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Associated hypergeometric-type functions and coherent states

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Abstract: We present in a unified and explicit way the systems of orthogonal polynomials defined by hypergeometric-type equations, the associated special functions and corresponding systems of coherent states. This general formalism allows us to extend some known results to a larger class of functions.

Key-Words: Orthogonal polynomials, Associated special functions, Coherent states, Raising and lowering operators, Creation and annihilation operators, Hypergeometric-type equations.

1 Introduction

Many problems in quantum mechanics and mathematical physics lead to equations of the type

$$\sigma(s)y''(s) + \tau(s)y'(s) + \lambda y(s) = 0 \qquad (1)$$

where $\sigma(s)$ and $\tau(s)$ are polynomials of at most second and first degree, respectively, and λ is a constant. These equations are usually called *equations of hypergeometric type* [15], and each can be reduced to the self-adjoint form

$$[\sigma(s)\varrho(s)y'(s)]' + \lambda\varrho(s)y(s) = 0 \qquad (2)$$

by choosing a function ϱ such that

$$[\sigma(s)\varrho(s)]' = \tau(s)\varrho(s). \qquad (3)$$

The equation (1) is usually considered on an interval (a, b), chosen such that

$$\begin{array}{rll} \sigma(s) > 0 & \text{for all} & s \in (a,b) \\ \varrho(s) > 0 & \text{for all} & s \in (a,b) \\ \multicolumn{3}{c}{\lim_{s\to a} \sigma(s)\varrho(s) = \lim_{s\to b} \sigma(s)\varrho(s) = 0.} \end{array} \qquad (4)$$

Since the form of the equation (1) is invariant under a change of variable $s \mapsto cs + d$, it is sufficient to analyse the cases presented in table 1. Some restrictions must be imposed on α, β in order for the interval (a, b) to exist.

The literature discussing special function theory and its application to mathematical and theoretical physics is vast, and there are a multitude of different conventions concerning the definition of functions. A unified approach is not possible without a unified definition for the associated special functions. In this paper we define them as

$$\Phi_{l,m}(s) = \left(\sqrt{\sigma(s)}\right)^m \frac{\mathrm{d}^m}{\mathrm{d}s^m}\Phi_l(s) \qquad (5)$$

where Φ_l are the orthogonal polynomials defined by equation (1). The table 1 allows one to pass in each case from our parameters α, β to the parameters used in different approach.

In our previous papers [6, 7], we presented a systematic study of the Schrödinger equations exactly solvable in terms of associated special functions following Lorente [14], Jafarizadeh and Fakhri [12]. In the present paper, our aim is to extend this unified formalism by including a larger class of creation/annihilation operators and some temporally stable coherent states of Gazeau-Klauder

Table 1: Particular cases (in each case $\tau(s) = \alpha s+\beta$)

$\sigma(s)$	$\varrho(s)$	α, β	(a, b)
1	$e^{\alpha s^2/2+\beta s}$	$\alpha < 0$	$(-\infty,\infty)$
s	$s^{\beta-1}e^{\alpha s}$	$\alpha<0$ $\beta>0$	$(0, \infty)$
$1-s^2$	$(1+s)^{-(\alpha-\beta)/2-1}\times$ $\times(1-s)^{-(\alpha+\beta)/2-1}$	$\alpha<\beta$ $\alpha+\beta<0$	$(-1, 1)$
s^2-1	$(s+1)^{(\alpha-\beta)/2-1}\times$ $\times(s-1)^{(\alpha+\beta)/2-1}$	$0<\alpha+\beta$ $\alpha < 0$	$(1, \infty)$
s^2	$s^{\alpha-2}e^{-\beta/s}$	$\alpha<0$ $\beta>0$	$(0, \infty)$
s^2+1	$(1 + s^2)^{\alpha/2-1}\times$ $\times\, e^{\beta\arctan s}$	$\alpha < 0$	$(-\infty,\infty)$

type [1, 2, 8, 9, 10, 13].

