frac{5}{4} w(f,\delta_n), \quad (3)$$

where $\delta_n = \dfrac{1}{\sqrt{n+1}}$, $n \in N$.

Now we have the following.

Theorem 2.1. Let $f \in C[0,1]$, then we have

$$\|B_n(f;x)-f(x)\|_{C[0,1]} \le 2K(f,\delta_n),$$

where $\delta_n = \dfrac{1}{16(n+1)}$ and $K(f,\delta_n)$ is the Peetre-K functional.

Proof. By the definition of the Peetre-K functional, we first take $g \in C^2[0,1]$. Applying the operators B_n to the Taylor expansion of $g(s)$ at the neighborhood of x we then get that

$$B_n(g;x) = g(x) + B_n((s-x);x)g'(x) + \frac{1}{2}B_n((s-x);x)g''(x)$$

From the last equation, it follows that

$$|B_n(g;x)-g(x)| \le \frac{1}{2}\left|B_n\left((s-x)^2;x\right)\right||g''(x)|. \quad (4)$$

Easy calculations show that

$$\left|B_n\left((s-x)^2;x\right)\right| \le \left|B_n(s^2;x)-x^2\right| + 2x|B_n(s;x)-x| \le \frac{|x(1-x)|}{n+1}.$$

Taking (4) and the last inequality into account, then it can be deduced that

$$\|B_n(g;x)-g(x)\|_{C[0,1]} \le \frac{1}{8(n+1)}\|g\|_{C^2[0,1]}. \quad (5)$$

Using the linearity of B_n we have the following.

$$|B_n(f;x)-f(x)| \le |B_n(f-g;x)| + |f(x)-g(x)| + |B_n(g;x)-g(x)|.$$

Taking maximum of each side of the last inequality over [0,1] and using the fact that $B_n(1;x)=1$, then we arrive at

$$\|B_n(f;x)-f(x)\|_{C[0,1]} \le 2\|f-g\|_{C[0,1]} + \|B_n(g,x)-g(x)\|_{C[0,1]}.$$

Considering (5) in the last inequality, it then follows that

$$\|B_n(f;x)-f(x)\|_{C[0,1]} \le 2\left\{\|f-g\|_{C[0,1]} + \frac{1}{16(n+1)}\|g\|_{C[0,1]}\right\}. \quad (6)$$

Taking infimum over $g \in C^2[0,1]$ from each side of (6) and by choosing $\delta_n = \dfrac{1}{16(n+1)}$, theorem follows.

Recall that the well-known Lipschitz class of order α, $Lip_M(\alpha)$, $0<\alpha\le 1$, $M>0$, is defined as follows: For $t,x \in [0,1]$,

$$Lip_M(\alpha) = \left\{f : |f(t)-f(x)| \le M|t-x|^{\alpha}\right\}.$$

The following theorem gives the rate of convergence of the linear positive operator B_n by means of the Lipschitz class.

Theorem 2.2. Let $f \in Lip_M(\alpha)$, then we have

$$\|B_n(f;x)-f(x)\|_{C[0,1]} \le \frac{M}{2^{\alpha}}\delta_n^{\alpha},$$

where $\delta_n = \dfrac{1}{\sqrt{n+1}}$ which is the same as in (3).

Proof. Let $f \in Lip_M(\alpha)$. Since B_n are linear and monotone, then we have

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$$\left|B_n(f;x)-f(x)\right| \le B_n\left(\left|f(t)-f(x)\right|;x\right) \le M\int_0^1 |t-x|^{\alpha}\psi_{n,x}(t)dt,$$

by the definition of $Lip_M(\alpha)$, where

$$\psi_{n,x}(t) := \frac{t^{nx-1}(1-t)^{n(1-x)-1}}{B(nx,n(1-x))}. \qquad (7)$$

Applying Hölder's inequality to the last integral, it readily follows that

$$\left|B_n(f;x)-f(x)\right| \le M\left\{\int_0^1 (t-x)^2\psi_{n,x}(t)dt\right\}^{\alpha/2} \le M\left(\frac{x(1-x)}{n-1}\right)^{\alpha/2}, \qquad (8)$$

where $\psi_{n,x}(t)$ is the function given by (7).
Taking maximum over [0,1] from each side of (8), we arrive at

$$\left\|B_n(f;x)-f(x)\right\|_{C[0,1]} \le \frac{M}{2^{\alpha}}\delta_n^{\alpha},$$

where $\delta_n = \frac{1}{\sqrt{n+1}}$, which proves the theorem.

3. A generalization of $r-th$ order of B_n.

Let $C^r[0,1]$, $r=0,1,2,...$, denote the space of r-times continuously differentiable functions defined on [0,1]. We introduce the following generalization, in the sense of Kirov (see[4]), of the linear positive operators B_n, $n\in N$.

$$B_n^{[r]}(f;x) = \int_0^1 \sum_{k=0}^{r} f^{(k)}(t)\frac{(x-t)^k}{k!}\psi_{n,x}(t)\,dt, \quad (9)$$

where $\psi_{n,x}(t)$ is the function given by (7), $f\in C^r[0,1]$, $r=0,1,2,...$. The operators $B_n^{[r]}$ are named as $r-th$ order generalization of the operators B_n. Clearly $B_n^{[0]} = \mathrm{B_n}$, $n\in N$.

The subject of this section is to investigate approximation properties of the operators $B_n^{[r]}$, $r=0,1,2,...$, $n\in N$. For this purpose we first give the following.

