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Analyses of Smooth Functions for Spectral Simulation of Traffic Flow

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Abstract: - This paper studies spectral methods for vehicular traffic flow simulation. According to related studies, high-order macroscopic models cannot ensure the anisotropic property of traffic flow. Fortunately, the first-order model, *i.e.*, LWR model, satisfies the anisotropic property in single-lane traffic. Although LWR model is the simplest macroscopic model, it is still a nonlinear model. For uninterrupted traffic flow, the initial and boundary conditions are continuous. The LWR model can be solved by general numerical algorithms. However, if interrupted traffic flow of signalized roads in an urban is considered, high-order numerical algorithm must be employed to approximate the discontinuous conditions. Besides, a high-order numerical algorithm takes more computing time and needs more strict constraints for convergence. If the number of simulated roads is huge, it is impossible to provide real-time information and control strategies of traffic flow. Spectral method, which is developed in 1970s, is a numerical solution method of partial differential equations. The method has the property of infinite differentiable and can be computed rapidly. It is considered as an alternative of finite difference method and finite element method. Spectral method employs a given smooth function to approximate the unknown equation. Generally, Fourier series、Chebyshev、Legendre、Laguerre or Hermite polynomials are considered candidates of smooth functions. Different forms of differential equations and conditions should be solved by differential polynomials. In this study, the convergence of the smooth function is analysed and compared numerically. Eventually, the suitable method will be suggested to simulate traffic flow.

Key-Words: - traffic flow model, smooth function, spectral method.

1 Introduction

Traffic congestion generates the interest in traffic flow researches. Traffic flow theory [1-19] is a new science, which has addressed questions related to understanding traffic processes and optimising these processes through proper design and control. The former questions could be described as basic research and the latter as applied research. Dynamic traffic management, such as ramp metering, congestion tolls, entrance/exit control and so on, is an efficient tool to solve traffic congestion. Traffic flow model is the fundamental theory of dynamic traffic management. There are four main modelling methodologies: car-following models [1-2], kinetic models [3-13], Boltzmann-like models [14-15] and cellular automation (CA) [18-19]. Microscopic models can describe individual behaviors of vehicles well. However, they need a lot of computing time and resource. Also, microscopic models need suitable assumptions to aggregate the individual results to macroscopic phenomena. Macroscopic models can be analyzed, but they cannot ensure the anisotropic property of traffic flow except the first order model [16-17]. The first order model is presented by Lighthill, Whitham and Richards in 1955 [3-4]. With different boundary conditions, the LWR model is good enough to describe traffic flow under different control strategies. Therefore, LWR model with different boundary conditions is considered in this study.

Traffic flow theory provides the information of speed, density and volume on a link. When planning, designing, operating and optimizing a traffic network, we have to simulate traffic flow on all links included in a network. Therefore, a robust and efficient numerical is needed while planning and implementing signal control strategies especially on-line control. Spectral methods involve seeking the solution to a differential equation in terms of a series of known, smooth functions. They have recently emerged as a viable alternative to finite difference and finite element methods for numerical solution of partial differential equations [20-25]. The key recent advance was the development of transform methods for the efficient implementation of spectral equations. Mostly, Fourier series, Chebyshev, Legendre, Laguerre, Hermite polynomials are considered as possible smooth functions for spectral method. In this study, analysis and numerical comparison are presented. According to the result, Fourier series is the most proper one for simulating traffic flow problem, because the boundary conditions may be periodic and non-periodic. Also, the convergence of Fourier series is the best among the candidate polynomials. The remaining content of this study is given as follows. Sec. 2 briefly reviews dynamic traffic flow models. Sec. 3 shows the spectral methods and smooth functions. Sec. 4 presents the computational method and numerical results. Finally, Sec. 5 draws the conclusion.

2 Historical Evolution of Traffic Flow Models

In this section, methodologies of dynamic traffic flow researches are reviewed briefly. Firstly, car-following theory is mentioned. Car-following theory is based on the dependency among vehicles traveling a platoon. It is microscopic dynamic model. The model, which represents the behavior of individual cars as they fight for survival and a place in a line of cars moving along a highway, was first considered by Reuschel [1] and Pipes [2] in the early 1950s. Every driver who finds himself in a single-lane traffic situation is assumed to react mainly to a stimulus from his immediate environment. The stimulus was assumed to be a function of the position of the car, the position of its neighbors, and the time-derivatives of these positions. It was conjectured, and verified experimentally, that the strongest stimulus was the relative speed of the car with respect to the car in front.

