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Sure Overall Orders to Identify Scalar Component Models

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Abstract: The identification stage of Vector Autoregressive Moving Average (VARMA) models plays an important role in multivariate time series analysis and it has been discussed from different approaches in literature. In particular, the notion of Scalar Component Model (SCM) is highly ingenious and emerges in [7] as a fairly natural way of modelling multivariate time series. The idea of SCM is of enormous benefit because the effect is the reduction of parameters in VARMA representation and it could simplified drastically the complexity in estimation. In the procedure to identify SCMs ([7]), the authors use subjective choices of a parameter denoted by h. Varying the value of h is likely to result in substantially different orders for the SCM. Then, of great interest is the question about how robust is the identification procedure with regard to the choice of h. Therefore, the aim of this paper is to find necessary and sufficient conditions to choose the minimum value of h, such that the procedure to identify SCMs will not lead to theoretical errors.

Key-Words: VARMA models, Identification Stage, SCM, Overall Order

1 Introduction

Vector Autoregressive Moving Average (VARMA) models are widely used in econometrics to model multivariate time series. They have been extensively studied by a large number of researchers, see e.g. [2], [3], [4], [6] and their references. This work is centered on the specification stage.

In [7] it is discussed an approach for modelling vector time series through Scalar Component Models (SCMs). The primary goal of the authors was to provide a practical model identification procedure. For a given process, they intended to find an overall parsimonious model and to simplify structures.

It is worthwhile pointing out that at the end of [7] a section is included in which many experts make their comments and they raise key questions about its practical and its theoretical contributions. An issue discussed by these experts triggered our paper, specifically, the ambiguity in the choice of a parameter, denoted by h, which controls the dimension of certain matrix, the effective sample size in estimation, the certainty in the theoretical specification of the model, etc.

The aim of this paper is to find necessary and sufficient conditions to choose h with the minimum value, such that the identification of SCMs will not lead to theoretical errors.

The paper is organized as follows. Section 2 introduces the idea of *Scalar Component Models* to describe a component structure in a multivariate framework. Moreover we provide a summary of results in [7] in which the analysis is focused. In Section 3 we propose the definition of *sure overall orders*. It is crucial in this paper and its utility will be clear in the main results, Propositions 1 and 2 of th section. Finally, in Section 4 we give conclusions and new directions for future research.

2 Definitions and Notations

In order to lay the theoretical foundations where essential points will be posed and solved, this section provides a summary of results in which this analysis is focused. Most precisely, it consists of a description of the table proposed in [7] to identify a pair of overall orders for the VARMA representation.

Let $z_t=(z_{1t}, z_{2t}, \ldots, z_{kt})^t$ be a k-dimensional process following the VARMA(p,q) model

$$\phi(B)z_t = \theta(B)a_t \tag{1}$$

where $\phi(B)=I-\phi_1B-\ldots-\phi_pB^p$, $\theta(B)=I-\theta_1B-\ldots-\theta_qB^q$, the ϕs and the θs are kxk matrices, B is the usual backshift operator and a_k is a sequence of independent k-variate random vectors with mean zero and definite covariance matrix Σ. The process is stationary if all the zeros of $|\phi(B)|$ are outside the unit circle and invertible if all the zeros of $|\theta(B)|$ are outside the unit circle.

This model is a generalization of ARIMA models in [1]. However, this direct generalization creates two mayor difficulties: the overflow of parameters (the estimates of which are often highly correlated) and the lack of identifiable models because exchangeable models appear (with the same or different pair of minimal orders).

Exchangeable models are special features of vector time series that do not occur in the univariate case. Two VARMA models are *exchangeable* if they are of finite order and give the same probability distribution of z_t. Since they have the same probability distribution, they give the same covariance structure and provide the same inference.

