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Splitting Method for Variational Formulation of Nonlinear Differential Operators

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Abstract: - This paper deals with the inverse problem of the calculus of variations. A new method for reach a variational formulation of non-potent operators is introduced. A variational formulation to complete Navier-Stokes equations has been developed.

Key-Words: - Calculus of variations, Inverse problem, Adjoint of a nonlinear operator, Navier-Stokes equation.

1 Introduction

Variational principles are mainly used today in the following contexts:
a) For obtaining the differential equations for a physical problem, together with corresponding boundary conditions;
b) For the study of symmetry and conservation laws under infinitesimal transformation groups;
c) For providing that a boundary problem is solvable (i.e., for showing the existence of solutions for nonlinear equations);
d) For obtaining solutions of linear and nonlinear problems using direct methods of variational calculus.

The use of variational principles in cases “c” or “d” above, leads to the so called “inverse problem of the calculus of variation”, i. e. the existence and formulation of functional $F[u]$, whose variation, being vanished, supplies the boundary problem in consideration.

The pioneer work of Vainberg [1] (whose first results were obtained in 1954), based on concepts of functional analysis, remained inaccessible to many applied mathematicians and engineers, until its importance was recognized by Enzo Tonti [2, 3].

Tonti, in his turn, changed the Vaimberg’s work in a practical device, by developing a procedure to derive many operational formulae that make it possible to determine whether a given operator is potential or not.

According to [4] and [5], it is not always simple to give a variational formulation to a mathematical problem. In particular, in the absence of symmetry of the operator governing the problem, with respect to a suitable bilinear form, it is impossible to construct a relevant variational formulation. This difficulty has prompted, already in the early 1950’s, the study of symmetrisation methods.

In [6] the authors make a very didactical classification of the methods of symmetrisation in four main classes, as summarized bellow:
a) Method of additional variables or dual principles;
b) Method of integrating operator;
c) Method of transformation of variables; and,
d) Method of modifications.

The essence of the method of additional variables or dual principles is that one or more variables and equations are introduced into the problem, thus extending the corresponding function space in such a way that the original operator will be part of a potent operator. The most important problem of this method is the meaningless of the new variables and equations. This method was first proposed, for linear operators, by Morse and Feshbach [7], under the name of “adjoint operator method”. This method was extended to the general, nonlinear case by Finlayson [8].

The method of integrating operator consists in considering, instead of the operator governing the problem, a composite operator created in a special way.

The method of transformation of variables means that we change the independent variable into an operator, more precisely, instead of the original operator of the given problem, we consider a composite operator. The best known example of this method is the variational principle for the Maxwell's equations in vacuum. These equations are non-potent in their original variables $(\vec{E},\vec{B})$, but introducing the scalar and the vector potential $(\phi,\vec{A})$, the resulting wave equations are potent. This special example motivates that the introduced new variables are called potentials [9].

The method of modifications is, sometimes, referred as quasi-variational principle, or restricted variational principle, and it can be subdivided into two approaches: modified operators and modified function space. Initially one, the operator is modified in such a way that the transformed operator will be potent. The domain of the modified operator is the same as the original one. Variational potentials exist only for the modified operator, not for the original one. Variational principles coming from this method are usually believed to be valid in a more general sense than they really are. For instance, the resulting Euler-Lagrange equations are transformed to get back the original operator.

The method of modified function spaces means that the domain of the original operator is restricted, so that the originally non-potent operator becomes potent on the restricted domain. Thus the operator is modified by restricting its domain instead of its shape.

We shall say that the classification above is only didactical, and a mixture of these methods can be done in order to derive a variational formulation of an operator. A detailed discussion on classification of symmetrisation methods can be find in [8].

In this paper we present a new method to derive a variational formulation of linear and nonlinear operators, which is based in splitting an non-potent operator into two operators: a symmetric one and a skew-symmetric one. With respect to the symmetric operator we follow the procedure established by Tonti [2, 3]. For the skew-symmetric operator, we define a bilinear form, according with the operator will be symmetric.

