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Approximations of an Hankel Transform of the Product of Two Bessel *J*-Functions

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Abstract: - A Hankel transform of the product of two Bessel *J*-functions under different restrictions on its parameters was approximated by infinite series expansions to the product of two hypergeometric functions, which in turn usually reduce to polynomials and thus, can be generated recursively. Such approximation makes the general transform very suitable for automatic computation. The transform concerned arise in the theory of the light changes of eclipsing binary star systems.

Key-Words: -Bailey's Theorem, Bessel *J*-functions, hypergeometric functions, Appell hypergeometric functions, Pochhammer symbol, Jacobi polynomial.

1. Introduction

The integral as the Hankel transform of two Bessel J-functions under consideration is given by

$$I = \int_0^\infty t^{\rho-1} J_\kappa(\alpha t) J_\lambda(\beta t) J_\mu(\gamma t)\, dt \qquad (1.1)$$

where all the parameters are real and J denotes the Bessel functions of the first kind. A special case of this integral when $\lambda = \mu$ was evaluated by Macdonald [1], as

$$\int_0^\infty t^{1-\kappa} J_\kappa(\alpha t) J_\lambda(\beta t) J_\lambda(\gamma t)\, dt = \frac{(\beta\gamma)^{\kappa-1}(\sin A)^{\kappa-\frac{1}{2}}}{\sqrt{2\pi}\,\alpha^\kappa} P^{\frac{1}{2}-\kappa}_{\lambda-\frac{1}{2}}(\cos A) \qquad (1.2)$$

which is valid when $\kappa > -\frac{1}{2}$, $\lambda > -\frac{1}{2}$; α,β,γ are the sides of a triangle and A is the angle between β and γ.

A general evaluation of the integral (1.1) was given by Bailey [2] of the form

$$I = \frac{2^{\rho-1}\alpha^\kappa\beta^\lambda\Gamma(\frac{\kappa+\lambda+\mu+\rho}{2})}{\gamma^{\kappa+\lambda+\rho}\Gamma(\kappa+1)\Gamma(\lambda+1)\Gamma(1-\frac{\kappa+\lambda-\mu+\rho}{2})} \times F_4\left(\frac{\kappa+\lambda-\mu+\rho}{2}, \frac{\kappa+\lambda+\mu+\rho}{2}; \kappa+1, \lambda+1; \frac{\alpha^2}{\gamma^2}, \frac{\beta^2}{\gamma^2}\right) \qquad (1.3)$$

where α,β,γ are positive quantities, $-(\kappa+\lambda+\mu) > \rho > \frac{5}{2}$ and the Appell generalized hypergeometric series of the form

$$F_4(a,b,c,c';x,y) = \sum_{m=0}^\infty \sum_{n=0}^\infty \frac{(a)_{m+n}(b)_{m+n}}{m!n!(c)_m(c')_n} x^m y^n \qquad (1.4)$$

occurring on the right hand side of (1.3) converges whenever $\gamma \geq \alpha+\beta$ and $|x|+|y| \leq 1$. The $(\xi)_k$ in (1.4) denotes the Pochhammer symbol defining by

$$(\xi)_k = \xi(\xi+1)...(\xi+k-1)$$

for $k > 0$ and $(\xi)_0 = 1$. Note that $I = 0$ if $\frac{\kappa+\lambda-\mu+\rho}{2}$ is a positive integer, since in this case Gamma function $\Gamma\left(1-\frac{\kappa+\lambda-\mu+\rho}{2}\right)$ in the denominator of (1.3) becomes infinity. Moreover, if $\frac{\kappa+\lambda-\mu+\rho}{2}$ becomes a negative integer, the function F_4 reduces into a polynomial. However, under many conditions, for example $\alpha-\beta < \gamma < \alpha+\beta$, the above restrictions on the parameters $\alpha,\beta,\gamma,\rho,\kappa,\lambda$ and μ can not be met and thus Bailey's Theorem (1.3) ceases to be applicable.