Secondly, macroscopic dynamic traffic flow models are discussed. In 1955, Lighthill, Whitham [3] and Richards [4] firstly proposed the LWR model, which is a continuity equation. The model, which is given as Eq. (1), is derived from the conservation of vehicle numbers.

$$\partial k/\partial t + \nabla \cdot Q = 0 , \tag{1}$$

where k is density and Q is flow. u is introduced as speed and the relation among the three variables is $Q = ku$. The assumption of LWR model is that speed changes instantaneously as density changes, i.e., speed is only a function of density. It is certainly not valid in some traffic flow situations. To overcome the steady state assumption of velocity, Payne [5] used a motion equation to obtain time variant speed and proposed a second order model, which was named as PW model [5-7]. PW model is given by coupling Eqs (1) and (2).

$$\frac{\partial u}{\partial t} + u(\nabla \cdot u) = -\frac{1}{k}\nabla \cdot P_e(k) + \frac{1}{\tau}\left(u_e(k) - u\right), \tag{2}$$

where $P_e(k)$ is the equilibrium traffic pressure, $u_e(k)$ is the equilibrium speed and τ is the relaxation time. $P_e(k)$ and $u_e(k)$ are functions of density. The term $-\nabla \cdot P_e(k)/k$ is an anticipation term, which takes it into account that drivers beware the preceding traffic condition. However, this kind of models has a lot of arguments, so families of gas-kinetic models [8-13] are presented. The systematic partial differential equations can be derived from the macroscopic traffic phenomena or the Boltzmann equation [8-13]. The systematic model, which includes continuity, motion and variance equations, is coupling by Eqs (1) ~ (3).

$$\frac{\partial \theta}{\partial t} + u(\nabla\theta) = -2\theta(\nabla \cdot u) + 2\frac{\mu}{k}(\nabla \cdot u)^2 + \frac{2}{\tau}(\theta_e(k) - \theta) + \frac{\mu}{k}\nabla\cdot(\nabla u) + \frac{\kappa}{k}\nabla\cdot(\nabla\theta), \tag{3}$$

where θ is speed variance, μ and κ are coefficients. θe is introduced as the equilibrium variance, which only depends on density. The systematic model describes the traffic situations that the variation of density, velocity and variance are significant. Three results are observed from the numerical simulations: (1) The section of high density induces low speed and small speed variance; (2) The section of low density induces high speed and large speed variance; (3) The largest speed variance takes place at the highest speed behind a platoon.

Vehicular Boltzmann equation is a further modeling methodology. This kind of model is first employed to describe traffic flow by Prigogine, Herman and their colleagues [14] and is referred to the Boltzmann-like model. Prigogine described a traffic fluid with a probability density for the velocity (v) of an individual car, $f(x,v,t)$, which may vary with a function of time t and the coordinate x along the highway. This density is assumed to satisfy the equation

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} = \left(\frac{\partial f}{\partial t}\right)_{relaxation} + \left(\frac{\partial f}{\partial t}\right)_{interaction} . \tag{4}$$

The first term of the right-hand element of Eq. (4) stems from the fraction that $f(x,v,t)$ differs from some desired velocity distribution $f^0(v)$. The second term describes that a fast car will slow down owing to the influence of a slow car. The interaction term has been criticized. It has been argued that the collision term is only valid in the situation, which describes the incoming vehicle passing each single car in the queue independently. Therefore, Paveri-Fontana [15] proposed an improved model to overcome the shortcoming of Prigogine's approach. He generalized the phase-plane distribution as $\tilde{f}(x, v, v^0, t)$, where v^0 is the individual desired velocity. $f(x, v, t) = \int \tilde{f}(x, v, v^0, t) dv^0$. Nevertheless, his model still considered that the maneuver on the lanes of a multilane road is the same and his model does not take queuing effects into account. Transitory and stationary solutions and numerical simulations are proposed by researches.

A more recent addition to the development of vehicular traffic flow theory is cellular automation (CA) or particle hopping method [18-19]. In CA, a road is represented as a string of cells, which are either empty or occupied by exactly one vehicle. Movement takes place by hopping between cells. All vehicles execute in parallel the following steps:

(1) Let g (gap) equal the number of empty sites ahead.

(2) If $u > g$ (too fast), then slow down to $u = g$ (rule 1); otherwise if u<g (enough headway) and $u < u_{maz}$, then accelerate by one $u=u+1$ (rule 2).