2 Associated special functions, raising and lowering operators

It is well-known [15] that for $\lambda = \lambda_l$, where

$$\lambda_l= -\frac{\sigma''(s)}{2}l(l-1)-\tau'(s)l= -\frac{\sigma''}{2}l(l-1)-\alpha\, l \quad (6)$$

and $l \in \mathbb{N}$, the equation (1) admits a polynomial solution $\Phi_l = \Phi_l^{(\alpha,\beta)}$ of at most l degree

$$\sigma(s)\Phi_l'' + \tau(s)\Phi_l' + \lambda_l\Phi_l = 0. \quad (7)$$

If the degree of the polynomial Φ_l is l then it satisfies the Rodrigues formula [15]

$$\Phi_l(s) = \frac{B_l}{\varrho(s)}\frac{d^l}{ds^l}[\sigma^l(s)\varrho(s)] \quad (8)$$

where B_l is a constant. Based on the relation

$$\{\ \delta \in \mathbb{R} \mid \lim_{s\to a}\sigma(s)\varrho(s)s^\delta = \lim_{s\to b}\sigma(s)\varrho(s)s^\delta = 0\ \}$$

$$= \begin{cases} [0,\infty) & \text{if} \quad \sigma(s) \in \{1,\ s,\ 1-s^2\} \\ [0,-\alpha) & \text{if} \quad \sigma(s) \in \{s^2-1,\ s^2,\ s^2+1\} \end{cases}$$

one can prove [7] that the system of polynomials $\{\Phi_l \mid l < \Lambda\}$, where

$$\Lambda=\begin{cases} \infty & \text{for} \quad \sigma(s) \in \{1,\ s,\ 1-s^2\} \\ \frac{1-\alpha}{2} & \text{for} \quad \sigma(s) \in \{s^2-1,\ s^2,\ s^2+1\} \end{cases} \quad (9)$$

is orthogonal with weight function $\varrho(s)$ in (a, b). This means that equation (1) defines an infinite sequence of orthogonal polynomials

$$\Phi_0,\ \ \Phi_1,\ \ \Phi_2,\ \ ...$$

in the case $\sigma(s) \in \{1,\ s,\ 1-s^2\}$, and a finite one

$$\Phi_0,\ \ \Phi_1,\ \ ...,\ \ \Phi_L$$

with $L = \max\{l \in \mathbb{N} \mid l < (1-\alpha)/2\}$ in the case $\sigma(s) \in \{s^2-1,\ s^2,\ s^2+1\}$.

The polynomials $\Phi_l^{(\alpha,\beta)}$ can be expressed in terms of the classical orthogonal polynomials as

$$\Phi_l^{(\alpha,\beta)}(s) =$$

$$= \begin{cases} H_l\left(\sqrt{\frac{-\alpha}{2}}\,s-\frac{\beta}{\sqrt{-2\alpha}}\right) & \text{if} \quad \sigma(s)=1 \\ L_l^{\beta-1}(-\alpha s) & \text{if} \quad \sigma(s)=s \\ P_l^{(-(\alpha+\beta)/2-1,\ (-\alpha+\beta)/2-1)}(s) & \text{if} \quad \sigma(s)=1-s^2 \\ P_l^{((\alpha-\beta)/2-1,\ (\alpha+\beta)/2-1)}(-s) & \text{if} \quad \sigma(s)=s^2-1 \\ \left(\frac{s}{\beta}\right)^l L_l^{1-\alpha-2l}\left(\frac{\beta}{s}\right) & \text{if} \quad \sigma(s)=s^2 \\ i^l P_l^{((\alpha+i\beta)/2-1,\ (\alpha-i\beta)/2-1)}(is) & \text{if} \quad \sigma(s)=s^2+1 \end{cases} \quad (10)$$

where H_n, L_n^p and $P_n^{(p,q)}$ are the Hermite, Laguerre and Jacobi polynomials, respectively.