Theorem 3.1. Let $f^{(r)}\in Lip_M(\alpha)$ and $f\in C^r[0,1]$, then we have

$$\left\|B_n^{[r]}(f;x)-f(x)\right\|_{C[0,1]} \le \frac{M}{(r-1)!}\frac{\alpha}{\alpha+r}B(\alpha,r)\left\|B_n\left(|t-x|^{\alpha+r};x\right)\right\|_{C[0,1]}, \quad (10)$$

where $B(\alpha,r)$ is the usual Beta function and $r=0,1,2,...$, $n\in N$.

Proof. From (9) we obtain the following.

$$f(x)-B_n^{[r]}(f;x) = \int_0^1 \left[f(x)-\sum_{k=0}^{r} f^{(k)}(t)\frac{(x-t)^k}{k!}\right]\psi_{n,x}(t)dt, \quad (11)$$

where $\psi_{n,x}(t)$ is the function given by (7). Using the integral from of the remainder term in (11) it follows that (see[4])

$$f(x)-\sum_{k=0}^{r} f^{(k)}(t)\frac{(x-t)^k}{k!} = \frac{(x-t)^r}{(r-1)!}\int_0^1 (1-z)^{r-1}\left[f^{(r)}(t+z(x-t))-f^{(r)}(t)\right]dt. \quad (12)$$

Since $f^{(r)}\in Lip_M(\alpha)$, then we have

$$\left|f^{(r)}(t+z(x-t))-f^{(r)}(t)\right| \le Mz^{\alpha}|t-x|^{\alpha}. \quad (13)$$

From the well-known property of the Beta function it follows that

$$\int_0^1 (1-t)^{r-1}t^{\alpha}dt = B(1+\alpha,r) = \frac{\alpha}{\alpha+r}B(\alpha,r). \quad (14)$$

Substituting (13) and (14) into (12), we arrive at

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$$\left| f(x) - \sum_{k=0}^{r} f^{(k)}(t) \frac{(x-t)^k}{k!} \right|$$
$$\leq \frac{M}{(r-1)!} \frac{\alpha}{\alpha + r} B(\alpha, r) |t - x|^{r+\alpha}. \quad (15)$$

Using (11) and (15) we reach to (10). Hence theorem follows.

Now let us define the function $g \in C[0,1]$ by $g(s) = |s - x|^{\alpha + r}$. Since $g(x) = 0$, then it can be easly seen from (2) that

$$\|B_n(g; x)\|_{C[0,1]} \to 0 \quad as \quad n \to \infty.$$

From Theorem 3.1 it readily follows that

$$\|B_n^{[r]}(f; x) - f(x)\|_{C[0,1]} \to 0 \quad as \quad n \to \infty$$

for $f \in C^r[0,1]$ such that $f^{(r)} \in Lip_M(\alpha)$.

As a result, taking the formula (3), Theorem 2.1 and Theorem 2.2 into account we can reach to the following respective results from Theorem 3.1.

Corollary 3.2. Let $f \in C^r[0,1]$ such that $f^{(r)} \in Lip_M(\alpha)$, then we have

$$\|B_n^{[r]}(f; x) - f(x)\|_{C[0,1]}$$
$$\leq \frac{M}{(r-1)!} \frac{\alpha}{\alpha + r} B(\alpha, r) \frac{5}{4} w(g; \delta)$$

by (3).

Corollary 3.3. Let $f \in C^r[0,1]$ such that $f^{(r)} \in Lip_M(\alpha)$, then we have

$$\|B_n^{[r]}(f; x) - f(x)\|_{C[0,1]}$$
$$\leq \frac{M}{(r-1)!} \frac{\alpha}{\alpha + r} B(\alpha, r) 2K(g; \delta),$$

where δ_n is the same as in Theorem 2.1 and K is the functional of Peetre.

Corollary 3.4. Let $f \in C^r[0,1]$ such that $f^{(r)} \in Lip_M(\alpha)$, then we have

$$\|B_n^{[r]}(f; x) - f(x)\|_{C[0,1]}$$
$$\leq \frac{M}{(r-1)!} \frac{\alpha}{\alpha + r} B(\alpha, r) \frac{1}{2^\alpha} \delta_n^\alpha,$$

where δ_n is the same as in (3) and we take $g \in Lip_1(\alpha)$ in Theorem 2.2.

4. An application to differential equations

In this section we obtain a differential equation that the linear positive operators B_n satisfy. One can find many differential equations that B_n satisfy.

Theorem 4.1. Let $g(t) = \ln \frac{t}{1-t}$, $t \in (0,1)$. For each $x \in [0,1]$ and $f \in C[0,1]$, $B_n(f; x)$ satisfy the following functional differential equation

$$\frac{d}{dx} B_n(f; x)$$
$$= -B_n(f; x) \frac{d}{dx} \ln[B(nx, n(1-x))]$$
$$+ nB_n(fg; x)$$

Proof. Since $f \in C[0,1]$ and the integral in (1) is convergent for each n, then differentiating each side of (1) we get the following.

$$\frac{d}{dx} B_n(f; x) = -\frac{\frac{d}{dx} B(nx, n(1-x))}{B(nx, n(1-x))} B_n(f; x)$$
$$+ \frac{n}{B(nx, n(1-x))} \int_0^1 f(t) \ln \frac{t}{1-t} t^{nx-1} (1-t)^{n(1-x)-1} \, dt.$$

Using the assumption $g(t) = \ln \frac{t}{1-t}$ in the last equation, the proof is reached.

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