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Dynamical Systems and Step-2 Nilpotent sub-Riemannian Geometry

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Abstract: We consider a dynamical system determined by a finite family of smooth vector fields. By using variational techniques, we obtain the equations of motion and the corresponding normal curves. We approach the problem as the sub-Riemannian geodesic problem on a step-2 nilpotent Lie group and present a detailed study for distributions of vector fields of order two and three, which lead to orthogonal Lie groups of dimensions three and six respectively. For the later case we compute the unit sub-Riemannian sphere and the wave front.

Key–Words: Euler-Lagrange equations, Step-2 nilpotent Lie group, sub-Riemannian geodesics, sub-Riemannian unit sphere, wave front.

1 Introduction

Nonlinear dynamical systems defined by means of finite families of vector fields have been studied in both the theory of dynamical systems (e.g. [1], [2]), and the differential geometric control theory, see for instance [3] and [4]. More recently has been shown that systems defined by a distribution of smooth vector fields determine intrinsically a geometric structure on the underlying manifold that goes under the name of singular, sub-Riemannian or Carnot-Caratheodory geometry, and provide a natural framework for studying certain problems in physics, see for instance [5], [6] and [7]. Some general aspects of the sub-Riemannian geometry for distributions of arbitrary degree on step-2 nilpotent Lie groups are discussed in our forthcoming paper [8].

We study in this paper, a nonlinear dynamical system given by means of a distribution Δ of smooth vector fields $\{X_1, \dots, X_n\}$. The flow of such a system is given by the solution of the nonlinear differential equation

$$\dot{q} = u_1 X_1(q) + \cdots + u_n X_n(q), \qquad (1)$$

where $u = (u_1, \dots, u_n)$ is a vector valued measurable and bounded function and the state variable q belongs to certain smooth manifold $\mathcal{M}$. When considering u as the control parameter one can think of (1) as a *controlled* dynamics on $\mathcal{M}$, and use the control theoretic techniques for deriving the properties of the flow. In contrast, we approach the problem from the point of view of variational calculus with non-holonomic constraints.

Apart from this introduction the paper contains three sections, in section 2 we derive the equations of motion in the standard way, i.e. through Euler-Lagrange formalism. In section 3 we present first a general idea about a hierarchy of dynamical systems determined by means of non-holonomic constraints defined by families of non-integrable 1-forms. These set of ideas were presented by R.W Brockett and L. Dai in [9]. We shall pursue the study of such hierarchy elsewhere. We then continue establishing a relationship of the dynamical system with the sub-Riemannian geodesic problem for the case of step-2 nilpotent Lie groups. In section 4 we integrate explicitly the extremal equations for the normal geodesics and discuss in detail some low dimensional cases, providing, in particular, a parametrization of the sub-Riemannian exponential mapping. At the end, we consider the sub-Riemannian unit sphere and the wave fronts for the $(3, 6)$ case, some commented pictures for level surfaces of these geometric objects are presented.

2 The Equations of Motion

Assume that $\mathcal{M} = \mathbb{R}^n \times \mathbb{R}$ with coordinates $q = (x, y)$, and assume also that there is a sufficiently smooth vector valued function $\xi = (\xi_1, \dots, \xi_n)$ such the dynamical system is written as follows

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$$\dot{x} = u, \quad (2)$$
$$\dot{y} = \langle u, \xi \rangle. \quad (3)$$

equation (1) directly implies

$$X_i = \frac{\partial}{\partial x_i} + \xi_i \frac{\partial}{\partial y}.\ i = 1, \ldots, n$$

We consider the distribution $\Delta = \{X_1, \ldots, X_n\}$. It is well known that Δ together with all its Lie brackets generate a Lie algebra that we shall denote by $\mathfrak{g}$.

If ξ is analytic then it is given by its Taylor series expansion around the origin as follows,

$$\xi_j(x) = \sum_{\nu > 0} \frac{1}{\nu!} \frac{\partial^\nu \xi_j}{\partial x^\nu}\bigg|_{x=0} x^\nu$$

where ν is the multi-index $(i_1, \ldots, i_n)$, with nonnegative entries, $\nu! = i_1! i_2! \cdots i_n!,\ x^\nu = x_1^{i_1} \cdots x_n^{i_n}$, and

$$\frac{\partial^\nu}{\partial x^\nu} = \frac{\partial^{i_1}}{\partial x_{i_1}} \cdots \frac{\partial^{i_n}}{\partial x_{i_n}}$$

The non-zero Lie brackets

$$X_{ij} := [X_i, X_j] = F_{ij} \partial y,\ i \neq j$$

determine a family of analytic functions F_{ij}.

Let G be the simply connected Lie group associated with the Lie algebra $\mathfrak{g}$, in such a way that the X_i are *left invariant* vector fields. For each $q \in G$ one defines on each plane $\Delta(q) = \mathrm{span}\{X_1, \ldots, X_n\}$ a smoothly varying inner product $\langle \cdot, \cdot \rangle_{\Delta(q)}$ by declaring the vectors $\{X_i(q)\}$ orthonormal. An absolutely continuous curve $q : [0, t_q] \to G$, is said to be *horizontal*, provided $\dot{q}(t) \in \Delta(q)$, almost everywhere.