(3) Randomization: if after the above steps the velocity is larger than zero (u > 0), then, with probability p, reduce u by one (rule 3).

(4) Vehicle propagation: Each vehicle moves u sites ahead (rule 4).

We can make some conclusions and comments of traffic flow researches herein. As the LWR-like, PW or gas-kinetic models are employed, one has to give externally the relation between speed, flow, density and speed variance [3-15]. Microscopic simulation models usually been adopted for the simulation of relatively small or simple systems and macroscopic simulation can be used to simulated the large network or complex system. Numerous runs of microscopic simulations may be necessary to achieve such a long time. This limitation makes applications of these simulation models problematic for the provision of real-time traffic information. Higher order macroscopic models should be modeled carefully or unreasonable situation will be induced, such as wrong way traveling flow [16-17]. In addition, the higher order models also need numerous computing with convergent problems. Boltzmann-like equation, which is the mesoscopic model, brilliant though it is, has not attracted many followers, probably because of its exacting requirements in mathematical sophistication. Consequently, LWR model, which is a simple and sufficient model [16], is suggested to simulate dynamic traffic flow in this study.

3 Spectral Method and Smooth Functions

Generally, finite different and finite element methods are most used numerical methods for solving partial differential equation. Finite difference methods (FDM) approximate the unknown function by a sequence of low-order overlapping polynomials, which interpolate the unknown function at a set of grid points. The derivative of the local interpolant is used to approximate the derivative of the unknown function. The result takes the form of a weighted sum of the values of the unknown function at the interpolation points. Finite element methods (FEM) chop the space into a number of sub-intervals and choose basis functions, which are polynomials of fixed degree, which are non-zero only over a couple of sub-intervals. Spectral methods use global basis functions of high order degree, which are non-zero, except at isolate points, over the entire computational domain [20-26].

Spectral methods are benefit in two ways. First, the interval between two grid points becomes smaller would cause the error to rapidly decrease even if the order of the method were fixed. The error $\approx O[(1/N)^N]$, where N is the number of grid points in a given interval. Although the order of FDM and FEM are not fixed, the error of both methods depends on the order of polynomial rather than the number of grid points. Spectral methods converge exponentially. The second advantage is that the high accuracy of spectral methods is memory minimizing. Problems that require high resolution can often be done satisfactorily by spectral methods when a three-dimensional second order FDM code would fail because the need for eight or ten times as many grid points would exceed the core memory of the available computer. Therefore, spectral methods are introduced to traffic flow simulation in this study.

The history of spectral methods can be dated back to 1822 in the days of the French physician and mathematician Jean-Baptiste-Joseph Fourier (1768-1830). Fourier discovered that any periodic function fulfilling certain simple conditions could be represented as a finite series. The foundation for using spectral methods was laid in 1965 when Cooley and Tukey presented the fast Fourier transform (FFT) algorithm. This turned the spectral methods into a competitive alternative to existing numerical methods, e.g. finite-difference methods, since the computational effort could be reduced significantly. Therefore, the popularity of spectral methods used for solving partial differential equations increased in the early 1970s. Five most popular smooth polynomials are analyzed and compared in this section [20, 22, 25-26].

The first one is Chebyshev polynomial which is denoted by $T_n(x)$. Let the unknown function is $k(x)$ and a degree n approximation of $k(x)$ is given by $k_n(x)$. Then,

$$k_n(x) = \sum_{j=0}^{n} a_j^c T_j(x), \tag{5}$$

where

$$a_j^c = \frac{2}{c_j \pi} \int_0^{\pi} \frac{k(x)}{\sqrt{1-x^2}} T_j(x) dx, \tag{6}$$

$$c_j = \begin{cases} 1, & j \neq 0. \\ 2, & j = 0. \end{cases} \tag{7}$$

Chebyshev polynomial, $T_n(x)$, can be solved by the following recursive relations.

$$T_0(x) = \cos(0) = 1, \tag{8}$$

$$T_1(x) = \cos\left(\cos^{-1}(x)\right) = x, \tag{9}$$

…

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x). \tag{10}$$

The Legendre polynomial is denoted by $P_n(x)$. The degree n approximation is

$$k_n(x) = \sum_{j=0}^{n} a_j^{le} P_j(x), \tag{11}$$

where

$$a_j^{le} = \frac{2j+1}{2} \int_{-1}^{1} k(x) P_j(x) dx. \tag{12}$$

Legendre polynomial, $P_n(x)$, can be solved by the following recursive relations.

