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A different approach for pricing European options

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Abstract: - In this paper we analyze the value of European options on a dividend-paying asset from a point of view different to that of Black-Scholes. The method that we propose uses partial differential equations and the Mellin transform, and can be applied to financial practices. Our starting point will be the partial differential equation, which gives us the value of the option. We then solve the problem using integral transform theory, thus obtaining an explicit integral solution different from the one given by Black-Scholes-Merton. Lastly, we use various practical examples to show that our formula and the classical one are in perfect agreement.

Key-Words: - Option pricing, Black-Scholes equation, European options, Mellin transform.

1 Introduction

One of the fundamental elements of a modern financial market is the options contract. Options are not the result of a recent financial innovation; in fact, they were conceived thousands of years ago. We know that the Phoenicians, the Greeks, and the Romans negotiated contracts with options clauses on the merchandise carried aboard their vessels. However, some market historians credit the famous Greek philosopher, mathematician, and astronomer Thales, who made a considerable profit investing in options on the olive harvest and the use of oil mills, with the discovery. The first loosely organized options market appeared in Holland in the 17th century, when options on tulip bulbs began trading. In England, in the early 18th century, the stocks of the main commercial companies began trading. In the fall of 1720, the sharp drop in the prices of the "South Sea Company", motivated in part by speculation on the stock options of the company, produced such a scandal that the options market was declared illegal. This prohibition was in force until the start of the 20th century, although options trading continued clandestinely.

In the United Status, options on the purchase of stocks began trading in the 18th century in unregulated markets, but its spectacular growth did not begin until April 26, 1973, with the opening of the CBOE (Chicago Board Options Exchange), the first organized financial derivatives market in the world.

A derivative is a financial product based on an asset. The owner of an option has the right (but not the obligation) to buy or sell a certain asset on a future date at a fixed price. One of the simplest types of options gives the right to buy an asset, known as a call option. It is important to keep in mind that the owner of a call option may choose not to exercise it, thus gaining no profit from it. So, in exercising the option, the owner benefits from a favorable movement in the price of the asset, and if he does not, the losses are limited. On the other hand, the seller of the call option is obligated with fulfilling the conditions of the contract should the owner wish to exercise the option

In 1973, the publication of the famous work by Black and Scholes revolutionized the world's financial markets. By considering a simple model for the price of a financial resource, they obtained an analytical formula for the value of a European call option on a stock. This type of option is an example of a financial derivative which gives the holder the right, but not the obligation, to buy a unit of an asset at a fixed moment (the expiration date) at a fixed price K (option striking price). If we use C and S to denote the call premium and the stock price at expiration, respectively, the option holder will receive $C = \max(0, S - K)$. Black and Scholes assumed that there was no arbitrage in the market and obtained the

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one price for the option that would let a bank take that money and, using a hedging strategy, guarantee payment of the option.

Different models for pricing the value of options on specific underlying assets have been studied, many based on Black-Scholes' groundbreaking work. It can be said that the topic of asset valuation has become a priority at many financial research centers.

The method we propose for pricing European options on dividend-paying stocks uses partial differential equations and the Mellin transform. This method can be applied to other types of options and has given satisfactory results (see [3]).

Recently, [6] also used the Mellin transform to price European options on non dividend-paying stocks. However, the explicit expression obtained depends on the inverse Mellin transform which, as noted in [6, Remark, p. 32], cannot always be expressed with a closed formula and would then require a numerical estimate.

Additionally, the solution obtained, which we reproduce here [6, (12) p. 31]:

$$C(S,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} S^{-(\alpha+i\tau)} f^*(\alpha+i\tau)e^{p(\tau)(t-T)}d\tau ,$$

poses some practical difficulties with the term $f(\alpha+i\tau)$, that is, with the final condition of the Cauchy problem that gives the value of the option, since it may not be Mellin transformable.

In this paper we show that a closed formula different from that given in [6, (12) p. 31] can be obtained and that it has practical applications. Our formula is also valid for European options on dividend-paying stocks.

To this end, in section 2 we will give a brief exposé on the Mellin transform, presenting the necessary formulas and results for understanding section 3, where we proceed to price European options and obtain a closed formula. We finish in section 4 by implementing two small routines with the Mathematica symbolic calculation software, and with various practical examples which show that the results given by our formula and that of Black-Scholes coincide perfectly.