The procedure proposed in [7] can recognize exchangeable models when they exist. The authors introduced the concept of *Scalar Component Model* (SCM) as follows:

Definition 1.- Given the VARMA(p,q) model (1), we say that a non-zero linear combination $y_{it}=v_0^t z_t$, where v_0 is a k-vector, follows a *Scalar Component Model* with orders (p_i,q_i), SCM(p_i,q_i), if v_0 has the properties:

$$v_0^t\phi_{p_i} \neq 0^t \text{ where } 0\le p_i\le p$$

$$v_0^t\phi_j = 0^t \text{ for } j=p_i+1,\ldots,p$$

$$v_0^t\theta_{q_i} \neq 0^t \text{ where } 0\le q_i\le q$$

$$\text{and } v_0^t\theta_j = 0^t \text{ for } j=q_i+1,\ldots,q$$

Since $v_0^t\phi(B)z_t = v_0^t\theta(B)a_t$ the structure of y_{it} can be written as

$$y_{it} + \sum_{j=1}^{p_i} v_j^t z_{t-j} = v_0^t a_t + \sum_{j=1}^{q_i} h_j^t a_{t-j}$$

where $v_j^t = -v_0^t\phi_j$ and $h_j^t = -v_0^t\theta_j$.

By allowing v_0 to be an arbitrary non-zero vector, the idea of an SCM is a direct generalization of the model of each component z_{it} in the VARMA framework so that the model structure can be simplified. The SCM is a device designed to capture the structure of a component within a vector model. It is not a univariate model. Note that the orders (p_i,q_i) of the SCM are lesser than or equal to the orders (p,q) of the overall VARMA representation. Therefore, given a VARMA process z_t, the transformation $y_t=V_0z_t$ -where V_0 is a kxk non-singular matrix associated with k SCMs of orders (p_i,q_i), i=1,…,k- could lead to considerable parsimony in parameterization of the model.

Since the choice of the components, their orders and their SCM structures are not unique, [7]'s goal is to obtain components which have the following minimal order property.

Definition 2.- Let y_{it} follow the SCM(p_i,q_i) structure and write $o_i=p_i+q_i$. Let $OR(y_t)=\{o_{(1)}\le o_{(2)}\le \ldots \le o_{(k)}\}$ be the set of the ordered o_is. We say that a vector of k linearly independent (l.i.) scalar components y_t is of *minimal order* if there exists no other vector y_t^* of k l.i. components, with $OR(y_t^*) = \{o_{(1)}^* \le \ldots \le o_{(k)}^*\}$ such that $o_{(i)}^* \le o_{(i)}$ for $1\le i\le k$ and a strict inequality holds for some i.

In [7], the concept of overall orders is basic and it is infered in the following way:

Definition 3.- Given a set of k l.i. SCMs with orders (p_i,q_i), i=1,2,...,k, its pair of *overall orders* is (p,q), where $p=\max\{p_i\}$, $q=\max\{q_i\}$.

Nevertheless, as it will be seen later, it will be necessary to take up again and improve the definition of this concept, so that the essential matters of this work can be solved.

2.1 The Increment Pattern of Zeros

The authors in [7] use properties of autocovariance matrices of z_t in finding SCMs.

The rank properties and the eigenstructure of the sample covariance matrices are the mathematical tools they use to justify their procedure.

Let A be and rxs real matrix and x be an s-dimensional vector. We say that x is a *right vector corresponding to a zero of A* if Ax=0. It is well known the rank(A)=s-v where v is the number of zeros associated with l.i. right vectors of A.

Next, for h≥0, m≥0, j≥0, let

$$\Gamma(m,h,j)=\begin{pmatrix} \Gamma_{j+1} & \Gamma_{j} & \cdots & \Gamma_{j+1-m} \\ \Gamma_{j+2} & \Gamma_{j+1} & \cdots & \Gamma_{j+2-m} \\ \vdots & \vdots & \cdots & \vdots \\ \Gamma_{j+1+h} & \Gamma_{j+h} & \cdots & \Gamma_{j+1+h-m} \end{pmatrix}$$

be the k(h+1)xk(m+1)-dimensional matrix where $\Gamma_i=E(z_{t-i}\ z_i^{'})$ is the lag i autocovariance matrix of z_t.