Let $l \in \mathbb{N}$, $l < \Lambda$, and let $m \in \{0, 1, ..., l\}$. The functions

$$\Phi_{l,m}(s) = \kappa^m(s)\frac{d^m}{ds^m}\Phi_l(s) \quad (11)$$

where

$$\kappa(s) = \sqrt{\sigma(s)}$$

are called the *associated special functions*. If we differentiate (7) m times and then multiply the obtained relation by $\kappa^m(s)$ then we get the equation

$$H_m\Phi_{l,m} = \lambda_l\Phi_{l,m} \quad (12)$$

where H_m is the differential operator

$$H_m = -\sigma(s)\frac{d^2}{ds^2} - \tau(s)\frac{d}{ds} + \frac{m(m-2)}{4}\frac{(\sigma'(s))^2}{\sigma(s)}$$

$$+\frac{m\tau(s)}{2}\frac{\sigma'(s)}{\sigma(s)} - \frac{1}{2}m(m-2)\sigma''(s) - m\tau'(s). \quad (13)$$

The relation

$$\langle f, g\rangle = \int_a^b \overline{f(s)}\,g(s)\varrho(s)ds \quad (14)$$

defines a scalar product on the space

$$\mathcal{H}_m = \mathrm{span}\{\Phi_{l,m} \mid m \le l < \Lambda\}$$

spanned by $\{\Phi_{l,m} \mid m \le l < \Lambda\}$. For each $m < \Lambda$, the special functions $\Phi_{l,m}$ with $m \le l < \Lambda$ are orthogonal with weight function $\varrho(s)$ in (a,b), and the functions corresponding to consecutive values of m are related through the raising/lowering operators [6, 7, 12]

$$\begin{aligned} A_m &= \kappa(s)\tfrac{d}{ds} - m\kappa'(s) \\ A_m^+ &= -\kappa(s)\tfrac{d}{ds} - \tfrac{\tau(s)}{\kappa(s)} - (m-1)\kappa'(s) \end{aligned} \tag{15}$$

namely,

$$A_m\Phi_{l,m} = \begin{cases} 0 & \text{for} \quad l=m \\ \Phi_{l,m+1} & \text{for} \quad m<l<\Lambda \end{cases}$$

$$A_m^+\Phi_{l,m+1} = (\lambda_l - \lambda_m)\Phi_{l,m} \quad \text{for} \quad 0 \le m < l < \Lambda. \tag{16}$$

In addition, we have the relations [5, 11]

$$\Phi_{l,m} = \frac{A_m^+}{\lambda_l - \lambda_m}\frac{A_{m+1}^+}{\lambda_l - \lambda_{m+1}}\cdots\frac{A_{l-1}^+}{\lambda_l - \lambda_{l-1}}\Phi_{l,l} \tag{17}$$

for $0 \le m < l < \Lambda$, and

$$H_m - \lambda_m = A_m^+ A_m \qquad H_{m+1} - \lambda_m = A_m A_m^+ \tag{18}$$

$$H_m A_m^+ = A_m^+ H_{m+1} \qquad A_m H_m = H_{m+1} A_m \tag{19}$$

for $m+1 < \Lambda$.

The functions

$$\Psi_{l,m} = \Phi_{l,m}/||\Phi_{l,m}|| \tag{20}$$

where

$$||f|| = \sqrt{\langle f, f\rangle} \tag{21}$$

are the *normalized associated special functions*. Since [6, 7]

$$||\Phi_{l,m+1}|| = \sqrt{\lambda_l - \lambda_m}\, ||\Phi_{l,m}|| \tag{22}$$

they satisfy the relations

$$\begin{aligned} &A_m\Psi_{l,m} = \begin{cases} 0 & \text{for} \quad l=m \\ \sqrt{\lambda_l - \lambda_m}\Psi_{l,m+1} & \text{for} \quad m<l<\Lambda \end{cases} \\ &A_m^+\Psi_{l,m+1} = \sqrt{\lambda_l - \lambda_m}\Psi_{l,m} \quad \text{for} \quad 0 \le m < l < \Lambda \\ &\Psi_{l,m} = \tfrac{A_m^+}{\sqrt{\lambda_l - \lambda_m}}\tfrac{A_{m+1}^+}{\sqrt{\lambda_l - \lambda_{m+1}}}\cdots\tfrac{A_{l-1}^+}{\sqrt{\lambda_l - \lambda_{l-1}}}\Psi_{l,l}. \end{aligned} \tag{23}$$