2 Integral Transform Theory

In this section we give an introduction to the Mellin transform ([4], [9]). This transform is named after the Finn Robert Hjalmar Mellin (1854 - 1933), and the Mellin transform of $C(S,t)$ is defined by ([9, p. 273]):

$$\hat{C}(p,t)=M\left\{C(S,t):S,p\right\}=\int_0^{\infty} S^{p-1}C(S,t)dS \quad (\operatorname{Re} p>0)$$
(2.1)

the inverse of which is given by:

$$C(S,t)=M^{-1}\left\{C(p,t):p,S\right\}=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\hat{C}(p,t)S^{-p}dp\,, \quad (2.2)$$

Additionally, the Mellin convolution [9, p.276] is denoted by * and is:

$$\left(f*g\right)(x)=\int_0^{\infty} f\left(\frac{x}{y}\right)g(y)\frac{1}{y}dy. \quad (2.3)$$

As with other integral transforms, the Melllin transform has some useful properties with respect to the derivative [4, (1954), (11) p. 307]:

$$M\left\{\left(S\frac{d}{dS}\right)^2 C(S,t);S,p\right\}=p^2\cdot M\left\{C(S,t);S,p\right\} \quad (2.4)$$

$$M\left\{\left(S\frac{d}{dS}\right)C(S,t);S,p\right\}=-p\cdot M\left\{C(S,t);S,p\right\} \quad (2.5)$$

3. European Option Pricing

3.1. CALL and PUT Pricing

Black-Scholes-Merton ([1], [8]) showed that, under certain market assumptions, the value of a European call option $C\equiv C(S,t)$ satisfies the following Cauchy problem ([5], [10], [11]):

$$\begin{cases}\dfrac{\partial C}{\partial t}+\dfrac{1}{2}\sigma^2S^2\dfrac{\partial^2 C}{\partial S^2}+(r-d)S\dfrac{\partial C}{\partial S}-rC=0, 0<S<\infty, 0\le t<T,\\ C(S,T)=f(S)=(S-K)^+,\\ C(0,t)=0,\\ C(S,t)\sim S\,e^{-d(T-t)}, \text{when } S\to\infty,\end{cases}$$
(3.1)

where the value of the option depends on the price of the underlying asset S, the volatility σ, the strike price K, the expiration date T, the dividends d

which are paid out continuously, and the constant interest rate r.

For the case of a European put option the Cauchy problem is as follows:

$$\begin{cases} \frac{\partial P}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 P}{\partial S^2}+(r-d)S\frac{\partial P}{\partial S}-r\,C=0, 0<S<\infty, 0\le t<T, \\ P(S,T)=f(S)=(K-S)^+, \\ P(0,t)=K\,e^{-r(T-t)}, \\ P(S,t)\to 0, \text{when } S\to\infty, \end{cases}$$
(3.2)

First we shall price the call option. The process for the put option is similar, the difference being in the final condition of the Cauchy problem.

We want to solve (3.1) using (2.4)-(2.5). For this we need to carry out some simple operations so that the PDE in (3.1) takes the form:

$$\frac{\partial}{\partial t}C(S,t)+\frac{1}{2}\sigma^2\left(S\frac{\partial}{\partial S}\right)^2 C(S,t)+ \left(r-d-\frac{1}{2}\sigma^2\right)\left(S\frac{\partial}{\partial S}\right)C(S,t)-r\,C(S,t)=0 \qquad (3.3)$$

Next, we apply the Mellin transform to the PDE in (3.3) and, using (2.4)-(2.5), we get

$$\frac{\partial}{\partial t}\hat{C}(p,t)+\left(\frac{1}{2}\sigma^2 p^2-\left(r-d-\frac{1}{2}\sigma^2\right)p-r\right)\hat{C}(p,t)=0,$$
(3.4)

with the final condition $\hat{C}(p,T)=\hat{f}(p)$.

The differential equation in (3.4) is of first order and easily solved:

$$\hat{C}(p,t)=\hat{f}(p)\cdot e^{\left(\frac{1}{2}\sigma^2 p^2-\left(r-d-\frac{1}{2}\sigma^2\right)p-r\right)(T-t)} = \hat{f}(p)\cdot e^{\frac{1}{2}\sigma^2(T-t)\left((p+\alpha)^2-\alpha^2-\frac{2r}{\sigma^2}\right)}, \qquad (3.5)$$

where $\alpha=\frac{d}{\sigma^2}+\frac{1}{2}-\frac{r}{\sigma^2}$.

Applying the inverse Mellin transform to (3.5) we get

$$C(S,t)=f(S)*M^{-1}\left\{e^{\frac{1}{2}\sigma^2(T-t)\left((p+\alpha)^2-\alpha^2-\frac{2r}{\sigma^2}\right)};p,S\right\}$$

$$=e^{\beta(T-t)}f(S)*M^{-1}\left\{e^{\frac{1}{2}\sigma^2(T-t)(p+\alpha)^2};p,S\right\}, \qquad (3.6)$$

where $\beta=-\frac{1}{2}\sigma^2\left(\alpha^2+\frac{2r}{\sigma^2}\right)$.

