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About the shape parameters of the family of Laplace distribution functions

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Abstract: In this paper we consider the distribution function of Laplace to find the shape parameter such as kurtosis, skew ness and sharply peaked parameter. By considering two types of kurtosis measures we show the difference between two kurtosis measures, then we estimate the parameters. We use the Mont-Carlo simulation methods for estimating one of the kurtosis measures.

Key-words: Kurtosis measure, Laplace distribution function, Sharply peaked parameter, Skew ness, Monte Carlo simulation methods.

1 Introduction

The shape of probability distribution function is often summarized by the distribution's skew ness and kurtosis measure. We will show that it is better to consider other shape of parameter which we will name it sharply peaked parameter.

This paper provides an introductory about skew Laplace distribution function and skew log-Laplace distribution function. For finding kurtosis measure of Laplace density function we note two types of kurtosis measure.

$$\beta_2 = \frac{E(X-\alpha)^4}{E^2(X-\alpha)^2}$$

Where α is the mean of a distribution.

$$\gamma_2 = E(\frac{X-\mu}{\sigma})^4$$

Where μ and σ are location and scale parameters.

We select one of them by considering the sharply peaked parameter.

2 Laplace function

Let X be a random variable with probability density function $f_\lambda(x)$ which is given by:

$$f_\lambda(x) = \frac{\lambda}{2\sigma} e^{-\lambda\left|\frac{x-\mu}{\sigma}\right|} \qquad x,\mu \in R, \qquad \sigma,\lambda > 0$$

We show X ~ $Lap(\lambda,\mu,\sigma)$.

Where μ,σ and λ are location, scale and sharply peaked parameter respectively.

Definition: The class of distribution which does not have derivative in their modes has been called sharply peaked distribution.

$$\gamma_2 = E(\frac{X-\mu}{\sigma})^4$$
$$= 2\int_{-\infty}^{\mu}\left(\frac{x-\mu}{\sigma}\right)^4 \frac{\lambda}{2\sigma} e^{\lambda\left(\frac{x-\mu}{\sigma}\right)} dx$$

By taking $\frac{x-\mu}{\sigma} = z$ we have

$$\gamma_2 = \int_{-\infty}^{0} \lambda z^4 e^{\lambda z} dz = \frac{24}{\lambda^4}$$

and

$$\beta_2 = \frac{E(X-\alpha)^4}{E^2(X-\alpha)^2} = \frac{\frac{24}{\lambda^4}}{\left(\frac{2}{\lambda}\right)^4} = 6$$

The value of β_2 is equal to 6 for all values of λ. For $\lambda < 0.798$ the density function is shorter than standard normal density function.

But γ_2 is a non-decreasing function of λ. So we prefer to use γ_2 rather than β_2.

3 Estimation of γ_2 based on estimation of scale parameter for $Lap(1,0,\sigma)$

1 - $$S = \sqrt{\frac{\sum (X_i - \overline{X})^2}{n-1}}$$

2 - $$R = X_{\max} - X_{\min}$$

3 - $$QD_p = \frac{Q_{(1-p)-Q_p}}{2}\text{ -}$$

4 - $$AAD = \frac{\sum |X_i - \overline{X}|}{n}$$

5 - $$MAD = med(|X_i - med(X_i)|)$$

In application the standard fourth moment, γ_2, must be estimated.

$$\gamma_2 = E\left(\frac{X-\mu}{\sigma}\right)^4 = \frac{E(X^4)}{\sigma^4}$$

because $\mu = 0$.

If σ is unknown we have to estimate it by considering the above estimators.

For example if we want to estimate σ by AAD we use the following distribution.

A sample $x_1, x_2, ..., x_n\ (n = 1000)$ from Lap(0,1) is generated and

$$\hat{\gamma}_{2,n} = \frac{\frac{1}{n}\sum_{i=1}^{n}(x_i - \overline{x})^4}{\left(\frac{1}{n}\sum_{i=1}^{n}|x_i - \overline{x}|\right)^4}$$

is computed. The values of $\hat{\gamma}_{2,n}$ for the estimators which is given by 1 – 5 are respectively.

$\hat{\gamma}_{2,n}$ based on AAD has been preferred because we know that $\sigma = 1$ and AAD is closer to 1 rather than the others.

4 Skew Laplace density function

Azzalini (1985) proves the following lemma.

Lemma: If f_0 is a one-dimensional probability density function symmetric about 0, and G is a one-dimensional distribution function such that G' exists and is a density symmetric about 0, then

$$f(z) = 2 f_0(z) G(w(z))$$

is a density function for any odd function w(.).

Skew Laplace function:

$$f_X(x) = c\Phi(\alpha x) f_\lambda(x) \qquad \lambda, \alpha \in R \qquad (1)$$

where α and λ are skew and sharply peaked parameter and $\Phi(.)$ is a distribution function which is called $SL(\alpha)$. We know α and λ are shape parameters. By taking $\alpha > 0$ positive skew ness will be appear and by taking $\alpha < 0$ negative skew ness will be appear. So by (1)

$$f_X(x) = \begin{cases} \dfrac{\lambda}{2\sigma} e^{\frac{\lambda}{\sigma}} e^{\frac{\lambda}{\sigma}x(\alpha+1)} & x < 0 \\ \dfrac{\lambda}{\sigma} e^{-\frac{\lambda}{\sigma}x} - \dfrac{\lambda}{2\sigma} e^{-\frac{\lambda}{\sigma}x(\alpha+1)} & x \ge 0 \end{cases}$$

We consider $f_X(x)$ when $\lambda = \mu = \sigma = 1$, so $SL(\alpha)$ for $\alpha = 1,-1$ has been plotted in figure 1.