$$P_0(x) = 1, \tag{13}$$

$$P_1(x) = x, \tag{14}$$

…

$$P_{n+1}(x) = \frac{1}{n+1}\left[(2n+1)xP_n(x) - nP_{n-1}(x)\right]. \tag{15}$$

The Laguerre polynomial is denoted by $L_n(x)$. The degree n approximation is

$$k_n(x) = \sum_{j=0}^{n} a_j^{la}(t) L_j(x), \tag{16}$$

where

$$a_j^{la} = \frac{1}{2j+1} \int_0^{\infty} x e^{-x} k(x) L_j(x) dx. \tag{17}$$

Laguerre polynomial, $L_n(x)$, can be solved by the following recursive relations.

$$L_0(x) = 1, \tag{18}$$

$$L_1(x) = 1 - x, \tag{19}$$

…

$$L_{n+1}(x) = \frac{1}{n+1}\left[nL_{n-1}(x) - (2n+1-x)L_n(x)\right]. \tag{20}$$

The Hermite polynomial is denoted by $H_n(x)$. The degree n approximation is

$$k_n(x) = \sum_{j=0}^{n} a_j^h H_j(x), \tag{21}$$

where

$$a_j^h = \frac{1}{2^j j! \sqrt{\pi}} \int_{-1}^{1} e^{-x^2} k(x) P_j(x) dx, \tag{22}$$

Hermite polynomial, $H_n(x)$, can be solved by the following recursive relations.

$$H_0(x) = 1, \tag{23}$$

$$H_1(x) = 2x, \tag{24}$$

…

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x). \tag{25}$$

The last one is Fourier series, which is given as

$$k_n(x) = a_0 + \sum_{j=1}^{n} a_n \cos(jx) + \sum_{j=1}^{n} b_n \sin(jx), \tag{26}$$

where

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} k(x) dx, \tag{27}$$

$$a_j = \frac{1}{2\pi}\int_{-\pi}^{\pi} k(x)\cos(jx)dx, \tag{28}$$

$$b_j = \frac{1}{2\pi}\int_{-\pi}^{\pi} k(x)\sin(jx)dx. \tag{29}$$

Definition 1. (Convergence)

Let $k_n(x)$ is the approximation function of $k(x)$. When $n \to \infty$, $\forall x \in [0, L]$, $k_n(x)$ satisfies

$$\|k(x) - k_n(x)\| \to 0, \tag{30}$$

then $k_n(x)$ is said to be convergent to $k(x)$ 。

Riemann-Lebesgue Lemma

If $f(x)$ is integrable on $[-\pi, \pi]$, then

$$\lim_{t\to\infty}\int_{-\pi}^{\pi} f(x)\sin(tx)dx = 0, \tag{31}$$

and

$$\lim_{t\to\infty}\int_{-\pi}^{\pi} f(x)\cos(tx)dx = 0. \tag{32}$$

Before analyzing of functions, the Fourier series is rewritten as

$$k_n(x) = \sum_{j=0}^{n} a_j^f e^{ijx}, \tag{33}$$

where

$$a_j^f = \frac{1}{2\pi}\int_0^{2\pi} e^{-ijx} k(x)dx. \tag{34}$$

Substituting the approximation of each polynomials into Eq. (30), we have

$$E_n = \|k(x) - k_n(x)\| = \left|\sum_{j=n+1}^{\infty} a_j^{poly} F(x)\right|, \tag{35}$$

where *poly* represents different coefficients of polynomials, i.e., *poly* = *f* is Fourier series, *poly* = *c* is Chebyshev polynomial, *poly* = *le* is Legendre polynomial, *poly* = *la* is Laguerre polynomial and *poly* = *h* is Hermite polynomial. $F(x)$ represents different polynomials, *i.e.*, $F(x) = e^{ijx}$ is Fourier series, $F(x) = T(x)$ is Chebyshev polynomial, $F(x) = P(x)$ is Legendre polynomial, $F(x) = L(x)$ is Laguerre polynomial and $F(x) = H(x)$ is Hermite polynomial. The analysis starts with Fourier series. According to Eqs (33) and (35), we have

$$E_n = \|k(x) - k_n(x)\| = \left\|\sum_{j=n+1}^{\infty} a_j^f e^{ijx}\right\| \approx O\left(\left|a_n^f\right|\right). \tag{36}$$

