Proceedings of the 8th WSEAS International Conference on APPLIED MATHEMATICS, Tenerife, Spain, December 16-18, 2005 (pp379-385)

Some Numerical Methods of Rational Characterization in Causal Time Series Models

GONZÁLEZ-CONCEPCIÓN, CONCEPCIÓN
GIL-FARIÑA, MARÍA CANDELARIA
PESTANO-GABINO, CELINA
Department of Applied Economics
University of La Laguna
Campus de Guajara, s/n
38701 La Laguna. Tenerife. Canary Islands
SPAIN

Abstract: The systematic study of data to obtain specific properties from long (or short) data series is a main objective. The use of rational models and related numerical methods can be useful to predict the behaviour of relevant economic variables.
This paper is a continuation of González-Gil [16] which is concerned with illustrating the application of several numerical methods, among them, the corner method, epsilon-algorithm, rs-algorithm and qd-algorithm to time series modelling. These methods which are closely related to theoretical research in Padé Approximation have been proposed to identify some type of rational structure associated to economic data in different contexts (financial, marketing, farming…). Now, we present the study of the statistical significance for the four mentioned methods. Two examples will be considered, namely, a simulated ARMA model and a Transfer Function Model for the sales series M given in Box-Jenkins [7] and Tsay [27].

Key-Words: Padé Approximation, economics, numerical methods, time series modelling.

1 Introduction

Over the last two decades, several research activities have helped to obtain new procedures and techniques to characterize dynamic relations associated to data series.
In the context of time series analysis, several authors (Lii [18], Claverie et al. [9], Berlinet-Francq [5]…) have considered the rational theory of series in econometric modelling. From this perspective, several techniques closely related to Padé Approximation (PA) have been proposed to identify possible rational structures associated to chronological data. As the covariance structure of underlying processes exhibits features connected with the order of models, it is possible to use numerical algorithms (*corner method*, *epsilon-algorithm*, *rs-algorithm and qd-algorithm*) linked with Hankel and determinants to estimate the unknown orders from available observations.
The contribution of this paper is the study of the statistical significance of two of these numerical methods, that is, the *rs-algorithm* and *qd-algorithm* as a continuation of the work made in González-Gil [16].
Consideration is given to both the univariate and the multivariate multivariate cases.
In the univariate case, the identification of ARMA models has been extensively considered in the last two decades (Beguin et al [2], Mareschal-Mélard [21], Claverie et al [9], Berlinet-Francq [5]).
As for the multivariate case, some results have been given to identify a VARMA model (Tiao-Tsay [26], Reinsel [24], Lütkepohl-Poskitt [20], Pestano-González [22], Berlinet-Francq [6]...) and, as a particular case, a Transfer-Function (TF) model (Liu-Hanssens [19], Lii [18], González et al [15]...).
Next, we show the theoretic characterization of these techniques in a TF model with one output and one or multiple inputs in a causal way.
The empirical work is carried out in the context of the Box-Jenkins´s [7] guidelines. Both proposals are illustrated in both univariate and multivariate cases, considering a simulated ARMA model and a Transfer-Function Model for the sales series M given in Box-Jenkins [7] and Tsay [27].

2 The Univariate Case: Some Methods of Rational Characterization in ARMA Models

Let us consider a minimal stationary and invertible Autoregressive Moving Average (ARMA) model of order (p,q) defined as

$$\Phi_p(L)X_t = \Theta_q(L)a_t, \quad \forall t \in Z$$

where L is the backward-shift operator, that is, $L^m X_t = X_{t-m}, \forall t \in Z$, $\Phi_p(L)$, $\Theta_q(L)$ are polynomials of degree p and q respectively and $\{a_t; t = 0, \pm 1, \pm 2, ...\}$ is a sequence of independently and identically distributed random variables with mean zero and variance σ^2_a. It is assumed that $\Phi_p(L)$ and $\Theta_q(L)$ have no common factors.
Various methods related to PA have been proposed to identify the orders p and q. For instance, the *C-Table method* (Baker-Graves-Morris [1]) from its properties it can be obtained the *corner method* in econometric literature (Beguin et al [2]). Many later papers have also considered the *corner method* in ARMA modelling, trying to get to the maximum of their power (Mareschal-Mélard [21]). Also, Beguin et al [2] studied the statistical significance of the *C-table*. Later, Tsay [27]

and Lii [18] proposed to consider an estimator of the asymptotic variance in terms of the partial derivatives of the entries in the *C-table*.