3 Creation and annihilation operators

Let m be a fixed natural number and γ a fixed real number. In the case $\sigma(s) \in \{1, s, 1-s^2\}$, the only considered in sequel, the sequence

$$\Psi_{m,m}, \quad \Psi_{m+1,m}, \quad \Psi_{m+2,m}, \ \ldots$$

is a complete orthonormal sequence in the Hilbert space

$$\mathcal{H} = \left\{ f : (a,b) \longrightarrow \mathbb{C} \;\middle|\; \int_a^b |f(s)|^2 \varrho(s)\, ds < \infty \right\}$$

with scalar product (14). The linear operators

$$\begin{aligned} &a_m,\ a_m^+ : \mathcal{H}_m \longrightarrow \mathcal{H}_m \\ &\qquad a_m = U_m^{-1} A_m \qquad a_m^+ = A_m^+ U_m \end{aligned} \tag{24}$$

defined by using the unitary operator

$$\begin{aligned} &U_m : \mathcal{H}_m \longrightarrow \mathcal{H}_{m+1} \\ &\qquad U_m\Psi_{l,m} = \mathrm{e}^{-\mathrm{i}\gamma(\lambda_{l+1}-\lambda_l)}\Psi_{l+1,m+1} \end{aligned} \tag{25}$$

are mutually adjoint (see figure 1),

$$\begin{aligned} &a_m\Psi_{l,m} = \begin{cases} 0 & \text{for } l=m \\ \sqrt{\lambda_l - \lambda_m}\,\mathrm{e}^{\mathrm{i}\gamma(\lambda_l - \lambda_{l-1})}\Psi_{l-1,m} & \text{for } l>m \end{cases} \\ &a_m^+\Psi_{l,m} = \sqrt{\lambda_{l+1} - \lambda_m}\,\mathrm{e}^{-\mathrm{i}\gamma(\lambda_{l+1} - \lambda_l)}\Psi_{l+1,m} \quad \text{for } l \ge m \end{aligned} \tag{26}$$

and

$$H_m - \lambda_m = a_m^+ a_m \tag{27}$$

$$[a_m^+, a_m]\Psi_{l,m} = (\lambda_l - \lambda_{l+1})\Psi_{l,m} = (\sigma'' l + \alpha)\Psi_{l,m}.$$

Since the operator $R_m = [a_m^+, a_m]$ satisfies the relations

$$[R_m, a_m^+] = \sigma'' a_m^+ \qquad [R_m, a_m] = -\sigma'' a_m \tag{28}$$

the Lie algebra $\mathcal{L}_m$ generated by $\{a_m^+, a_m\}$ is finite dimensional.

Theorem 1.

$$\mathcal{L}_m \textit{ is isomorphic to } \begin{cases} h(2) & \textit{if} \quad \sigma(s) \in \{1,\ s\} \\ su(1,1) & \textit{if} \quad \sigma(s) = 1 - s^2. \end{cases}$$

$$
\begin{tikzcd}
\vdots & \vdots & \vdots & \vdots \\
\Psi_{3,0} \arrow[r, shift right, "A_0"'] \arrow[d, shift left, "a_0"] & \Psi_{3,1} \arrow[l, shift right, "A_0^+"'] \arrow[r, shift right, "A_1"'] \arrow[d, shift left, "a_1"] & \Psi_{3,2} \arrow[l, shift right, "A_1^+"'] \arrow[r, shift right, "A_2"'] \arrow[d, shift left, "a_2"] & \Psi_{3,3} \arrow[l, shift right, "A_2^+"'] \\
\Psi_{2,0} \arrow[u, shift left, "a_0^+"] \arrow[ur, "U_0"] \arrow[r, shift right, "A_0"'] \arrow[d, shift left, "a_0"] & \Psi_{2,1} \arrow[u, shift left, "a_1^+"] \arrow[ur, "U_1"] \arrow[l, shift right, "A_0^+"'] \arrow[r, shift right, "A_1"'] \arrow[d, shift left, "a_1"] & \Psi_{2,2} \arrow[u, shift left, "a_2^+"] \arrow[ur, "U_2"] \arrow[l, shift right, "A_1^+"'] & \\
\Psi_{1,0} \arrow[u, shift left, "a_0^+"] \arrow[ur, "U_0"] \arrow[r, shift right, "A_0"'] \arrow[d, shift left, "a_0"] & \Psi_{1,1} \arrow[u, shift left, "a_1^+"] \arrow[ur, "U_1"] \arrow[l, shift right, "A_0^+"'] & & \\
\Psi_{0,0} \arrow[u, shift left, "a_0^+"] \arrow[ur, "U_0"] & & &
\end{tikzcd}
$$

Fig. 1: The operators A_m, A_m^+, a_m, a_m^+ and U_m relating the functions $\Psi_{l,m}$.