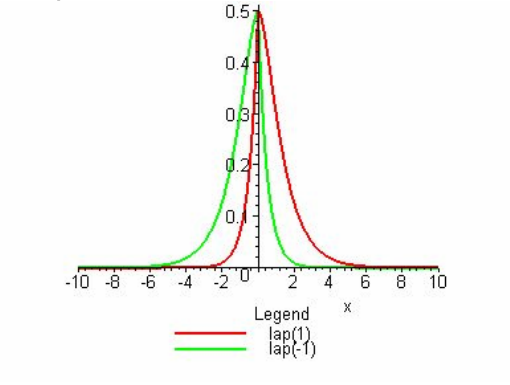

fig.1

The function $f_X(x)$ is continuous at x. The graph of f does not have a tangent line at the x = 0. In figure 1 there is a sketch of the graph of a function satisfying this condition observe that there is a sharp turn in the graph at x = 0, because

$$f'_X(0^-) = \frac{\lambda^2}{2\sigma^2}(\alpha+1)$$

and

$$f'_X(0^+) = \frac{-\lambda^2}{\sigma^2}\left(\frac{1}{2}(\alpha+1)-1\right)$$

for example for Lap(1,0,1) c is equal to 2 and in this case by taking $\alpha = 1$ we have

$f'_X(0^-) = 1$

$f'_X(0^+) = 0$

$SL(\alpha)$ has the following properties.

1 – If $\alpha = 0$, we obtain the L(0,1,0) density.

2 – If $Z \sim SL(\alpha)$, then $-Z \sim SL(-\alpha)$.

3 – As $\alpha \to \infty$, (4.1) converges pointwise to the half- Laplace density, namely $2\Phi(z)$ for $z \geq 0$.

5 Computing skewness and kurtosis for $SL(\alpha)$

The moment generating function is :

$$M_X(t) = E(e^{tX})$$
$$= \frac{\lambda}{2\lambda(\alpha+1)+2\sigma t} - \frac{\lambda}{-\lambda+\sigma t} + \frac{\lambda}{-2\lambda(\alpha+1)+2\sigma t}$$

We know that :

$$E(X^k) = \left.\frac{\partial^k M_X(t)}{\partial t^k}\right|_{t=0}$$

So we have

$$E(X) = V(X) = \frac{\sigma}{\lambda}$$

The kurtosis measure of the density function is:

$$\beta_2 = \frac{E(X-E(X))^4}{(V(X))^2} = 9$$

The skew ness of the distribution function is computed by the following formula:

$$\beta_1 = \frac{E(X-E(X))^3}{\left(\sqrt{V(X)}\right)^3}$$

For the considered density function the value of skew ness is:

$\beta_1 = 4$

6 Log Laplace density function

If $X \sim L(\lambda,\mu,\sigma)$ then the density of $Y = e^X$ is given by (2) with $e^\mu = \theta$, which is named by $Y \sim LL(\lambda,\mu,\sigma)$

$$f_Y(y) = \frac{1}{2\sigma y}\begin{cases} e^{\lambda\left(\frac{\ln y-\mu}{\sigma}\right)} & 0 < y < e^\mu \\ e^{-\lambda\left(\frac{\ln y-\mu}{\sigma}\right)} & y \geq e^\mu \end{cases}$$

$$= \frac{1}{2\sigma y}\begin{cases} e^{\frac{\lambda}{\sigma}(\ln y-\ln\theta)} & 0 < y < \theta \\ e^{-\frac{\lambda}{\sigma}(\ln y-\ln\theta)} & y \geq \theta \end{cases}$$

$$= \frac{1}{2\sigma\theta}\begin{cases} \left(\frac{y}{\theta}\right)^{\frac{\lambda}{\sigma}-1} & 0 < y < \theta \\ \left(\frac{\theta}{y}\right)^{\frac{\lambda}{\sigma}+1} & y \geq \theta \end{cases} \qquad (2)$$

We find the density function of SLL by (1). If $Z \sim SLL(\alpha, \lambda, \mu, \sigma)$ then the density function is given by :

$$f_Z(z) = c\begin{cases} \frac{\alpha^{\frac{\lambda}{\sigma}}}{4\lambda\sigma\theta}\left(\frac{z}{\theta}\right)^{2\frac{\lambda}{\sigma}-1} & 0 < z < \theta \\ \frac{1}{2\lambda\sigma\theta}\left(\frac{\theta}{z}\right)^{\frac{\lambda}{\sigma}+1} - \frac{1}{2\lambda\sigma\theta\alpha^{\frac{\lambda}{\sigma}}}\left(\frac{\theta}{z}\right)^{2\frac{\lambda}{\sigma}+1} \end{cases}$$

where

$$c = \frac{8\lambda^2}{\alpha^{\frac{\lambda}{\sigma}} + 4 - 2\alpha^{-\frac{\lambda}{\sigma}}}$$

7 Discussion

In this paper the Laplace density function has been considered. Two types of kurtosis measures β_2 and γ_2 have been computed and the use of γ_2 has been preferred. So for estimating γ_2, the scale parameter would be estimated by considering the estimators of scale parameter. AAD for Laplace has been suggested. Then the skew log laplace distribution has been made. They can be considered in application work.

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