The relation of this method with the Hankel determinants and PA has stimulated the study of other algorithms in ARMA models. For instance, we can mention the *epsilon-algorithm* (Wynn [28]), proposed by Berlinet [3]. Its relation with PA and the *corner method* can be seen in Brezinski [8] and the characterization for an ARMA process in Berlinet-Francq [5]. They proposed statistical properties of the entries in the *epsilon-algorithm*, based on the same statistical and assumptions than Beguin et al [2].

We can also refer to the *rs-algorithm* (Pye-Atchison [23]), proposed by Gray et al. [17] for ARMA models and whose relation with PA can be seen in Brezinski [8]. This algorithm which is linked to determinants of the Hankel matrices associated with the sequence of autocorrelations of X_t, $\rho \equiv \{\rho_i\}_{-\infty}^{\infty}$, that is, $C_{i,j}(\rho) = \det(\rho_{i-h+k})_{h,k=1}^{j}$, is defined as

$$\forall n \in Z,\quad s_0^n(\rho) = 1,\quad r_1^n(\rho) = \rho_n$$

$$\forall k \in N, \forall n \in Z,\quad s_k^n(\rho) = s_{k-1}^{n+1}(\rho)\left(\left(r_k^n(\rho)\right)^{-1} r_k^{n+1}(\rho) - 1\right)$$

$$r_{k+1}^n(\rho) = r_k^{n+1}(\rho)\left(\left(s_k^n(\rho)\right)^{-1} s_k^{n+1}(\rho) - 1\right)$$

It can be proved that

$$\forall k \in N, \forall n \in Z,\quad r_k^n(\rho) = -\frac{C_{n+k-1,k}(\rho)}{{}_1C_{n+k-2,k-1}(\rho)},$$

$$s_k^n(\rho) = \frac{{}_1C_{n+k-1,k}(\rho)}{C_{n+k-1,k}(\rho)}$$

where ${}_1C_{ij}(\rho)$ is the determinant of the Hankel matrix associated with $L\rho$, and it is deduced that

$$r_j^{i-j+1} = 0 \Leftrightarrow C_{i,j} = 0 .$$

The study of the statistical significance of the algorithm is given in González [11] computing the values of the t-Student $\frac{s_k^n(\rho)}{\sqrt{v(s_k^n(\rho))}}$ (respectively, $\frac{r_k^n(\rho)}{\sqrt{v(r_k^n(\rho))}}$) where $v(s_k^n(\rho))$ (respectively $v(r_k^n(\rho))$) represents the variance estimated. Following Tsay [27] and Berlinet [4], the last one can be approximately be represented by

$$v(s_k^n(\rho)) \cong F_k^{n'}(\rho) M_k^n(\rho) F_k^n(\rho) \text{ (respectively,}$$

$$v(r_k^n(\rho)) \cong F_k^{n'}(\rho) M_k^n(\rho) F_k^n(\rho)$$

where $M_k^n(\rho)$ is the sample covariance matrix for the sequence $(\rho_n, \ldots, \rho_{k+n})$, that is,

$$M_k^n(\rho) = (m_{ij}^{k,n})_{i,j=1,\ldots,k+1} \;/\; m_{ii}^{k,n} = v(\rho_{k+i-1}), m_{ij}^{k,n} = \\ = \mathrm{cov}(\rho_{n+i-1}, \rho_{n+j-1})$$

and $F_k^n(\rho)$ is the sequence $({}^{n}s_k^n(\rho), \ldots, {}^{n+k}s_k^n(\rho))$ (respectively, $({}^{n}r_k^n(\rho), \ldots, {}^{n+k}r_k^n(\rho))$) where

$${}^{i}s_k^n(\rho) \equiv \frac{\partial s_k^n}{\partial \rho_i}(\rho) =$$

$${}^{i}s_{k-1}^{n+1}(\rho)\frac{r_k^n(\rho)\,{}^{i}r_k^{n+1}(\rho) - r_k^{n+1}(\rho)\,{}^{i}r_k^n(\rho)}{\left(r_k^n(\rho)\right)^2} + {}^{i}s_{k-1}^{n+1}(\rho)\left(\frac{r_k^{n+1}(\rho)}{r_k^n(\rho)} - 1\right),$$

if $i = n, n+1, \ldots, n+k$ and 0, otherwise

$${}^{i}r_{k+1}^n(\rho) \equiv \frac{\partial r_{k+1}^n}{\partial \rho_i}(\rho) =$$

$${}^{i}r_k^n(\rho)\frac{s_{k-1}^n(\rho)\,{}^{i}s_{k-1}^{n+1}(\rho) - s_{k-1}^{n+1}(\rho)\,{}^{i}s_{k-1}^n(\rho)}{\left(s_{k-1}^n(\rho)\right)^2} + {}^{i}r_k^{n+1}(\rho)\left(\frac{s_{k-1}^{n+1}(\rho)}{s_{k-1}^n(\rho)} - 1\right)$$

if $i = n, n+1, \ldots, n+k$ and 0, otherwise

where ${}^{i}s_0^n(\rho) = 0$ and ${}^{i}r_1^n(\rho) = \begin{cases} 1 & i = n \\ 0 & i \neq n \end{cases}$ ($\forall$n,i≥0) are the initial values.