Proof. In the case $\sigma(s) \in \{1, s\}$ the operator R_m is a constant operator, namely, $R_m = \alpha$. Since $\alpha < 0$, the operators $P_+ = \sqrt{-1/\alpha}\, a_m^+$, $P_- = \sqrt{-1/\alpha}\, a_m$ and I form a basis of $\mathcal{L}_m$ such that

$$[P_+, P_-] = -I \qquad [I, P_\pm] = 0$$

that is, $\mathcal{L}_m$ is isomorphic to the Heisenberg-Weyl algebra $h(2)$.

If $\sigma(s) = 1 - s^2$ then $K_+ = a_m^+$, $K_- = a_m$ and $K_0 = R_m$ form a basis of $\mathcal{L}_m$ such that

$$[K_+, K_-] = -2K_0 \qquad [K_0, K_\pm] = \pm K_\pm. \qquad \blacksquare$$

The operator a_m can be regarded as an annihilation operator, and a_m^+ as a creation operator.

4 Coherent states

Let $\sigma(s) \in \{1, s, 1-s^2\}$, and let $m \in \mathbb{N}$ be a fixed natural number. The functions $|0\rangle$, $|1\rangle$, $|2\rangle$, $\cdots$, where

$$|n\rangle = \Psi_{m+n,m} \tag{29}$$

satisfy the relations

$$\begin{aligned} a_m|n\rangle &= \sqrt{e_n}\, \mathrm{e}^{\mathrm{i}\gamma(e_n - e_{n-1})}|n-1\rangle \\ a_m^+|n\rangle &= \sqrt{e_{n+1}}\, \mathrm{e}^{-\mathrm{i}\gamma(e_{n+1} - e_n)}|n+1\rangle \\ (H_m - \lambda_m)|n\rangle &= e_n|n\rangle \end{aligned} \tag{30}$$

where

$$\begin{aligned} e_n &= \lambda_{m+n} - \lambda_m \\ &= \begin{cases} -\alpha n & \text{if} \quad \sigma(s) \in \{1, s\} \\ n(n + 2m - \alpha - 1) & \text{if} \quad \sigma(s) = 1 - s^2. \end{cases} \end{aligned} \tag{31}$$

By using the confluent hypergeometric function

$${}_0F_1(c;z) = 1 + \frac{1}{c}\frac{z}{1!} + \frac{1}{c(c+1)}\frac{z^2}{2!} + \frac{1}{c(c+1)(c+2)}\frac{z^3}{3!} + \cdots \tag{32}$$

and the modified Bessel function

$$K_\nu(z) = \frac{\pi}{2}\frac{I_{-\nu}(z) - I_\nu(z)}{\sin(\nu\pi)} \tag{33}$$

where

$$I_\nu(z) = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}z\right)^{\nu+2n}}{n!\,\Gamma(\nu + n + 1)} \tag{34}$$

we prove (following [3]) that $\{\ |z, \gamma\rangle \mid z \in \mathbb{C}\}$, where

$$|z, \gamma\rangle = \sum_{n=0}^{\infty} \frac{z^n\, \mathrm{e}^{-\mathrm{i}\gamma e_n}}{\sqrt{\varepsilon_n}}|n\rangle \tag{35}$$

and

$$\varepsilon_n = \begin{cases} 1 & \text{if} \quad n = 0 \\ e_1 e_2 \ldots e_n & \text{if} \quad n > 0 \end{cases} \tag{36}$$

is a system of coherent states continuous in z.