In section 2, we will introduce the method. In section 3 will be presented an example of a linear operator. In section 4, will be presented an application of the method for a nonlinear operator. The chosen example was the complete Navier-Stokes equations. Finally, to finish this paper, section 5 we will discuss the remarkable aspects of the results of the application of the method.

2 Method of Splitting

Let $T(\cdot)$ be an operator, which is non-potent. According to this method, if $T(\cdot)$ has an adjoint, $\tilde{T}(\cdot)$, it is possible to decompose the operator $T(\cdot)$ into two operators: a symmetric one, $T_S(\cdot)$, and other skew-symmetric $T_{SKEW}(\cdot)$, such as

$$T\left(\cdot\right)=T_S\left(\cdot\right)+T_{SKEW}\left(\cdot\right) \tag{1}$$

The symmetric and skew-symmetric parts can be reached following the procedure bellow

$$T_S\left(\cdot\right)=\frac{T\left(\cdot\right)+\tilde{T}\left(\cdot\right)}{2} \tag{2}$$

$$T_{SKEW}\left(\cdot\right)=\frac{T\left(\cdot\right)-\tilde{T}\left(\cdot\right)}{2} \tag{3}$$

The next step is to define the bilinear forms for the symmetric and skew-symmetric part of the operator. The bilinear form for the symmetric operator can be the usual inner product, while for the skew-symmetric operator; it should be defined according to the nature of the operator. Finally supposing the operator T acting on u, $T(u)$, we can write the functional $T[u]$, as follows

$$T[u]=\int_0^1\left\langle T_S\left(\lambda u\right),u\right\rangle_S d\lambda+\int_0^1\left\langle T_{SKEW}\left(\lambda u\right),u\right\rangle_{SKEW} d\lambda \tag{4}$$

3 The RLC Serial Circuit

This is the case of a linear operator:

$$T(q)=Lq_{tt}+Rq_t+\frac{1}{C}q$$

extracted from a RLC serial circuit, where L is the inductance of the circuit, C the capacitance, R the resistance, q the charge flowing through the circuit and q_t, q_{tt} are the first and second, respectively, derivatives of the charge with respect to the time. This operator can be rewritten as

$$T\left(q\right)=\left(LD_{tt}+RD_{t}+\frac{1}{C}\right)\left(q\right)\therefore T\left(\cdot\right)= \\ =\left(LD_{tt}+RD_{t}+\frac{1}{C}\right)\left(\cdot\right) \quad (5)$$

The adjoint of (5) is

$$\tilde{T}\left(\cdot\right)=\left(LD_{tt}-RD_{t}+\frac{1}{C}\right)\left(\cdot\right) \quad (6)$$

According to (2) and (3), we can write

$$T_{S}\left(\cdot\right)=\left(LD_{tt}+\frac{1}{C}\right)\left(\cdot\right)$$

and

$$T_{SKEW}\left(\cdot\right)=RD_{t}\left(\cdot\right) \quad (7)$$

Now, let us define bilinear forms

$$\left\langle u,v\right\rangle_{S}=\int_{t}u\cdot vdt=\left\langle v,u\right\rangle_{S} \\ \left\langle u,v\right\rangle_{SKEW}=\int_{t}u\cdot\frac{dv}{dt}dt=-\left\langle v,u\right\rangle_{SKEW} \quad (8)$$

The functional for $T(q)$ can be written as

$$T[q]=\int_{0}^{1}\left\langle T_{S}\left(\lambda q\right),q\right\rangle_{S}d\lambda+ \\ +\int_{0}^{1}\left\langle T_{SKEW}\left(\lambda q\right),q\right\rangle_{SKEW}d\lambda \quad (9)$$

From (8), (9) becomes

$$T[q]=\frac{1}{2}\int_{t}\left(-Lq_{t}^{2}+\frac{1}{C}q^{2}\right)dt+\frac{1}{2}\int_{t}Rq_{t}^{2}dt \quad (10)$$

4 The Navier-Stokes Equations

In order to exemplify this method we have chosen the Navier-Stokes equations, because, according to [8] and [10] there isn't a potential, and consequently a variational formulation for them.