With regard to the *qd-algorithm* (Rutishaüser [25]), it has been considered by Berlinet [3] to study the partial autocorrelation function in an ARMA model and by González [11] and González and Gil [16] to model identification. This algorithm is defined as

$$\forall n \in Z,\quad d_0^n(\rho) = 0,\quad q_1^n(\rho) = \frac{\rho_{n+1}}{\rho_n}$$

$$\forall k \in N, \forall n \in Z,\quad d_k^n(\rho) = q_k^{n+1}(\rho) - q_k^n(\rho) + d_{k-1}^{n+1}(\rho)$$

$$q_{k+1}^n(\rho) = \frac{d_k^{n+1}(\rho) q_k^{n+1}(\rho)}{d_k^n(\rho)}$$

Its relation with the PA is not direct (Brezinski [8]) and it can be proved that

$$\forall k \in N, \forall n \in Z,$$

$$q_k^n(\rho) = \frac{C_{n+k,k}(\rho) C_{n+k-2,k-1}(\rho)}{C_{n+k-1,k}(\rho) C_{n+k-1,k-1}(\rho)},$$

$$d_k^n(\rho) = \frac{C_{n+k-1,k-1}(\rho) C_{n+k,k+1}(\rho)}{C_{n+k-1,k}(\rho) C_{n+k,k}(\rho)}$$

It is deduced that

$$C_{i-2,j-1}(\rho) \neq 0, C_{i-1,j}(\rho) \neq 0, C_{i-1,j-1}(\rho) \neq 0,$$

$$C_{i,j}(\rho) = 0 \Rightarrow q_j^{i-j}(\rho) = 0$$

$$C_{i-1,j-2}(\rho) \neq 0, C_{i-1,j-1}(\rho) \neq 0, C_{i,j-1}(\rho) \neq 0,$$

$$C_{i,j}(\rho) = 0 \Rightarrow d_{j-1}^{i-j+1}(\rho) = 0$$

In order to study the statistical significance for the elements of the *qd-algorithm*, the same statistical and similar notations are used. Partial derivatives are computed following the next iterative procedure

$${}^{i}d_0^n(\rho) = 0, \forall n, i \geq 0$$

$${}^{i}q_1^n(\rho) = \begin{cases} -\frac{r_{n+1}}{(r_n)^2} & i = n \\ \frac{1}{r_n} & i = n+1 \\ 0 & i \neq n, n+1 \end{cases}$$

For k>0:

$$^{i}q_{k+1}^{n}(\rho) \equiv \frac{\partial q_{k+1}^{n}}{\partial \rho_i}(\rho) = \frac{(d_k^n(\rho)^i d_k^{n+1}(\rho) - d_k^{n+1}(\rho)^i d_k^n(\rho))q_k^{n+1}(\rho) + d_k^n(\rho) d_k^{n+1}(\rho)^i q_k^{n+1}(\rho)}{\left(d_k^n(\rho)\right)^2}$$

if $i = n, n+1, \ldots, n+k+1$ and 0, otherwise

$$^{i}d_k^n(\rho) \equiv \frac{\partial d_k^n}{\partial \rho_i}(\rho) = {}^{i}q_k^{n+1}(\rho) - {}^{i}q_k^n(\rho) + {}^{i}d_{k-1}^{n+1}(\rho)$$

if $i = n, n+1, \ldots, n+k+1$ and 0, otherwise

These methods can be used to find a parsimonious approximation or reduce possible competing models to only a few for further testing. Other techniques can be found, for example, in Berlinet-Francq [5].

This proposal is illustrated following the model $X_t - 0.7X_{t-1} = a_t + 0.5a_{t-1}, \quad \forall t \in Z$ simulated by Berlinet-Francq [5], where a_t is a white noise process with media zero and variance 1. Initial values were taken equal to zero; 200 values were generated but only the last 100 values were considered.