An other evaluation of the integral I was given by Watson[3, p. 413, Eq.7] of the form

$$I=\frac{2^{\rho-1}\alpha^{\kappa}\beta^{\lambda}\gamma^{\mu}}{\Gamma(\mu+1)[\Gamma(\lambda+1)]^2}\sum_{n=0}^{\infty}\frac{(-1)^n(\kappa+\lambda+2n+1)\Gamma(\lambda+n+1)}{n!\Gamma(\kappa+n+1)}$$
$$\times\frac{\Gamma(\kappa+\lambda+n+1)\Gamma(n+(\kappa+\lambda+\mu+\rho+1)/2)}{\Gamma(n+2+(\kappa+\lambda-\mu-\rho)/2)}$$
$$\times{}_2F_1\left(\left.\begin{matrix}n+\frac{\kappa+\lambda+\mu+\rho+1}{2},-n-1-\frac{\kappa+\lambda-\mu-\rho}{2}\\ \mu+1\end{matrix}\right|\gamma^2\right)$$
$$\times\left[{}_2F_1\left(\left.\begin{matrix}-n,\kappa+\lambda+n+1\\ \lambda+1\end{matrix}\right|\beta\right)\right]^{-2}, \qquad (1.5)$$

where the conditions for applicability $-(\kappa+\lambda+\mu)>\rho>\frac{5}{2}$, $0\le\alpha,\beta,\gamma\le1$ and $\alpha+\beta=1$.

The second hypergeometric series on the right hand side of (1.5) is a Jacobi polynomial, the first one also reduces into a Jacobi polynomial if $(\mu+\rho-\kappa-\lambda)/2$ becomes unity, zero or a negative integer. Thus, for $(\mu+\rho-\kappa-\lambda)/2=1,0,-1,-2,...$ the right hand side of (1.5) can be generated recursively with the aid of the recurrence relations for the Jacobi polynomials. With this latter solution to the integral *I*, the restrictions on the parameters are altered, for example we can now evaluate it when $\alpha-\beta<\gamma<\alpha+\beta$ if $\alpha+\beta=1$. Thus, the solution domain of the integral (1.1) is slightly extended by (1.5).

In the present paper three other general approximations for the integral *I* have been presented. All these expansions together with those given by (1.3) and (1.5) differ in form but all converge (at different rate in different domains) to the same answer under different restrictions on the parameters. Thus, at least one expansion is always suitable for an application. The expansions given which are all highly oscillatory are very suitable for a fast automatic computation. However, great care should be taken in the numerical applications because of the loss of accuracy by propagating round-off error. Some special cases of the integral *I* arise in finding the loss of light caused by the mutual eclipses of close binary star systems were given in Section 3.

2. New Approximations

We shall present in this section three different approximations in series expansions form for the integral *I* which were obtained by making use of some certain expansions for the first kind of Bessel functions. For the first expression let us resort to a Neumann-type expansion for J_μ (cf., e.g., Erdélyi et al. [4, p.99, Eq. 3]) of the form

$$J_\mu(\gamma t)=\sqrt{\frac{2}{\pi t}}\gamma^{\mu}\Gamma(\mu+\frac{1}{2})\sum_{n=0}^{\infty}\frac{\Gamma(n+\frac{1}{2})(\mu+2n+\frac{1}{2})}{\Gamma(n+\mu+1)}C_{2n}^{(\mu+1/2)}(\sqrt{1-\gamma^2})$$
$$\times J_{\mu+2n+1/2}(t) \qquad (2.1)$$

where $C_{2n}^{(\alpha)}(x)$ denotes the ultraspherical (Gegenbauer) polynomials of even order and can be written in terms of the ordinary hypergeometric functions as

$$C_{2n}^{(\alpha)}(x)=\frac{(-1)^n(\alpha)_n}{n!}{}_2F_1\left(\left.\begin{matrix}-n,n+\alpha^2\\ 1/2\end{matrix}\right|x^2\right).$$