If $k(x)$ is *d*-times differentiable and $k^{(d)}(x)$ is integrable, then we differentiate Eq.(34) *d* times and have

$$a_n^f = \frac{1}{2\pi(in)^d}\int_0^{2\pi} k^{(d)}(x)e^{-inx}dx. \tag{37}$$

Since $k^{(d)}(x)$ is integrable, the Riemann-Lebesgue Lemma tells us

$$a_n^f << 1/n^d, \tag{38}$$

that is, $E_n = O(1/n^d)$ when $n \to \infty$. For example, if $k(x)$ is a second order differentiable function, then $E_n = O(1/n^2)$.

The analysis of Chebyshev polynomial is the same as Fourier series. According to the same procedure, we have

$$a_n^c << 1/n^d, \tag{39}$$

that is, when $n \to \infty$, $E_n = O(1/n^d)$.

From Davis's study in 1975, the convergence of Legendre polynomial is worse than Chebyshev polynomial. When $n \to \infty$, a $\sqrt{2/(\pi n)}$ difference exists, i.e.,

$$a_n^{le} \approx 1/n^d \sqrt{2/(n\pi)}. \tag{40}$$

Therefore, $E_n = O(1/n^{d-1/2})$, when $n \to \infty$. Figure 1 compares Chebyshev and Legendre coefficients. Both sets asymptote to parallel straight lines, but Chebyshev coefficients are smaller. The convergence of Legendre polynomial is just worse than Chebyshev polynomial.

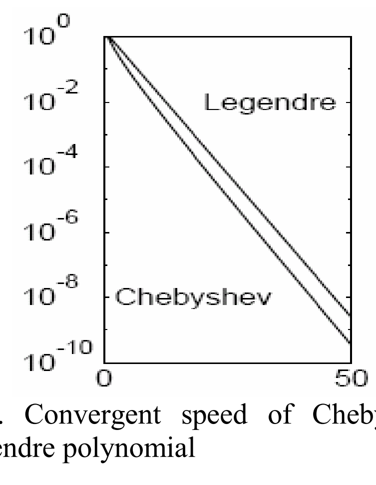

Figure 1. Convergent speed of Chebyshev and Legendre polynomial

The convergence of Laguerre polynomial depends on the unknown function. Here, sin$x$ is considered as an example.

$$\sin x = \sum_{n=0}^{\infty} \frac{1}{2^{(n+1)/2}} \cos\left[\frac{\pi}{4}(n+1)\right] L_n(x). \tag{41}$$

Therefore, when $n \to \infty$

$$L_n(x) \approx \frac{1}{\sqrt{\pi}} e^{x/2} x^{-1/4} \pi^{-1/4} \cos\left[2\sqrt{nx} - \frac{1}{4}\pi\right]. \tag{42}$$

If $n >> x$, then

$$E_n \approx \frac{e^{x/2}}{2^{n/2}(nx)^{1/4}}. \tag{43}$$

That means E_n is small only when $n \geq 1.44x$. If the unknown function is periodic, then $n \geq 1.44x$ must be satisfied in one period so as to achieve acceptable accuracy. Hence, to achieve the same accuracy, the Laguerre expansions require many more terms than Chebyshev and Legendre expansions.

Hermite polynomial can be analyzed as the same way of Laguerre polynomial. Also, the expansion of sin$x$ is considered as an example.

$$\sin x = \sum_{n=0}^{\infty} \frac{1}{2^{2n+1}(2n+1)!} H_{2n+1}(x). \tag{44}$$

When $n \to \infty$,

$$E_n(x) \approx \frac{n!}{(n/2)!} e^{x^2/2} \cos\left(\sqrt{2n+1}x - \frac{1}{2}n\pi\right). \tag{45}$$

E_n will close to zero rapidly at x only if $n \geq x^2 \log x$,. The result is very bad. To obtain high accuracy, Laguerre and Hermite polynomials require too many terms. Consequently, the two polynomials are not suggested to be the smooth function of spectral method in this study.

Therefore, Fourier series and Chebyshev polynomials are suggested to be the smooth functions of spectral methods herein. Both Fourier series and Chebyshev polynomials can be computed by Fast Foutier Transform (FFT). Firstly, we compare the numerical results of Fourier series and Chebyshev polynomial. If the space is discretized by uniform grid, Fourier series performs accurate results and repaid computation. On the other hand, approximation of Chebyshev polynomial produces oscillations. To improve the result, Chebyshev grid should be considered. Figure 2 illustrates the comparison of unispaced and Chebyshev points. Thus, using Chebyshev polynomial needs some pre-processes and pre-analyzes so as to design Chebyshev grid. The application of Fourier series is more convenient than Chebyshev polynomial.