Using the *rs-algorithm*, the obtained results are

Critical Value	Accepted (p,q) models
1.28	(1,5) (2,1)
1.64	(1,1)
1.96	(1,1)
2.33	(1,1)
2.58	(1,1)
2.81	(1,1)
3.09	(1,1)
3.29	(1,1)
3.72	(1,1)
4.26	(1,1)

Using the *qd-algorithm*, results are given below:

Critical Value	Accepted (p,q) Models
1.28	(1,1)
1.64	(1,1)
1.96	(1,1)
2.33	(1,1)
2.58	(1,1) (0,2)
2.81	(1,1) (0,2)
3.09	(1,1) (0,2)
3.29	(1,1) (0,2)
3.72	(1,1) (0,2)
4.26	(1,1) (0,2)

Obtained results suggest that both methods are efficient alternatives to reproduce the simulated model.

3 The Multivariate Case: Some Methods of Rational Characterization in Causal TF Models

Let us consider a VARMA (p,q) process defined as

$$\Phi_p(L)\, Z_t = \Theta_q(L) u_t$$

where now $\Phi_p(L)$ and $\Theta_q(L)$ are matrix polynomials of dimension n and degrees p and q respectively, Z_t is a multiple process Z_t and u_t a vector of independent white noise processes. A structure of particular interest when

$$Z_t = \begin{pmatrix} X_t \\ Y_t \end{pmatrix}, \Phi_p = \begin{pmatrix} \varphi & 0 \\ \psi & \phi \end{pmatrix}, \Theta_q = \begin{pmatrix} \alpha & 0 \\ 0 & \theta \end{pmatrix}, u_t = \begin{pmatrix} c_t \\ a_t \end{pmatrix}$$

can be expressed as

$$\varphi(L)X_t = \alpha(L)c_t$$

$$\phi(L)Y_t + \psi(L)X_t = \theta(L)a_t, \quad \forall t \in Z$$

If ϕ is invertible, Y_t is given by

$$Y_t = -\phi^{-1}(L)\psi(L)X_t + N_t, \quad \forall t \in Z$$

In this expression, which is called the TF model, the output Y_t is a function of the contemporary and delayed effects of the input variable X_t. It is assumed a one-way causal relation $X_t \rightarrow Y_t$ and the presence of a disturbance series described as $N_t = \phi^{-1}(L)\theta(L)a_t$.

Here we refer to TF models with one output $Y_t \equiv y_t$ and one or multiple inputs $X_t \equiv (x_{it})_{i=1,\ldots,n}$, that is,

$$y_t = \sum_{i=1}^{n} \frac{\omega_{is_i}(L)}{\delta_{ir_i}(L)} L^{b_i} x_{it} + N_t$$

where $\omega_{is_i}(L) = \omega_{i0} + \omega_{i1}L + \ldots + \omega_{is_i}L^{s_i}$ and $\delta_{ir_i}(L) = \delta_{i0} + \delta_{i1}L + \ldots + \delta_{ir_i}L^{r_i}$, b_i is the delay in the response of y_t to x_{it} and a_t is a white noise process.

The Box-Jenkins´s guideline deals with modelling this type of input-output dynamic relations. It is based on the specification of the dynamic structures in a TF model from the sample available information.

In order to identify the values of b_i, s_i and r_i and obtain a satisfactory response of y_t for each input, several proposals have been considered, just based on algorithms related to the PA. Padé table computation offers consistent initial values, without previous identification of the noise structure.

We can write the following compact relation

$$y_t = \sum_{i=1}^{n} v_i(L)x_{it} + N_t; \qquad v_i(L) = \sum_{i=0}^{\infty} v_{ij}L^j$$

where $v_i(L)$ is the Impulse Response Function (IRF), which transforms x_{it} into y_t.

First, the weights v_{ij} for each input and the matrix covariance are computed using Ordinary Least Squares or maximising the Likelihood Function in accordance with the following expression

$$y_t \cong \sum_{i=1}^{n} \sum_{j=0}^{k_i} \hat{v}_{ij} L^j x_{it} + N_t^*$$

The lag structure for x_{it} is approximated by choosing a finite number k_i of terms. N_t^* is the reestimated noise term.

Next, we define the sequence of estimated relative weights $\hat{\eta}_i = (\hat{\eta}_{ij})_{j \in N}$ for x_{it} as

$$\hat{v}_{i,max} = \max_j |\hat{v}_{ij}|; \quad \hat{\eta}_{ij} = \frac{\hat{v}_{ij}}{\hat{v}_{i,max}}$$

that verifies the following linear difference equation of order r_i and rank b_i+s_i

$$\eta_{ij} - \delta_{i1}\eta_{i,j-1} - \delta_{i2}\eta_{i,j-2} - \ldots - \delta_{ir_i}\eta_{i,j-r_i} \begin{cases} = 0 & j > b_i + s_i \\ \neq 0 & j = b_i + s_i \end{cases}$$

This expression just constitutes a characterization for a TF model.