Let us consider a bidimensional compressible flowing off, isothermal with viscosity $\mu=\mu(x,y)$, and viscosity coefficient

$$\xi=\lambda+\frac{2}{3}\mu$$

with velocity $\vec{u}=(u,v)$, pressure P and specific mass ρ, described by the following equations:

$$\begin{cases}-\mu\left(u_{xx}+u_{yy}\right)-\xi\left(u_{xx}+v_{xy}\right)+P_{x}+\rho\left(u_{x}u+u_{y}v\right)=0\\ -\mu\left(v_{xx}+v_{yy}\right)-\xi\left(u_{xy}+v_{yy}\right)+P_{y}+\rho\left(v_{x}u+v_{y}v\right)=0\end{cases} \quad (11)$$

Equations (11) can be joined at an unique equation, and then it's possible to extract an operator, since now on, called $T(\cdot)$, as follows

$$T(\vec{u})=-\nabla^{2}\left(\mu\vec{u}\right)-\vec{\nabla}\left(\xi\vec{\nabla}\cdot\vec{u}\right)+\vec{\nabla}P+\rho\left(\vec{\nabla}\vec{u}\cdot\vec{u}\right), \quad (12)$$

where

$$\vec{u}=\begin{bmatrix}u\\ v\end{bmatrix}\ ;\ \vec{\nabla}=\begin{bmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial y}\end{bmatrix}\ ;\ \vec{\nabla}\vec{u}=\begin{bmatrix}u_{x}&u_{y}\\ v_{x}&v_{y}\end{bmatrix}\ ;$$

$$\vec{\nabla}P=\begin{bmatrix}P_{x}\\ P_{y}\end{bmatrix}\ ;\ u_{x}=\frac{\partial u}{\partial x}\ ;\ P_{x}=\frac{\partial P}{\partial x}$$

Now, consider the following two bilinear forms:

a) Symmetric bilinear form

$$\sigma_{1}\left\langle\begin{bmatrix}a_{1}\\ a_{2}\end{bmatrix},\begin{bmatrix}b_{1}\\ b_{2}\end{bmatrix}\right\rangle=\int_{\Omega}\left(a_{1}b_{1}+a_{2}b_{2}\right)dxdy \quad (13)$$

b) Skew-symmetric bilinear form

$$\sigma_{2}\left\langle\begin{bmatrix}a_{1}\\ a_{2}\end{bmatrix},\begin{bmatrix}b_{1}\\ b_{2}\end{bmatrix}\right\rangle_{SKEW-SYMMETRIC}=\int_{\Omega}\left(a_{1}b_{2}-a_{2}b_{1}\right)dxdy \quad (14)$$

The operator $T(\cdot)$ is a nonlinear operator. By inspection, it can be seen that it has a symmetric part $T_{S}(\cdot)$.

$$T_{S}\left(\vec{u}\right)=-\nabla^{2}\left(\mu\vec{u}\right)-\vec{\nabla}\left(\xi\vec{\nabla}\cdot\vec{u}\right)+\vec{\nabla}P \quad (15)$$

and a skew-symmetric part $T_{SKEW}(\cdot)$:

$$T_{SKEW}(\vec{u}) = \rho\left(\vec{\nabla}\vec{u}\cdot\vec{u}\right) \tag{16}$$