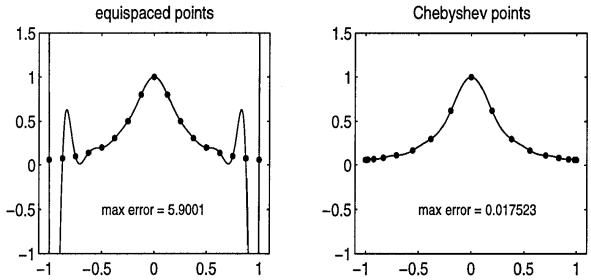

Figure 2. Comparison of (a) equispaced and (b) Chebyshev points for degree N intepolation of $u(x) = 1/(1+16x^2)$.

4 Spectral Method for Traffic Flow

The method begins by considering Eq. (1) with the Greenshield model, which is the most used speed-density relation $u(k) = u_f(1 - k/k_j)$. u_f is free flow speed and k_j is jam density. Both of them are constants and obtained by calibrating of empirical data. Then $Q(k) = ku(k) = u_f(k - k^2/k_j)$. Equation (1) is rewritten as

$$\partial k/\partial t = Lk + N(k), \tag{46}$$

where L is a linear operator and is equal to $-u_f(\partial/\partial x)$. $N(k)$ includes the nonlinearly and non-constant coefficient terms, which is equal to $2(u_f k/k_j)(\partial k/\partial x)$. By applying Fourier transform, Eq.(33) reduce to a ordinary differential equation

$$d\hat{k}/dt = \alpha(\omega)\hat{k} + \hat{N}(k), \tag{47}$$

where $\hat{k}$ and $\hat{N}$ are Fourier transform of $k(x)$ and $N(k)$, respectively. $\alpha(\omega) = -i\omega u_f$. Once the PDE is reduced to ODE, Eq.(47) can be solved by Runge-Kutta method.

Five cases with continuous and discontinuous initial and boundary conditions are simulated herein. A 1.5 km highway and 3 minutes time period are considered. Free flow speed u_f is 90 km/hour (kph) and jam density k_j is 120 vehicle/km. The first case describes platoon formation traffic. The inflow boundary condition is given as a constant, which is a jam situation. The downstream boundary is a Neumann condition, which allows vehicles leaving the highway section freely. Besides, the initial condition describes a gradual increasing traffic flow. Equation (48) gives the initial and boundary conditions.

Figure 3 illustrates the simulated result. Since the input boundary always keeps at jam density, platoon forms gradually and the highway becomes congestion. Case 2 is a platoon dissipative traffic. The entrance flow is assumed to be a moderate density and the initial condition is a high-low traffic situation. It may occur on a highway after an incident has just been removed. Equation (49) gives the initial condition and boundary conditions. Therefore, the pre-existing platoon dissipates as time goes by. Finally, the traffic flow becomes smooth and stable. The result is shown in Fig. 4. Traffic flow with a periodic initial condition is simulated in case 3. Assume the input flow is constant, which is smaller than the jam density, i.e., the simulated traffic flow is under free flow situation. Equation (50) gives the initial and boundary conditions. Figure 5 illustrates the result. Also, the platoon moves forward and dissipates and finally traffic density is equal to the input density. The traffic flow becomes stable. According to previous three cases, the spectral method can solve traffic flow problems with continuous conditions well.

$$\begin{cases} k = 80 + 40 \times Exp\left(0.2x - 1.2x^2\right) & t = 0 \quad I.C. \\ k = 120, & x = 0 \quad B.C., \\ \partial k / \partial x = 0, & x = 1.5 \quad B.C. \end{cases} \quad (48)$$