In [7] D(m,h,j) is defined to be:

a) the number of zeros of Γ(m,h,j) for m=0 or j=0.

b) the diagonal increment of number of zeros between Γ(m,h,j) and Γ(m-1,h,j-1) for m≥1 and j≥1.

D(m,h,j) is arranged in a two-way table according to (m,j), m≥0, j≥0.

If the process can be represented by a set of k l.i. SCM(p_i,q_i), i=1,2,...,k, having the minimal property, p=max{p_i}, q=max{q_i} and it has not any other *non-nested* exchangeable representation[1], in [7] it is affirmed that, for h≥m,

$$D(m,h,j)\ \begin{cases} = k & \text{if } m \geq p,\ j \geq q \\ < k & \text{otherwise} \end{cases}$$

In the case, the process has another VARMA(s,r) exchangeable representation whose SCMs also have the minimal order property, being (p,q) and (s,r) non-nested orders,

$$D(m,h,j)\ \begin{cases} = k & \text{if}\,(m \geq p,\ j \geq q) \cup (m \geq s,\ j \geq r) \\ < k & \text{otherwise} \end{cases}$$

[1] The orders (p,q) and (s,r) of two VARMA exchangeable representations are said to be *non-nested* if either (p<s, q>r) or (p>s, q<r).

This can be generalized to the case in which z_t has more than two minimal exchangeable representations.

An example given in [7] is:

$$z_t=\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} z_{t-1} + a_t$$

Pattern of number of zeros of Γ(m,h,j)

	0	1	2	3	4
0	1	2	2	2	2
1	2	3	4	4	4
2	2	4	5	6	6
3	2	4	6	7	8
4	2	4	6	8	9

Diagonal increments: D(m,h,j)

	0	1	2	3	4
0	1	2	2	2	2
1	2	2	2	2	2
2	2	2	2	2	2
3	2	2	2	2	2
4	2	2	2	2	2

The authors in [7] affirm that this table makes it possible to identify a VARMA overall order of z_t by searching for a lower right rectangular pattern of k in each place (m,j) for m≥p, j≥q. In the last example, taking into account that k=2, we could identify as a pair of overall orders any (p,q) such that p≥1 or q≥1.

In the last pages (pp. 197-213) of [7], several experts discuss the paper profoundly, underlining its contributions and raising questions about the limitations of the proposed method. For instance, Priestley comments: "*Of further interest is the question about how robust is the identification procedure with regard to the choice of m and h. (...) Is varying the value of m and/or h likely to result in substantially different orders for the component models*?"

Regarding the choice of h, some commentaries in [7, pp. 176-177] are ambiguous and could lead into errors. It seems appropriate to quote them:

"*In theory, h may assume any integer that is greater than or equal to m, and theorem 3 suggests that a high value of h is preferred. In* *practice, h controls the dimension of the matrix A(m,h,j) (…), it controls the effective sample size in the estimation (…). Thus, a low value of h would be preferable. As a compromise, we suggest using h=m in this first stage of the analysis to reduce computation. However, this choice of h has the risk of underspecifying MA models in the presence of skipping MA lags, e.g. seasonal MA models. (…) other values of h may be used when skipping lag is likely to occur.*

(…)

In searching for SCMs of minimal orders, it is reasonable to let h be guided by the specified overall order (p,q). For instance, in declaring that y_t has an SCM of order (0,0) (…) under the condition that the overall model is ARMA(p,q) (…). In general, we use h=m+q-j at the (m,j) position."

Therefore, we have no clear ideas to choose a suitable h. and we have focused this work on the specific objective that follows:

To find necessary and sufficient conditions to choose h in Γ(m,h,j) with the minimum value needed, so that, the theoretical affirmations in [7] will not lead to errors due to ambiguity.

3 Sure overall orders

The first step needed to analyze in depth this aspect is to detect and to outline the definition which is crucial in this analysis. As it has been mentioned, h can be guided by an specified overall order. However, the concept of overall order, as it is understood in [7] does not allow to answer the question posed. The concept of overall order in [7] is "unsure" and “non-optimum”. “Unsure” in the sense that sometimes h can be smaller than that required and “non-optimum” in the sense that sometimes it can be greater than that required. After what has just been said, it seems logical to propose the following definition whose utility will be clear in Propositions 1 and 2.