Several methods have been proposed for obtaining a identifiable TF model. Among them, the *corner method* (Liu-Hanssens [19], Tsay [27], Lii [18], Claverie et al [9]…), provides a generalisation of the one given in the univariate case. For this method, the study of the statistical significance can be also found in Tsay [27].

In the context of a TF model with multiple inputs, the *epsilon-algorithm* has been proposed by González-Cano [12,13] and González et al [14,15]...) The study of the statistical significance can be seen in Berlinet-Francq [5] and González et al [15].

We can also bring out the *rs-algorithm*, which has been proposed by González [11] and González-Gil [16] for a TF model in accordance with the following result.

Theorem 1.- $\hat{v}_i(L)$ has a rational representation with orders (b_i,s_i,r_i) if the following conditions are verified:

$$\text{a)} \begin{cases} s_{r_i}^{k-r_i}(\hat{\eta}_i) = C_1, & \forall k > b_i + s_i \\ s_{r_i}^{b_i+s_i-r_i}(\hat{\eta}_i) \neq C_1 \end{cases}$$

$$\text{b)} \begin{cases} r_{j+1}^{k-j+1}(\hat{\eta}_i) = 0, & \forall j, \ \forall k < b_i \\ r_{r_i+1}^{k-r_i+1}(\hat{\eta}_i) = 0, & \forall k > b_i + s_i \\ r_{r_i+1}^{b_i+s_i-r_i}(\hat{\eta}_i) \neq 0 \end{cases}$$

Displaying these values in a double-entry table, tabular structures for each input x_{it} can be obtained (González-Gil [16]).

In certain cases, some transformations in the sequence of relative weights could be necessary to avoid computational instability.

In the same way, the *qd-algorithm* has been proposed by González [11] and González-Gil [16] to identify a TF model in accordance with the following characterization.

Theorem 2.- If $\hat{v}_i(L)$ has a rational representation with orders (b_i,s_i,r_i), then one of the following statements is verified:

$$\text{a)} \begin{cases} q_j^{k-j}(\hat{\eta}_i) = 0, & \forall j, \forall k < b_i \\ q_j^{k-j}(\hat{\eta}_i) = 0, & \forall k > b_i + s_i, j > r_i \\ q_{r_i}^{k-r_i}(\hat{\eta}_i) \neq 0, & \forall k \geq b_i + s_i \\ q_j^{b_i+s_i-j}(\hat{\eta}_i) \neq 0, & \forall j \geq r_i \end{cases}$$

$$\text{b)} \begin{cases} d_{j-1}^{k-j+1}(\hat{\eta}_i) = 0, & \forall j, \forall k < b_i \\ d_{j-1}^{k-j+1}(\hat{\eta}_i) = 0, & \forall k > b_i + s_i, j > r_i \\ d_{r_i-1}^{k-r_i+1}(\hat{\eta}_i) \neq 0, & \forall k \geq b_i + s_i \\ d_{j-1}^{b_i+s_i-j+1}(\hat{\eta}_i) \neq 0, & \forall j \geq r_i \end{cases}$$

Displaying the entries in a double-entry table, tabular structures can be obtained for each input x_{it} (González-Gil [16]). Comments made in section 2 are again valid here to study the statistical significance.

To illustrate these methods a simulated model with two inputs (Liu-Hanssens [19]) is considered,

$$y_t = (2L^3 + 4L^4)x_{1t} + \frac{1.5L^2 + 3L^3}{1 - L + 0.24L^2} x_{2t} + N_t, t = 1,..,100$$

$$(1-1.3L+0.4L^2)N_t=a_t, \quad a_t \sim N(0,2)$$
$$(1-1.4L+0.48L^2)x_{1t}=c_t, \quad c_t \sim N(0,1)$$
$$(1-0.7L)x_{2t}=d_t, \quad d_t \sim N(0,2)$$

where a_t is independent of c_t and d_t, and c_t and d_t are contemporaneously correlated with correlation 0.7.

The identification pattern is clearly $b_1=3$, $s_1=1$, $r_1=0$, $b_2=2$, $s_2=1$ and $r_2=2$.