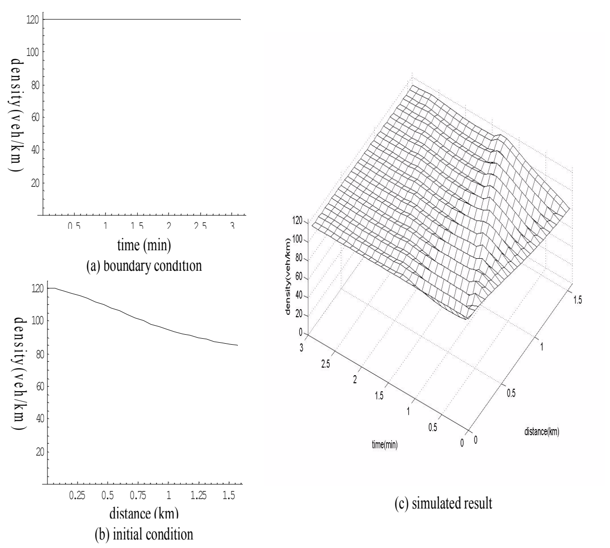

Figure 3. (a) boundary condition, (b) initial condition and (c) simulated result of case 1.

$$\begin{cases} k = 80 - 40 \times Exp\left(0.2x - 1.2x^2\right) & t = 0 \quad I.C. \\ k = 80, & x = 0 \quad B.C., \\ \partial k / \partial x = 0, & x = 1.5 \quad B.C. \end{cases} \quad (49)$$

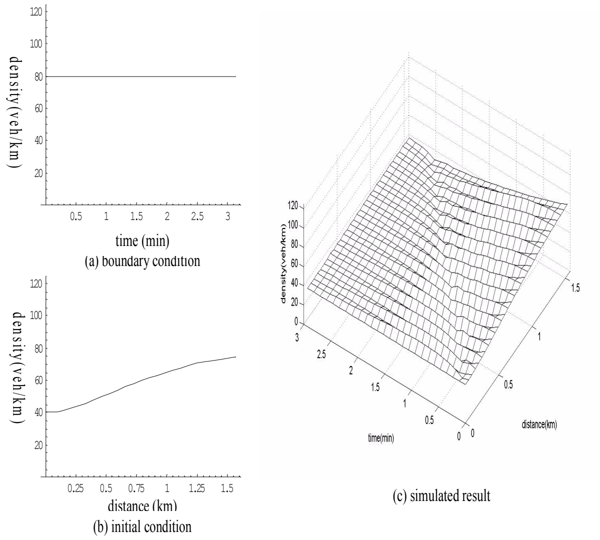

Figure 4. (a) boundary condition, (b) initial condition and (c) simulated result of case 2.

$$\begin{cases} k = 50 + 30 \times \cos(\pi x) & t = 0 \quad I.C. \\ k = 80, & x = 0 \quad B.C., \\ \partial k / \partial x = 0, & x = 1.5 \quad B.C. \end{cases} \quad (50)$$

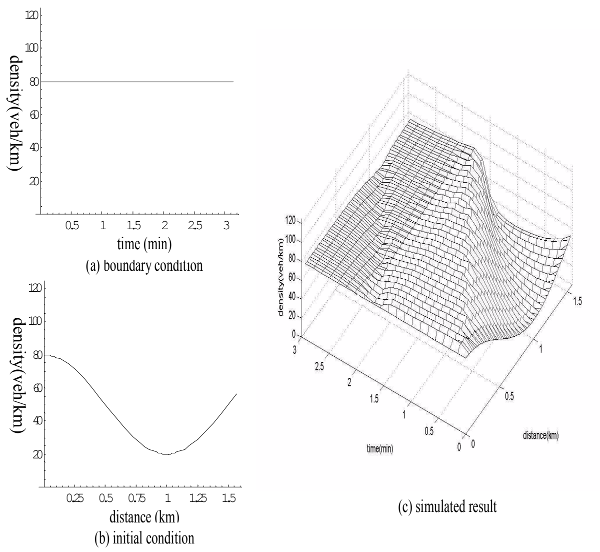

Figure 5. (a) boundary condition, (b) initial condition and (c) simulated result of case 3.

Cases 4 and 5 involve with discontinuous initial and boundary conditions. Figures 6 and 7 illustrate the simulated results, respectively.