Definition 4.- We say that (s,r) is a pair of *sure overall orders* if and only if

$$\text{rank}\,\Gamma(s-1,s-1,r-1)=\text{rank}\,\Gamma(s+u,s+u,r+u) \quad \forall\, u\geq 0$$

The following results contribute with subtle improvements, though important both in theory and in practice. Moreover, the proofs show how you can obtain all the sets of k l.i. SCMs with certain overall orders.

Proposition 1.- If (s,r) is a pair of sure overall orders, then for m≥s-1 and j≥r-1, the value rank Γ(m,h,j) is independent of h for h≥m. Therefore, it can be obtained all the possible SCMs with overall orders (s,r) from the matrix Γ(s,h,r), considering h=s-1.

Proof:
If (s,r) is a pair of sure overall orders ⇒
rank Γ(s-1,s-1,r-1)=rank Γ(s+u,s+u,r+u), ∀ u≥0
⇒ the first k(u+1) columns of Γ(s+u,s+u,r+u) are linearly dependent (l.d.) of the columns in Γ(s-1,s-1,r-1), and the last k(u+1) rows of Γ(s+u,s+u,r+u) are l.d. of the rows in Γ(s-1,s-1,r-1)
⇒ rank Γ(s-1,s-1,r-1) = rank Γ(s,s-1,r) =
rank Γ(s,s+u,r), ∀u≥0.
As rank Γ(s-1,s-1,r-1) = rank Γ(s,s-1,r), according to de Rouche Frobenius' Theorem, the system

$$\begin{pmatrix} \Gamma_r & \cdots & \Gamma_{r-s+1} \\ \vdots & & \vdots \\ \Gamma_{r+s-1} & \cdots & \Gamma_r \end{pmatrix}\begin{pmatrix} \phi_1 \\ \vdots \\ \phi_s \end{pmatrix}=\begin{pmatrix} \Gamma_{r+1} \\ \vdots \\ \Gamma_{r+s} \end{pmatrix} \qquad (2)$$

has a solution. Besides, as rank Γ(s-1,s-1,r-1) = rank Γ(s,s+u,r), ∀u≥0, any solution of (2) is also solution of

$$\Gamma_{i-1}\phi_1 + \ldots + \Gamma_{i-s}\phi_s = \Gamma_i \qquad \forall i\geq r+1$$

Consequently, z_t has a representation VARMA(s,r)

$$z_t - \phi_1 z_{t-1} - \ldots -\phi_s z_{t-s} = a_t - \theta_1 a_{t-1} - \ldots -\theta_r a_{t-r}$$

Obviously, $\forall V_0 \,/\, |V_0| \neq 0$, the vector $y_t=V_0 z_t$ can be represented as k SCMs l.i. with overall orders (p,q) such that (p,q)≤(s,r) because

$$V_0 z_t - V_0\phi_1 z_{t-1} - \ldots - V_0\phi_s z_{t-s} = V_0 a_t - V_0\theta_1 a_{t-1} - \ldots - V_0\theta_r a_{t-r}$$

Note that it could be happen that $V_0\phi_{p+1}=\ldots=V_0\phi_s=V_0\theta_{q+1}=\ldots=V_0\theta_r=0$. □

The following result tries to take into account the process where ∃ k SCMs l.i. with overall orders (p,q) and, however, (p,q) is not a pair of sure overall orders. From Theorem 2 and Proposition 2 in [5], if z_t follows a VARMA model there exists at least a pair of sure overall orders (s,r) and if (i,j)≥(s,r) then (i,j) is a pair of sure overall orders[2].

Proposition 2.- Given a pair of sure overall orders (s,r): ∃ k SCMs l.i. with overall orders (p,q) where (p,q)≤(s,r) if and only if
rankΓ(p-1,r+s-q-1,q-1)=rankΓ(p+u-1,p+u-1,q+u-1), where (p+u,q+u) is a pair of sure overall orders.