Theorem 2.
a) If $\sigma(s) \in \{1, s\}$ then $\{\ |z, \gamma\rangle \mid z \in \mathbb{C}\}$, where

$$|z, \gamma\rangle = \sum_{n=0}^{\infty} \frac{z^n\, \mathrm{e}^{-\mathrm{i}\gamma e_n}}{\sqrt{\varepsilon_n}}|n\rangle = \sum_{n=0}^{\infty} \frac{z^n\, \mathrm{e}^{-\mathrm{i}\gamma e_n}}{\sqrt{n!\,(-\alpha)^n}}|n\rangle \tag{37}$$

is a system of coherent states in $\mathcal{H}$ such that

$$\langle z, \gamma | z, \gamma\rangle = \mathrm{e}^{-\frac{|z|^2}{\alpha}} \qquad a_m|z, \gamma\rangle = z|z, \gamma\rangle \tag{38}$$

and

$$\frac{-1}{\pi\alpha}\int_{\mathbb{C}} \mathrm{e}^{\frac{|z|^2}{\alpha}}\, d(\mathrm{Re}\, z)\, d(\mathrm{Im}\, z)|z, \gamma\rangle\langle z, \gamma| = I. \tag{39}$$

b) If $\sigma(s) = 1 - s^2$ then $\{\ |z, \gamma\rangle \mid z \in \mathbb{C}\}$, where

$$\begin{aligned} |z, \gamma\rangle &= \textstyle\sum_{n=0}^{\infty} \frac{z^n\, \mathrm{e}^{-\mathrm{i}\gamma e_n}}{\sqrt{\varepsilon_n}}|n\rangle \\ &= \sqrt{\Gamma(2m - \alpha)}\textstyle\sum_{n=0}^{\infty} \frac{z^n\, \mathrm{e}^{-\mathrm{i}\gamma e_n}}{\sqrt{n!\,\Gamma(n+2m-\alpha)}}|n\rangle \end{aligned} \tag{40}$$

is a system of coherent states in $\mathcal{H}$ such that

$$\langle z, \gamma | z, \gamma\rangle = {}_0F_1(2m - \alpha; |z|^2) \tag{41}$$

$$a_m|z, \gamma\rangle = z|z, \gamma\rangle \tag{42}$$

and

$$\int_{\mathbb{C}} d\mu(z)\,|z,\,\gamma\rangle\langle z,\,\gamma| = I \tag{43}$$

where

$$d\mu(z) = \frac{2r^{2m-\alpha}}{\pi\Gamma(2m-\alpha)} K_{\frac{\alpha+1}{2}-m}(2r)\,dr\,d\theta \tag{44}$$

and $z = r\mathrm{e}^{\mathrm{i}\theta}$.

Proof. a) By denoting $t = -\frac{r^2}{\alpha}$ and using the integration by parts we get

$$\frac{-1}{\pi\alpha}\int_{\mathbb{C}} d(\mathrm{Re}\, z)\, d(\mathrm{Im}\, z)|z,\gamma\rangle\langle z,\gamma|$$

$$= \frac{-1}{\pi\alpha}\sum_{n,n'} \mathrm{e}^{-\mathrm{i}\gamma(e_n-e_{n'})}\times$$

$$\times\left(\int_0^\infty \mathrm{e}^{\frac{r^2}{\alpha}} \frac{r^{n+n'+1}}{\sqrt{n!\,n'!\,(-\alpha)^{n+n'}}} dr \int_0^{2\pi} \mathrm{e}^{\mathrm{i}(n-n')\theta} d\theta\right)|n\rangle\langle n'|$$

$$= \frac{-2}{\alpha}\sum_n \left(\int_0^\infty \mathrm{e}^{\frac{r^2}{\alpha}} \frac{1}{n!}\left(\frac{r^2}{-\alpha}\right)^n r\,dr\right)|n\rangle\langle n|$$

$$= \sum_n \left(\int_0^\infty \mathrm{e}^{-t}\frac{t^n}{n!}\,dt\right)|n\rangle\langle n| = \sum_n |n\rangle\langle n| = I.$$

b) Denoting $d\mu = \mu(r)\,dr\,d\theta$ we get

$$\int_{\mathbb{C}} d\mu(z)\,|z,\,\gamma\rangle\langle z,\,\gamma|$$

$$= \sum_{n=0}^\infty \frac{2\pi\Gamma(2m-\alpha)}{n!\,\Gamma(n+2m-\alpha)}\left(\int_0^\infty r^{2n}\mu(r)\,dr\right)|n\rangle\langle n|$$

and hence, we must have the relation (Mellin transformation)

$$2\pi\Gamma(2m-\alpha)\int_0^\infty r^{2n}\mu(r)\,dr = \Gamma(n+1)\,\Gamma(n+2m-\alpha). \tag{45}$$