Previous results for the *corner method* and the *epsilon algorithm* can be seen in Liu-Hanssens [19] and González et al [15] respectively. They don't differ substantially from the next ones given for the *rs-algorithm* and the *qd-algorithm*.

The IRF is now computed by using the Cochrane-Orcutt iterative method, one of the three ones considered in González et al [15]. The other methods and Least Ordinary Squares estimation provide similar results.

Table r
Statistical significance for the $\{(-1)^j\eta_{1j}\}$

	1	2	3	4	5	6
0	-.132					
1	.241	-.006				
2	-.176	-.036	.002			
3	1.566	-.052	.009	-.001		
4	2.523	-1.170	.503	-.020	.001	
5	1.265	-.090	.001	.000	.000	.000
6	1.050	.004	-.002	.000	.000	.000
7	.795	.062	-.001	.000	.000	
8	-.142	-.046	.006	.000		
9	.626	-.020	.000			
10	-.123	-.003				
11	.081					

Proceedings of the 8th WSEAS International Conference on APPLIED MATHEMATICS, Tenerife, Spain, December 16-18, 2005 (pp379-385)

Table r
Statistical significance for the $\{(-1)^j\eta_{2j}\}$

	1	2	3	4	5	6
0	-.215					
1	.212	-.010				
2	1.839	-.050	.006			
3	-6.227	1.316	-.029	.006		
4	5.428	-.643	-.010	.000	.000	
5	-3.890	.124	-.013	.000	.000	.000
6	2.378	.082	.002	.000	.000	.000
7	-1.562	-.093	.001	.000	.000	
8	1.480	-.167	.001	.000		
9	-.583	-.072	.000			
10	.897	-.056				
11	-.271					

Therefore, among different alternatives it can be obtained the identification pattern of the model.

Table q
Statistical significance for $\{(-1)^j\eta_{1j}\}$

	1	2	3	4	5	6
η_0=-.051						
0	-.138					
1	-.170	.068				
2	-.180	.254	.027			
3	1.205	-.211	-.238	-.031		
4	1.035	-.832	.421	.216	.093	
5	.723	-.203	-.347	.295	.302	-.276
6	.572	.046	-.055	.019	.004	-.027
7	-.146	-.443	.049	-.019	-.025	
8	-.148	.148	-.475	-.044		
9	-.126	.141	.024			
10	-.078	-.125				
11	-.084					

Table q
Statistical significance for $\{(-1)^j\eta_{2j}\}$

	1	2	3	4	5	6
η_0=-.031						
0	-.128					
1	.222	-.215				
2	-1.580	.327	-.061			
3	-3.380	-.594	.262	.087		
4	-2.632	-.265	.067	-.089	1.302	
5	-1.690	.534	-.164	.135	.075	-.005
6	-1.081	-.064	.084	.090	-.058	-.006
7	-.884	.331	-.046	-.058	.045	
8	-.473	.349	.124	.051		
9	-.412	1.112	.171			
10	-.232	-.123				
11	-.058					

The orders for the first input can be adequately identified. For the second one, a possible pattern is b_2=2, s_2=0, r_2=1.

4 An Application

Now we consider empirical results for a set of sales leading indicator data identified as series M in Box-Jenkins [7] and also studied in Tsay [27]. Data set are 150 pair of observations (x_t, y_t).

The FT model proposed by Box-Jenkins [7] is

$$\Delta y_t = 0.035 + \frac{4.82L^3}{1-0.72L}\Delta x_t + (1-0.54L)a_t$$
$$\Delta x_t = (1-0.32L)b_t$$

In this specification Δ=1-L is the operator that allow to obtain the rates of data variation and a_t and b_t are white noise processes. Therefore, b=3, s=0 and r=1, which confirms the model proposed by Box-Jenkins [7]. Tsay [27] carried out further examination studying the statistical significance of the corner table.

Starting from Berlinet-Francq [5] and Tsay [27], González et al [15] showed the statistical significance of null entries in the epsilon table to confirm the adequacy of the identified model.