Thus,

$$T(\cdot) = T_S(\cdot) + T_{SKEW}(\cdot) \tag{17}$$

Now, let us make the variational formulation for the symmetric operator $T_S(\cdot)$ in (15)

$$T_S[\vec{u}] = \sigma_1\left\langle \int_0^1 T_S(\lambda\vec{u})\,d\lambda, \vec{u}\right\rangle =$$
$$= \sigma_1\left\langle \int_0^1 \left(-\nabla^2(\mu\lambda\vec{u}) - \vec{\nabla}\left(\xi\vec{\nabla}\cdot\lambda\vec{u}\right) + \vec{\nabla}P\right)d\lambda, \vec{u}\right\rangle =$$
$$= \frac{1}{2}\left\langle -\nabla^2(\mu\vec{u})\cdot\vec{u} - \vec{\nabla}\left(\xi\vec{\nabla}\cdot\vec{u}\right), \vec{u}\right\rangle + \left\langle \vec{\nabla}P\cdot\vec{u}\right\rangle =$$
$$= \frac{1}{2}\left\langle -\nabla^2(\mu\vec{u}), \vec{u}\right\rangle + \frac{1}{2}\left\langle -\vec{\nabla}\left(\xi\vec{\nabla}\cdot\vec{u}\right), \vec{u}\right\rangle + \left\langle \vec{\nabla}P\cdot\vec{u}\right\rangle =$$
$$= \frac{1}{2}\left\langle -\vec{\nabla}\cdot\vec{\nabla}(\mu\vec{u}), \vec{u}\right\rangle + \frac{1}{2}\left\langle -\vec{\nabla}\left(\xi\vec{\nabla}\cdot\vec{u}\right), \vec{u}\right\rangle + \left\langle \vec{\nabla}P\cdot\vec{u}\right\rangle =$$
$$= \frac{1}{2}\left\langle \mu\vec{\nabla}\vec{u}, \vec{\nabla}\vec{u}\right\rangle + \frac{1}{2}\left\langle \xi\left(\vec{\nabla}\cdot\vec{u}\right), \left(\vec{\nabla}\cdot\vec{u}\right)\right\rangle + \left\langle \vec{\nabla}P\cdot\vec{u}\right\rangle$$

Then

$$T_S[\vec{u}] = \frac{1}{2}\int_\Omega \left(\mu\left(\vec{\nabla}\vec{u}\right)^2 + \xi\left(\vec{\nabla}\cdot\vec{u}\right)^2\right)d\Omega + \int_\Omega \vec{\nabla}P\cdot\vec{u}\,d\Omega \tag{18}$$

The variational formulation for the skew-symmetric operator (16), can be reached as follows

$$T_{SKEW}[\vec{u}] = \sigma_2\left\langle \int_0^1 T_{SKEW}(\lambda\vec{u})\,d\lambda, \vec{u}\right\rangle =$$
$$= \sigma_2\left\langle \int_0^1 \rho\left(\vec{\nabla}(\lambda\vec{u})\cdot(\lambda\vec{u})\right)d\lambda, \vec{u}\right\rangle =$$
$$= \frac{1}{3}\sigma_2\left\langle \rho\left(\vec{\nabla}(\vec{u})\cdot(\vec{u})\right), \vec{u}\right\rangle$$

According to (14.0),

$$T_{SKEW}[\vec{u}] = \frac{1}{3}\sigma_2\left\langle \rho\begin{bmatrix} u_x & u_y \\ v_x & v_y\end{bmatrix}\begin{bmatrix} u \\ v\end{bmatrix}, \begin{bmatrix} u \\ v\end{bmatrix}\right\rangle =$$
$$= \frac{1}{3}\sigma_2\left\langle \begin{bmatrix} \rho(u_x u + u_y v) \\ \rho(v_x u + v_y v)\end{bmatrix}, \begin{bmatrix} u \\ v\end{bmatrix}\right\rangle =$$
$$= \frac{1}{3}\left\langle \rho\left[u_x uv + u_y v^2 - v_x u^2 - v_y vu\right]\right\rangle$$

Integrating by parts, we have

$$T_{SKEW}[\vec{u}] = \frac{1}{3}\left\langle \rho\left[u_x uv + 2u_x uv - 2v_y vu - v_y vu\right]\right\rangle =$$
$$= \frac{1}{3}\left\langle \rho\left[3u_x uv - 3v_y vu\right]\right\rangle$$