The formula [4]

$$\int_0^\infty 2x^{\eta+\xi}K_{\eta-\xi}(2\sqrt{x})\,x^{n-1}dx = \Gamma(2\eta+n)\,\Gamma(2\xi+n)$$

for $x = r^2$, $\eta = \frac{1}{2}$, $\xi = m - \frac{\alpha}{2}$ becomes

$$4\int_0^\infty r^{2n}K_{\frac{\alpha+1}{2}-m}(2r)\,r^{2m-\alpha}dr = \Gamma(n+1)\Gamma(n+2m-\alpha). \tag{46}$$

The relations (45) and (46) lead to (44). ∎

If we consider the 'number' operator [2, 8]

$$N:\mathcal{H}\longrightarrow\mathcal{H} \qquad N|n\rangle = n|n\rangle \tag{47}$$

that is,

$$N = \sum_{n=0}^\infty n\,|n\rangle\langle n| \tag{48}$$

then the operator $H = H_m - \lambda_m$ can be written as

$$H = \sum_{n=0}^\infty e_n|n\rangle\langle n|$$

$$= \begin{cases} -\alpha N & \text{if} \quad \sigma(s)\in\{1,\,s\} \\ N(N+2m-\alpha-1) & \text{if} \quad \sigma(s) = 1-s^2. \end{cases}$$

The operators a_m and $a_m^{\perp}$, where [8]

$$a_m^{\perp} = \frac{N}{H}a_m^+ = \begin{cases} -\frac{1}{\alpha}a_m^+ & \text{if} \quad \sigma(s)\in\{1,\,s\} \\ \frac{1}{N+2m-\alpha-1}a_m^+ & \text{if} \quad \sigma(s) = 1-s^2. \end{cases}$$

satisfy the relations

$$[a_m, a_m^{\perp}] = I \qquad [N, a_m^{\perp}] = a_m^{\perp} \qquad [N, a_m] = -a_m.$$

Therefore, we can consider the non-unitary displacement operator [8]

$$D(z) = \exp(z\,a_m^{\perp} - \overline{z}\,a_m)$$

$$= \exp\left(-\tfrac{1}{2}|z|^2\right)\ \exp(z\,a_m^{\perp})\ \exp(-\overline{z}\,a_m)$$

and

$$|z,\,\gamma\rangle = D(z)\,|0\rangle \qquad \text{for any } z\in\mathbb{C}. \tag{49}$$

Since the Hermitian operators

$$X = \frac{1}{\sqrt{2}}(a_m^+ + a_m) \qquad P = \frac{\mathrm{i}}{\sqrt{2}}(a_m^+ - a_m) \tag{50}$$

satisfy the commutation relation

$$[X, P] = \mathrm{i}[a_m, a_m^+] \tag{51}$$

and $|z,\,\gamma\rangle$ are eigenstates of a_m, the coherent states $|z,\,\gamma\rangle$ minimize the uncertainty relation [8]

$$(\Delta X)^2(\Delta P)^2 \geq \frac{1}{4}\langle\mathrm{i}[X,P]\rangle^2. \tag{52}$$

The presence of the phase factor in definition of $|z,\,\gamma\rangle$ leads to the temporal stability of these coherent states

$$\mathrm{e}^{-\mathrm{i}tH}|z,\,\gamma\rangle = |z,\,\gamma+t\rangle. \tag{53}$$

5 Concluding remarks

The associated hypergeometric-type functions can be studied together in a unified formalism, and are directly related to the bound-state eigenfunctions of some important Schrödinger equations (Pöschl-Teller, Morse, Scarf, etc.). The raising/lowering operators, the creation/annihilation operators and the systems of coherent states used in quantum mechanics correspond to some operators and systems of functions from the theory of orthogonal polynomials and associated special functions.

It is useful to obtain fundamental versions (occurring at the level of associated special functions) for some methods and formulae from quantum mechanics because in this way one can extend results known in particular cases to other quantum systems. A large number of formulae occurring in various applications of quantum mechanics follow from a very small number of fundamental mathematical results.

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