Moreover, using [4, 7.2 (1), p.344] and making the variable substitution $p+\alpha=x$ we have

$$M^{-1}\left\{e^{\frac{1}{2}\sigma^2(T-t)(p+\alpha)^2};p,S\right\}=S^{\alpha}M^{-1}\left\{e^{\frac{1}{2}\sigma^2(T-t)(p+\alpha)^2};x,S\right\}$$

$$=S^{\alpha}\frac{1}{2}\pi^{-1/2}\left(\frac{1}{2}\sigma^2(T-t)\right)^{-1/2}e^{-\frac{1}{4}\left(\frac{1}{2}\sigma^2(T-t)\right)^{-1}(\ln S)^2}. \qquad (3.7)$$

Then, from (3.6), (3.7), and the expression for the Mellin convolution (2.3) we obtain:

$$C(S,t)=f(S)*M^{-1}\left\{e^{\frac{1}{2}\sigma^2(T-t)\left((p+\alpha)^2-\alpha^2-\frac{2r}{\sigma^2}\right)};p,S\right\}$$

$$=\frac{1}{\sigma\sqrt{2\pi(T-t)}}e^{\beta(T-t)}\int_0^{\infty}(S/y)^{\alpha}\,e^{-\frac{\left(\ln\frac{S}{y}\right)^2}{2\sigma^2(T-t)}}f(y)\frac{1}{y}dy$$

Lastly, if we impose the final condition $f(S)=(S-K)^+$, the value of the option $C(S,t)$ is given by:

$$C(S,t)=\frac{1}{\sigma\sqrt{2\pi(T-t)}}e^{\beta(T-t)}\int_K^{\infty}(S/y)^{\alpha}\,e^{-\frac{\left(\ln\frac{S}{y}\right)^2}{2\sigma^2(T-t)}}(y-K)\frac{1}{y}dy$$

where $\alpha=\frac{d}{\sigma^2}+\frac{1}{2}-\frac{r}{\sigma^2}$ and $\beta=-\frac{1}{2}\sigma^2\left(\alpha^2+\frac{2r}{\sigma^2}\right)$.

Thus we obtain an integral expression for the pricing of a European call option.

Note that by changing the final condition we easily obtain the value of a put option:

$$P(S,t)=\frac{1}{\sigma\sqrt{2\pi(T-t)}}e^{\beta(T-t)}\int_0^{K}(S/y)^{\alpha}\,e^{-\frac{\left(\ln\frac{S}{y}\right)^2}{2\sigma^2(T-t)}}(K-y)\frac{1}{y}dy$$

where $\alpha=\frac{d}{\sigma^2}+\frac{1}{2}-\frac{r}{\sigma^2}$ and $\beta=-\frac{1}{2}\sigma^2\left(\alpha^2+\frac{2r}{\sigma^2}\right)$.

3.2. Comparison with Black-Scholes and Binomial models. Examples.

In this section we provide some examples so as to compare the results obtained with our formula with those given by the known Black-Scholes-Merton formula ([1], [8]) and Binomial Option Pricing Model [2].

For this, we shall introduce in Mathematica the two formulas.

The classic Black-Scholes formula:

```
d1[S_, K_, σ_, r_, d_, t_, T_] =
  1/(σ Sqrt[T-t]) (Log[S/K] + (r - d + σ^2/2) (T - t));
d2[S_, K_, σ_, r_, d_, t_, T_] =
  d1[S, K, σ, r, d, t, T] - σ Sqrt[T-t];
Norma[z_?NumberQ] = N[0.5 + 0.5 Erf[z/Sqrt[2]]];
BSCall[S_, K_, σ_, r_, d_, t_, T_] :=
  S e^(-d (T-t)) Norma[d1[S, K, σ, r, d, t, T]] -
  K e^(-r (T-t)) Norma[d2[S, K, σ, r, d, t, T]]
```

And the one obtained in this paper.

```
α[S_, K_, σ_, r_, d_, t_, T_] = (d/σ^2 + 1/2 - r/σ^2);
β[S_, K_, σ_, r_, d_, t_, T_] =
  -1/2 σ^2 (α[S, K, σ, r, d, t, T]^2 + 2r/σ^2);
BSCallMellin[S_, K_, σ_, r_, d_, t_, T_] :=
  1/(σ Sqrt[2π (T-t)]) e^(β[S,K,σ,r,d,t,T] (T-t))
  ∫_K^∞ (y - K) (S/y)^α[S,K,σ,r,d,t,T] e^(-(Log[S/y])^2/(2σ^2 (T-t))) 1/y dy
```

Next we provide some examples. In the first we will verify that the Put-Call parity holds.

Example 1. Put-call parity.