If we substitute (2.1) in (1.1) and integrate with the aid of Bailey's formula (1.3) we obtain

$$I=\frac{2^{\rho-1}\alpha^{\kappa}\beta^{\lambda}\gamma^{\mu}}{\sqrt{\pi}\Gamma(\kappa+1)\Gamma(\lambda+1)}\Gamma(\mu+1/2)$$
$$\times\sum_{n=0}^{\infty}\frac{\Gamma(n+\frac{1}{2})(\mu+2n+\frac{1}{2})}{\Gamma(n+\mu+1)}$$
$$\times\frac{\Gamma((\kappa+\lambda+\mu+\rho)/2+n)}{\Gamma(n+1+(\mu+1-\kappa+\lambda-\rho)/2}C_{2n}^{(\mu+1/2)}(\sqrt{1-\gamma^2})$$
$$\times F_4(-n+\frac{\kappa+\lambda-\mu+\rho-1}{2},n+\frac{\kappa+\lambda+\mu+\rho}{2};\kappa+1,\lambda+1;\alpha^2,\beta^2) \qquad (2.2)$$

which is valid when $3>\rho>-(\kappa+\lambda+\mu)$, $0\le\alpha,\beta\le1$ and $-1\le\gamma\le1$ but -as an advantage- no relation is required to be satisfied by α,β and γ. It may be noted that if $(\kappa+\lambda-\mu+\rho-1)/2$ is zero or a negative integer Appell hypergeometric function F_4 on the right hand side of (2.2) reduces into a polynomial.

The second approximation for the integral *I* in terms of the F_4 series, alternative to (2.2) can be obtained with the aid of another Neumann-type expansion for the Bessel functions J_μ (cf. e.g. [4, p.64,Eq.6]) given by

$$\frac{J_\mu(\gamma t)}{(\gamma t)^{\mu-\nu}}=\frac{\gamma^{\mu}}{2^{\mu-\nu}\Gamma(\mu+1)}\sum_{n=0}^{\infty}\frac{(2n+\nu)\Gamma(n+\nu)}{n!}{}_2F_1\left(\left.\begin{matrix}-n,n+\nu\\ \mu+1\end{matrix}\right|\gamma^2\right)J_{\nu+2n}(t)$$
$$\nu,\mu,\mu-\nu\ne0,-1,-2,-3,... \,. \qquad (2.3)$$

On insertion of this expansion in (1.1) and integrating it again with the aid of Bailey's formula (1.3) we obtain

$$I = \frac{2^{\rho-1}\alpha^{\kappa}\beta^{\lambda}\gamma^{\mu}}{\Gamma(\kappa+1)\Gamma(\lambda+1)\Gamma(\mu+1)} \sum_{n=0}^{\infty} \frac{(2n+\nu)\Gamma(n+\nu)}{n!}$$
$$\times \frac{\Gamma(n+(\kappa+\lambda+\mu+\rho)/2)}{\Gamma(1+n+\nu-(\kappa+\lambda+\mu+\rho)/2)}\, {}_2F_1\left(\begin{matrix}-n, n+\nu \\ \mu+1\end{matrix}\middle| \gamma^2\right)$$
$$\times F_4\left(-n-\nu+\frac{\kappa+\lambda+\mu+\rho}{2}, n+\frac{\kappa+\lambda+\mu+\rho}{2}; \kappa+1, \lambda+1; \alpha^2, \beta^2\right), \quad (2.4)$$

where $\nu, \mu, \mu-\nu \neq 0, -1, -2, -3, \ldots; -(\kappa+\lambda+\mu) < \rho < \frac{5}{2}+\nu-\mu$, *and* $0 \le \alpha, \beta, \gamma \le 1$. For an other expansion of the integral *I* in series of the F_4- function, use will be made of a formula due to Bailey [5] of the form

$$\left(\frac{t}{2}\right)^{a-\kappa-\lambda} J_{\kappa}(\alpha t) J_{\lambda}(\beta t) = \frac{\alpha^{\kappa}\beta^{\lambda}}{\Gamma(\kappa+1)\Gamma(\lambda+1)} \sum_{n=0}^{\infty} \frac{(a+2n)\Gamma(a+n)}{n!}$$
$$\times F_4\left(-n, a+n; \kappa+1, \lambda+1; \alpha^2, \beta^2\right) J_{a+2n}(t) \quad (2.5)$$