Thus

$$T_{SKEW}[\vec{u}] = \int_\Omega \rho uv\left(u_x - v_y\right)dxdy \tag{19}$$

In varying (19), we should return to (16). Thus, let $\delta T_{SKEW}[\vec{u}, \delta\vec{u}] = 0$. Then let us consider $\delta\vec{u} = \begin{bmatrix} h \\ k\end{bmatrix}$, then

$$\delta T_{SKEW}[\vec{u}, \delta\vec{u}] = \int_\Omega \rho\Big\{v\left(u_x - v_y\right)h + uvh_x$$
$$+ u\left(u_x - v_y\right)k - uvk_y\Big\}dxdy$$

Integrating by parts, we obtain

$$\delta T_{SKEW}[\vec{u}, \delta\vec{u}] = \int_\Omega \rho\Big\{v\left(u_x - v_y\right)h - \left(u_x v + uv_x\right)h +$$
$$+ u\left(u_x - v_y\right)k + \left(u_y v + uv_y\right)k\Big\}dxdy =$$
$$= \int_\Omega \rho\left\{-\left(vv_y + uv_x\right)h + \left(uu_x + u_y v\right)k\right\}dxdy =$$
$$= \int_\Omega \rho\left\{\begin{bmatrix} uu_x + u_y v \\ uv_x + vv_y\end{bmatrix}, \begin{bmatrix} h \\ k\end{bmatrix}\right\}dxdy =$$
$$= \int_\Omega \rho\left\{\begin{bmatrix} u_x + u_y \\ v_x + v_y\end{bmatrix}\begin{bmatrix} u \\ v\end{bmatrix}, \begin{bmatrix} h \\ k\end{bmatrix}\right\}dxdy$$

$$= \int_\Omega \rho\left(\vec{\nabla}\vec{u}\cdot\vec{u}\right)\cdot\delta\vec{u}\,dxdy = 0 \therefore \rho\left(\vec{\nabla}\vec{u}\cdot\vec{u}\right) = 0 \tag{20}$$

We can observe that (20) matches with (16), what assure us that (19) is correct. Thus we can join (18) and (19) in a unique functional.

$$T\left[\vec{u}\right]=\frac{1}{2}\sigma_1\left\langle \mu\vec{\nabla}\vec{u},\vec{\nabla}\vec{u}\right\rangle+\frac{1}{2}\sigma_1\left\langle \xi\vec{\nabla}\cdot\vec{u},\vec{\nabla}\vec{u}\right\rangle+ \\ +\sigma_1\left\langle \vec{\nabla}P,\vec{u}\right\rangle-\sigma_2\left\langle \rho\left(\vec{u}\cdot\vec{\nabla}\right)\vec{u},\vec{u}\right\rangle \quad (21)$$

5 Conclusions

A new method for given a variational formulation to non-potential operators was presented. As an example of the application of this method, it was given a variational formulation to complete Navier-Stokes equations. Which, until now, was believed not have one, since it is a non-potent operator.

The necessary condition, for applying this method, is the existence of an adjoint for the operator to be decomposed.

For each operator in study, it will be necessary to define a skew-symmetric bilinear form, which, with respect to itself, becomes the skew symmetric part of the operator, symmetric.

It is important to note that, in terms of energy, the symmetric part of the operator corresponds to the conservative part of the system, while the skew-symmetric corresponds to dissipation.

Observing (10.0), it can be seen that the functional $T\left[q\right]$ can be divided into a part that deals with the electric and magnetic energies and a second part related to the power dissipation.

Interpreting (18.0), it can be seen that in the functional $T_S\left[\vec{u}\right]$ can be found two terms, one of them responsible for mechanical work while the other due to P. In $T_{SKEW}\left[\vec{u}\right]$ is found the work due to the vorticity.

An overall view of the method shows that it has an straight forward application, and permit us to deal with a large class of problems, reaching important results, allowing us to give an physical meaning to them.

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