Proof:
"⇒"
∃ k SCMs l.i. with overall orders (p,q) ⇔

$$\Gamma_{i-1}\phi_1 + ... + \Gamma_{i-p}\phi_p = \Gamma_i \qquad \forall i\geq q+1$$

⇔ the system

$$\begin{pmatrix} \Gamma_q & \cdots & \Gamma_{q-p+1} \\ \vdots & & \vdots \\ \Gamma_{r+s-1} & \cdots & \Gamma_{r+s-p} \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_p \end{pmatrix} = -\begin{pmatrix} \Gamma_{q+1} \\ \vdots \\ \Gamma_{r+s} \end{pmatrix} \qquad (3)$$

has solution ⇔
rank Γ(p-1,r+s-q-1,q-1) = rank Γ(p,r+s-q-1,q) and, moreover,
rankΓ(p-1,r+s-q+v,q-1)=rankΓ(p,r+s-q+v,q) ∀v≥-1.

Since ∃ k SCMs l.i. with overall orders (p,q) then ∃ k SCMs l.i. with overall orders (p+i,q+i), ∀i>0, then
rankΓ(p,r+s-q+v,q)=rankΓ(p+1,r+s-q+v,q+1) ∀v≥-2,
rankΓ(p+1,r+s-q+v,q+1)=rankΓ(p+2,r+s-q+v,q+2) ∀v≥-3,
...
rankΓ(p+u-2,r+s-q+v,q+u-2)=rankΓ(p+u-1,r+s-q+v,q+u-1) ∀v≥-u.

Choosing u such that p+u=s and q+u≥r, or, p+u≥s and q+u=r, then (p+u,q+u) is a pair of sure overall orders and p+u-1≥r+s-q-2.
Therefore
rankΓ(p-1,r+s-q,q-1)=rank Γ(p+u-1,p+u-1,q+u-1).
"⇐"
Since (p+u,q+u) is a pair of sure overall orders, rank Γ(p+u-1,p+u-1,q+u-1)= rank Γ(p+t,p+t,q+t), ∀t≥u. Therefore, since
rank Γ(p-1,r+s-q-1,q-1)=rank Γ(p+u-1,p+u-1,q+u-1), any solution of

$$\begin{pmatrix} \Gamma_q & \cdots & \Gamma_{q-p+1} \\ \vdots & & \vdots \\ \Gamma_{r+s-1} & \cdots & \Gamma_{r+s-p} \end{pmatrix} \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_p \end{pmatrix} = \begin{pmatrix} \Gamma_{q+1} \\ \vdots \\ \Gamma_{r+s} \end{pmatrix}$$

is a solution of

$$\Gamma_{i-1}\phi_1 + ... + \Gamma_{i-p}\phi_p = \Gamma_i \qquad \forall i\geq q+1$$

Therefore, z_t is a vector of k SCMs l.i. with overall orders (p,q). □

The most important implication of Proposition 2 is that:
if (p,q) is not a pair of sure overall orders but ∃ k l.i. SCMs with overall orders (p,q), if we choose a pair of sure overall orders (s,r) such that:

i) (s,r)≥(p,q) and

ii) s+r≤i+j if (i,j) is a pair of sure overall orders verifying (i,j)≥(p,q),

then:
it can be obtained all the possible SCMs with overall orders (p,q) from the matrix Γ(p,h,q), considering h=r+s-q-1.

4 Conclusions

This paper can be considered as a little refinement of the identification procedure in [7]. The definition of a pair of sure overall orders -instead of a pair of overall orders- improve the interpretation of the results and reduce the computation involved -dimension of matrices involved has been reduced. As a consequence, statistical properties will improve too.

As an extension of this paper, we would like to seek necessary and sufficient conditions to know whether the proposed k SCMs are identifiable or not. In the negative case, it would be necessary in the estimation stage to know which are all the redundant parameters.

The whole procedure lends itself to a future implementation in standard statistical packages and it could be anticipated a significant progress in multivariate time series analysis of economic data.

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