Discontinuous condition occurs while shock wave appears or at signalized intersections. Equation (51) is the initial and boundary conditions of case 3. The case describes dissipation of shock wave. The situation may occur behind a red-light signalized intersection. Oscillation of solution caused by discontinuous initial condition. Fortunately, it disappears with time. Equation (52) gives a more realistic condition. It simulates one cycle length of signal, which starts with red light. When the signal is red at the entrance of the road, the pre-existing platoon in the road dissipates. As the signal turns green, platoon moves forward with jam density. In this case, the waiting queue of vehicles is assumed to release during the green light. Oscillation occurs severely in this case. In one signal cycle, the oscillation won't cause divergence of solution. However, if two or more cycles are considered, it diverges. Therefore, post-processes [27-28] must be considered appropriately so as to smooth out the oscillation while applying spectral methods to interrupted traffic flow.

$$\begin{cases} \begin{matrix} k = 70, & x <= 0.5, \\ k = 90, & x > 0.5 \end{matrix} \quad t = 0 \quad I.C. \\ k = 70, \qquad x = 0 \quad B.C., \\ \partial k/\partial x = 0, \qquad x = 1.5 \quad B.C. \end{cases} \tag{51}$$

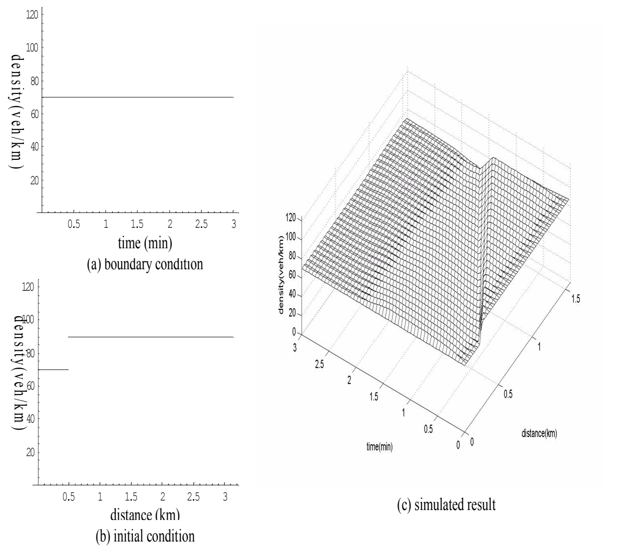

Figure 6. (a) boundary condition, (b) initial condition and (c) simulated result of case 4.

$$\begin{cases} k = 40 - 40 \times Exp\left(0.2x - 1.2x^2\right) & t = 0 \quad I.C. \\ k = 0, \quad x < 1, x >= 2.33 & \\ k = 100, \quad 1 <= x < 1.33 & x = 0 \quad B.C. \\ k = 100 \times \left(1 - (x - 1.33)\right), \quad 1.33 <= x < 2.33 & \\ \partial k/\partial x = 0, & x = 1.5 \quad B.C. \end{cases}, \tag{52}$$

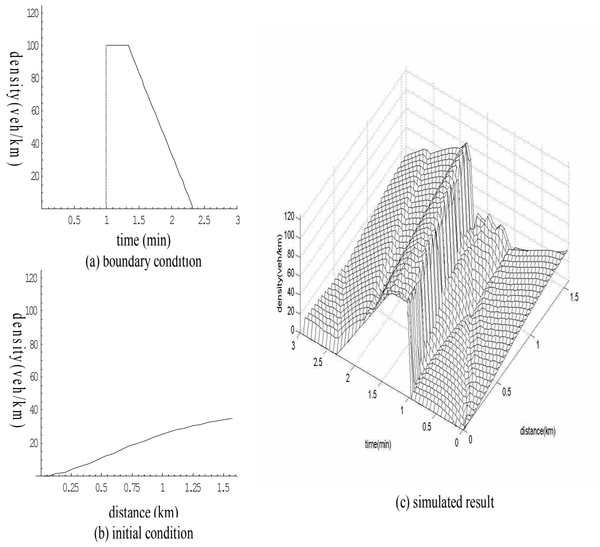

Figure 7. (a) boundary condition, (b) initial condition and (c) simulated result of case 5.

4 Conclusions

In this paper, the spectral method is applied to simulate traffic flow simulation successfully. Five smooth functions are analyzed and compared. Fourier series is considered as the best one. Continuous and discontinuous conditions are simulated. The spectral method performs well with continuous condition under congestion and noncongestion traffic situation. However, oscillation occurs in discontinuous problems. If the problem is simple, the oscillation is small and negligible for traffic flow simulation. If the problem is complicate, such as traffic flow behind a signalized intersection, the discontinuity is periodic. Oscillation of solution will propagate and the solution diverges. Tadmor [27-28] develops spectral viscosity method and shock capturing method to find out the location of discontinuity then approximate the discontinuity by Gegenbauser reconstruction method. The method not only provides accurate approximation but also converges rapidly. This part must be considered in further work.

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