In this sense, applying the *epsilon-algorithm* to the sequence $(-1)^i\eta_{i.}$ it can be deduced the following table of statistical significance

	0	2	4	6	8	10	12
0	.105						
1	-.105	-.053					
2	-.211	-.109	.103				
3	10.541	6.419	5.199	4.729			
4	-7.800	-.046	.025	-.010	-.117		
5	5.692	.025	.002	.024	-.082	-.066	
6	-4.360	-.011	.025	.042	-.032	.032	.058
7	3.130	-.123	-.083	-.032	-.039	.044	.051
8	-2.460	-.017	-.074	.033	.043	.025	
9	1.792	.189	.173	.059	.051		
10	-1.054	.144	.162	.024			
11	.843	-.086	-.117				
12	-.843	-.206					
13	.422						

With these results, the orders for the accepted models according to certain critical values are as follows

Critical value	Accepted (b,s,r) models
1.28	(3,6,0) (3,0,1)
1.64	(3,6,0) (3,0,1)
1.96	(3,5,0) (3,0,1)
2.33	(3,5,0) (3,0,1)
2.58	(3,4,0) (3,0,1)
2.81	(3,4,0) (3,0,1)
3.09	(3,4,0) (3,0,1)
3.29	(3,3,0) (3,0,1)
3.72	(3,3,0) (3,0,1)
4.26	(3,3,0) (3,0,1)

They confirm the model proposed by Box-Jenkins [7] and Tsay [27]. Other possible models can be also obtained although they are less parsimonious.

Now, applying the *rs-algorithm*, next results are obtained

Table r (Statistical significance)

	1	2	3	4	5	6
0	-.208					
1	.122	-.002				
2	.083	.000	-.000			
3	-10.998	.084	-.001	.000		
4	8.169	-.410	.020	.000	.000	
5	-5.533	-.176	.000	.000	.000	.000
6	4.239	-.177	.002	.000	.000	.000
7	-2.930	-.137	.001	.000	.000	
8	2.561	-.152	-.000	.000		
9	-1.733	.004	-.000			
10	1.155	.037				
11	-.976					

With these results, the orders for the accepted models according to certain critical values are as follows:

Critical value	Accepted (b,s,r) models
1.28	(3,6,0) (3,0,1)
1.64	(3,6,0) (3,0,1)
1.96	(3,5,0) (3,0,1)
2.33	(3,5,0) (3,0,1)
2.58	(3,4,0) (3,0,1)
2.81	(3,4,0) (3,0,1)
3.09	(3,3,0) (3,0,1)
3.29	(3,3,0) (3,0,1)
3.72	(3,3,0) (3,0,1)
4.26	(3,2,0) (3,0,1)

The results obtained with the *qd-algorithm* are the following:

Table q (Statistical significance)

	1	2	3	4	5	6
η_0 = -.017						
0	-.089					
1	.090	-.107				
2	-.083	.082	.416			
3	-5.513	-.078	-.048	.078		
4	-3.862	.351	-.197	3.282	-1.223	
5	-2.820	.651	-.191	.164	.352	-.241
6	-2.025	.187	-.143	-.153	.201	-.169
7	-1.601	.594	.152	-.130	.109	
8	-1.209	-.046	.559	-.138		
9	-.811	.016	-.015			
10	-.621	.292				
11	-.452					

The selected models are

Critical value	Accepted (b,s,r) models
1.28	(3,5,0) (3,0,1)
1.64	(3,4,0) (3,0,1)
1.96	(3,4,0), (3,0,1)
2.33	(3,3,0) (3,0,1)
2.58	(3,3,0) (3,0,1)
2.81	(3,3,0) (3,0,1)
3.09	(3,2,0) (3,0,1)
3.29	(3,2,0) (3,0,1)
3.72	(3,2,0) (3,0,1)
4.26	(3,1,0) (3,0,1)

The comparison among obtained results suggests to accept like probable better model the one corresponding to the orders (3,0,1).

5 Conclusions and Open Questions

This paper highlights the usefulness of several numerical methods which are closely related to PA to identify some rational structures associated to data series. This is illustrated in the context of causal time series models, that is, ARMA and TF Models.

The main contribution of this paper is the study of the statistical significance of the *rs-algorithm* and *qd-algorithm* as a continuation of the work made in González-Gil [16].

Empirical findings points out the role of the statistical significance for the numerical values in the mentioned algorithms. In general, different possible models will be obtained according to certain critical values.

For future research, the generalisation of the results obtained here to VARMA models, in general, is not evident. For example, for the *corner method*, consideration has to be given to the rank of matrices and non determinants (Pestano-González [22]). Also, the use of matrix *epsilon-algorithm* has only given partial results (Francq [10]). The generalisation of the *rs-algorithm* and *qd-algorithm* has not yet been considered.

6 References

References

[1] G. A. Baker Jr. and P. Graves-Morris, *Padé Approximants*. Encyclopaedia of Mathematics and its Applications, 53, Cambridge University Press. 2nd edition (1996).

[2] J. M. Beguin, C. Gourieroux and A. Monfort (1980), Identification of a Mixed Autoregressive-Moving Average Process: The Corner Method in *Time Series*, O. D. Anderson (ed.), North-Holland, Amsterdam (1980) 423-436.