Let us consider an option on a non dividend-paying asset with the following values: $S = 30\,€$, $K = 29\,€$, $r = 5\,\%$ annual continuous, $\sigma = 0.25$ annual, and a time until option expiration of 4 months.

a) Calculate the price of the option if it is a European call.

```
BSCall[30, 29, 0.25, 0.05, 0, 0, 4/12]
2.52515
BSCallMellin[30, 29, 0.25, 0.05, 0, 0, 4/12]
2.52515
```

We see that our result coincides with that given by the Black-Scholes formula.

b) Calculate the price of the option if it is a European put.

```
BSPut[30, 29, 0.25, 0.05, 0, 0, 4/12]
1.04582
BSPutMellin[30, 29, 0.25, 0.05, 0, 0, 4/12]
1.04582
```

We see that the formulas for the put option also match.

c) Verify that the put-call parity is satisfied.

```
{S = 30, K = 29, r = 0.05, T = 4/12};
c = BSCallMellin[30, 29, 0.25, 0.05, 0, 0, 4/12];
p = BSPutMellin[30, 29, 0.25, 0.05, 0, 0, 4/12];
p + S
31.0458
c + K e^(-r T)
31.0458
```

As expected, the put-call parity is also satisfied.

Example 2. *European options on dividend-paying assets.*

Calculate the value of a European call option with three months to go until expiration, on the “Standard and Poor’s 500” index (S&P 500), with a current price of \$250, a strike price of \$250, a continuously compounded interest rate of 2.07, a volatility of 18%, and a constant annual index dividend estimated at 3%

So the values are as follows: $S = 250$, $K = 250$, $r = 2.07$, $\sigma = 0.18$, $d = 0.03$.

```
BSCall[250, 250, 0.18, 0.0207, 0.03, 0, 3/12]
```

```
8.63068
```

```
BSCallMellin[250, 250, 0.18, 0.0207, 0.03, 0, 3/12]
```

```
8.63068
```

We see that they also coincide.

Example 3. Comparison with Binomial Option Pricing Model.

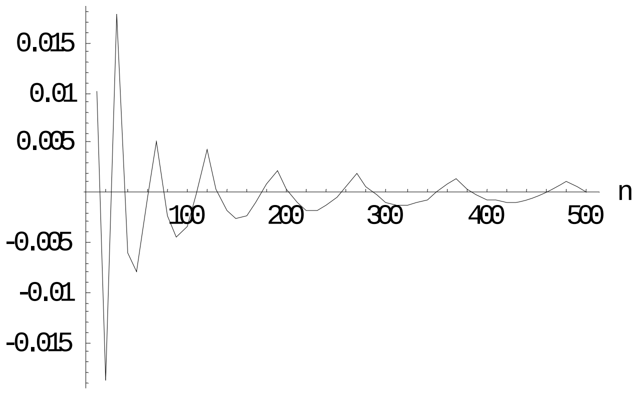
In this example we will consider an option on a non dividend-paying asset with the following values: $S = 50$€, $K = 45$€, $r = 5\,\%$ annual continuous, $\sigma = 0.25$ annual, and a time until option expiration of 4 months.

In Mathematica the binomial option pricing formula can be given as:

```
BinomialEuropeanOption[s_, σ_, T_, r_,
  exercise_Function, n_] :=
 Module[{
   u = N[Exp[Sqrt[T/n] * σ]],
   d = N[Exp[-Sqrt[T/n] * σ]],
   R = N[Exp[r * T/n]]},
  p = (R - d) / (R * (u - d));
  q = (u - R) / (R * (u - d));
  Sum[exercise[s * u^j * d^(n - j)] *
    Binomial[n, j] * p^j * q^(n - j), {j, 0, n}]]
BinomialEuropeanCall[s_, x_, σ_, T_, r_, n_] :=
  BinomialEuropeanCall[s, x, σ, T, r, n] =
   BinomialEuropeanOption[s, σ, T, r,
    Max[#1 - x, 0] &, n];
```

We can now show that this binomial pricing formula converges to our formula:

```
ListPlot[
 Table[
  {n, BSCallMellin[50, 45, 0.25, 0.05, 0, 0,
     4/12] - BinomialEuropeanCall[50, 45,
     0.25, 4/12, 0.05, n]}, {n, 10, 500, 10}],
 PlotJoined → True, PlotRange → All,
 AxesLabel → {n, ""}]
```



4 Conclusion

In this paper we have proposed a different method for pricing European options. Furthermore, this method can be applied to other financial derivatives. We have obtained a closed integral formula for pricing European options on dividend-paying or non paying assets. We have done this by using the theories of partial differential equations and integral transforms. Unlike other recent results, our method is consistent and is applicable to financial practices. Lastly, we have used several examples to show that the results coincide perfectly with those given by the Black-Scholes-Merton formula.

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