with a free parameter a, and provided that $\alpha+\beta=1$. This expansion is a generalized form of Bateman's [6] formula (which was utilised in derivating expression (1.5)) and it reduce to it for $a=\kappa+\lambda+1/2$. And, moreover, we shall consider (cf., e.g., [3]) that

$$\int_0^{\infty} t^{-\lambda} J_{\nu}(at) J_{\mu}(bt)\, dt = \frac{a^{\nu}\Gamma((\nu+m-\lambda+1)/2)}{2^{\lambda} b^{\nu-\lambda+1}\Gamma(\nu+1)\Gamma((-\nu+m+\lambda+1)/2)}$$
$$\times {}_2F_1\left(\begin{matrix}\frac{\nu+\mu-\lambda+1}{2}, \frac{\nu-\mu-\lambda+1}{2} \\ \nu+1\end{matrix}\middle| \frac{a^2}{b^2}\right), \quad (2.6)$$

for $\nu+\mu-\lambda+1>0,\ \lambda>-1$ and $0<a<b$.
Now, by substituting (2.5) in (1.1) and evaluating the remaining integral with the aid of (2.6) we are left with

$$I = \frac{2^{2a-2\kappa-2\lambda+\rho-1}\alpha^{\kappa}\beta^{\lambda}\gamma^{\mu}}{\Gamma(\kappa+1)\Gamma(\lambda+1)\Gamma(\mu+1)} \sum_{n=0}^{\infty} \frac{(a+2n)\Gamma(a+n)\Gamma(a+n+\frac{\mu-\kappa+\lambda+\rho}{2})}{n!\Gamma(1+n-\frac{\mu-\kappa-\lambda+\rho}{2})}$$
$$\times {}_2F_1\left(\begin{matrix}a+n+\frac{\mu-\kappa+\lambda+\rho}{2}, -n+\frac{\mu-\kappa+\lambda+\rho}{2} \\ \mu+1\end{matrix}\middle| \gamma^2\right)$$
$$\times F_4\left(-n, a+n; \kappa+1, \lambda+1; \alpha^2, \beta^2\right). \quad (2.7)$$

which is valid when
$\kappa+\lambda-\mu-2a<\rho<\kappa+\lambda-a+2,\ 0\le\alpha,\beta,\gamma\le 1$ *and* $\alpha+\beta=1$. Clearly F_4 series in (2.7) is a polynomial: the ${}_2F_1$ series also becomes a polynomial if $-n+(\mu-\kappa-\lambda+\rho)/2$ is zero or a negative integer.

3. Applications

It has been shown by Kopal [7] that the loss of light caused by the mutual eclipses of the components of close binary star systems can be described in terms of some special cases of the integral *I* defining by (1.1); namely, these special cases

$$\alpha_n^{\nu} = 2^{\nu}\Gamma(\nu) b \int_0^{\infty} \frac{J_{\nu}(ax)}{(ax)^{\nu}} J_1(bx) J_0(cx)\, dx, \quad (3.1)$$

$$I_{-1,n}^{0} = 2^{\nu-1} a^{\nu}\Gamma(\nu) \int_0^{\infty} \frac{J_{\nu}(ax)}{(b^2x)^{\nu-1}} J_0(bx) J_0(cx)\, dx, \quad (3.2)$$

and

$$I_{-1,n}^{1} = 2^{\nu-1} a^{\nu}\Gamma(\nu) \int_0^{\infty} \frac{J_{\nu}(ax)}{(b^2x)^{\nu-1}} J_1(bx) J_1(cx)\, dx, \quad (3.3)$$

where $\nu=(n+2)/2$ takes the values 1, 1.5, 2, 2.5, 3, ... etc. The quantities *a, b* and *c* in the integrands are the functions of the radii and the apparent separation of the companion stars satisfying that $0 \le a,b,c \le 1$ and $a+b=1$ (see Kopal [8]). Thus, the- so called – "eclipse functions" defining by (3.1)-(3.3) may be easily evaluated with the aid of the general series expansions given in the previous sections of this paper.

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