[3] A. Berlinet, Estimating the degrees of an ARMA model, *Physica-Verlag, Compstat. Lect.* 3 (1984) 61-94.

[4] A. Berlinet, Sequence Transformations as Statistical Tools, *Applied Numerical Mathematics* 1 (1985) 531-544.

[5] A. Berlinet and C. Francq, Identification of a Univariate Arma Model, *Computational Statistics* 9 (1994) 117-133.

[6] A. Berlinet and C. Francq, On the Identifiability of Minimal VARMA Representations, *Statistical Inference for Stochastic Processes* 1 (1998) 1-15.

[7] G. E. P. Box and G.M. Jenkins, *Time Series Analysis: Forecasting and Control*, Revised Edition, (Holden Day, San Francisco, 1976).

[8] C. Brezinski, *Padé-Type Approximants and General Orthogonal Polynomials* (Birkhäuser, Basel, 1980).

[9] P. Claverie, D. Szpiro and R. Topol, Identification des Modèles à Fonction de Transfert: La Méthode Padé-

Transformée en z, *Annales d'Economie et de Statistique* 17 (1990) 145-161.

[10] C. Francq, *Identification et Minimalité dans les Séries Chronologiques,* Thèse, Université des Sciences et Techniques du Languedoc, Montpellier II (1989).

[11] C. González, Algunas técnicas útiles para la identificación de estructuras racionales en series temporales, *Documento de Trabajo* 9, *Facultad de Ciencias Económicas y Empresariales*, *Univ. de La Laguna* (1997).

[12] C. González and V. Cano, Determinación de los órdenes de los polinomios de retardo en una función de transferencia: Comparación de algoritmos, *Rev. Acad. Canaria de Ciencias* 1 (1990a) 173-183.

[13] C. González and V. Cano, Especificación de una función de transferencia bajo limitación en el comportamiento de la variable dependiente, *Anales de Economía Aplicada*, Murcia (1990b) 337-344.

[14] C. González, V. Cano and C. Gil, Comparación de algoritmos para la identificación de una función de transferencia: una generalización al caso de varios inputs, *Rev. Española de Economía, Segunda Época* 10 (1993) 163-175.

[15] C. González, V. Cano and C. Gil, The epsilon-algorithm for the identification of a transfer-function model: some applications, *Numerical Algorithms* 9 (1995) 379-395.

[16] C. González and M.C. Gil (2003), Padé Approximation in economics, *Numerical Algorithms* 33:277-292, 2003.

[17] H.L. Gray, G.D. Kelley and D.D. Mac Intire, A new approach to ARMA modelling, *Commum. Stat. - Simul. and Comp.* B7 (1978) 1-77.

[18] K. Lii, Transfer Function Model Order and Parameter Estimation, *Journal of Time Series Analysis* 6 (3) (1985) 153-169.

[19] L.M. Liu and D. M Hanssens, Identification of Multiple Inputs Transfer Function Models, *Commun. Statist. - Theor. Math.* A11 (3) (1982) 297-314.

[20] H. Lütkepohl and D.S. Poskitt, Specification of Echelon-Form VARMA Models, *J. Bus. Econ. Statist.* 14 (1) (1996) 69-79.

[21] B. Mareschal and G. Mélard, The Corner Method for Identifying Autoregressive Moving Average Models, *Applied Statistics* 37(2) (1988) 301-316.

[22] C. Pestano and C. González, A new approach in Multivariate Time Series Specification, *International Advances in Economic Research,* vol. 4 (3) (1998) 229-242.

[23] W.C. Pye and T.A. Atchison, An algorithm for the computation of higher order G-transformation, *SIAM J. Numer. Anal.* 10 (1973) 1-7.

[24] G.C. Reinsel, *Elements of Multivariate Time Series Analysis* (Springer Verlag, New York, 1993).

[25] H. Rutishaüser, *Der Quotienten-Differenzen Algoritmus* (Birkhäuser-Verlag, Basel, 1957).

[26] G. C. Tiao and R.S. Tsay, Model Specification in Multivariate Time Series, *J. Roy. Statist. Soc.*, Series B 51 (1989) 157-213.

[27] R.S. Tsay, Model Identification in Dynamic Regression (Distributed Lag) Models, *J. Bus. Econ. Statist.* 3(3) (1985) 228-237.

[28] P. Wynn, On a device for computing the $e_m(s_n)$ transformation, *MTAC*, 10 (1956) 91-96.