We shall consider the variational problem on G consisting on the minimization of the kinetic energy action:

$$S_0 = \frac{1}{2} \int \|\dot{q}(t)\|^2 \, dt, \quad (4)$$

in the class of horizontal curves.

As customary, the standard variational method consists in the study of the Lagrangian

$$\mathcal{L} = \frac{\lambda_0}{2} \|\dot{q}(t)\|^2 + \lambda(\dot{y} - \langle \xi, \dot{x} \rangle) \quad (5)$$

The case for which $\lambda_0 = 0$ is usually called the abnormal or singular case. For $\lambda_0 \neq 0$, we get the normal case for which we can set $\lambda_0 = 1$. An easy but lengthy calculation gives the following

Proposition 2.1 *The Euler-Lagrange equations for the normal extremals are*

$$\ddot{x} = \lambda F \dot{x}, \quad \dot{y} = \langle \xi, \dot{x} \rangle \quad \dot{\lambda} = 0, \quad (6)$$

whereas the abnormal extrema satisfy

$$F \dot{x} = 0.$$

It is evident that in the normal case the Lagrange parameters λ_i are constants of motion, and for the even dimensional case there are no abnormal extremals.

3 The sub-Riemannian geodesic problem

The Lie algebra obtained from bracketing the distribution Δ is in general infinite dimensional and it is determined by the analyticity of function ξ. In this work, we consider finite expansions around the origin, approach that fits into the hierarchy introduced by R. Brockett and L. Dai in [9], for analyzing the nonlinear effects of mechanical systems. In the aforementioned reference, the authors consider polynomial vector fields, which can be seen, as finite Taylor series approximation of analytic vector fields, through particular examples they study the hierarchy and call the subsystem $\dot{x} = u$ the "level" 0. The higher levels of the hierarchy essentially correspond to the powers in the truncated Taylor series.

We shall pursue the study of such hierarchy and its implications in both dynamical systems and sub-Riemannian geometry elsewhere. Here, we want to consider in detail the first level only, and to show how this problem can be solved using step-2 nilpotent Lie algebras. For the first level case, the polynomials ξ_i^1 are linear and we can select as the family of non integrable forms $\{x_i \, dx_j\}$, for $i < j$. There are exactly $n(n-1)/2$ of these forms.

Similarly, we could add complete differentials and obtain the more symmetrical choice $\{x_i \, dx_j - x_j \, dx_i\}$. We select for this work this last choice, since the corresponding vector fields turn out to be the left invariant vector fields of the Lie group associated to the Lie algebra resulting from Δ.

At this level, trigonometric functions are sufficient to express the solutions of the extremal problem. Higher levels are much more complicated in a general approach and only some particular cases can be solved, for which elliptic and hyperelliptic functions arise in certain settings, see for instance [9], [6] and [7].

In what follows we shall present the general setting, let G be a step-2 nilpotent Lie group of dimension $n(n+1)/2$, and let $\Delta = \{X_1, \ldots, X_n\}$ be a rank

n, bracket generating and left invariant distribution on G. We assume that $\{X_1, \ldots, X_n\}$ is a nilpotent basis for Δ, with order of nilpotency one, that is to say, $ad^k_{X_i}(X_j) = 0, \quad$ for all $\quad k > 1$. Furthermore, we assume that $X_1, \ldots, X_n$ together with the non-zero Lie brackets

$$X_{ij} := [X_i, X_j], \quad 1 \leq i < j = 2, \ldots, n, \qquad (7)$$

determine a basis of left invariant vector fields for the Lie algebra $\mathfrak{g}$ of the group G. The Chow-Rashevskii's theorem guarantees that any two elements $g_i, g_f \in G$ can be connected by an horizontal curve, i.e., an arc-length parametrized absolutely continuous curve $g : [0, T_g] \to G$ satisfying $g(0) = g_i,\ g(T_g) = g_f$, and $\dot{g}(t) \in \Delta(g)$, a.e.. The class of horizontal curves shall be denoted by $\mathcal{H}$.

As mentioned in the introduction, a sub-Riemannian structure on the group G is naturally defined by declaring the vectors $X_1(g), \ldots, X_n(g)$ orthonormal, in order to define an smooth varying inner product $\langle \cdot, \cdot \rangle_g$ on each plane $\Delta(g) = \mathrm{span}\{X_1(g), \ldots, X_n(g)\}$. The *sub-Riemannian geodesic problem* consists of the minimization of the length functional

$$\ell(g) = \int_0^{T_g} \|\dot{g}(t)\|_g \, dt,$$

on the class $\mathcal{H}$. Incidentally the sub-Riemannian distance

$$d(g_i, g_f) = \inf_{g \in \mathcal{H}} \{\ell(g) \mid g(0) = g_i,\ g(T_g) = g_f\}$$

is well defined and finite. It reduces the amount of calculations the consideration of the functional of energy

$$\mathcal{E}(g) = \int_0^{T_g} \|\dot{g}(t)\|_g^2 \, dt,$$

instead of the functional ℓ, both variational problems are equivalent. Furthermore the orthonormality of the vector fields implies

$$\begin{aligned} \|\dot{g}\|_g^2 &= \langle u_1 X_1(g) + \cdots + u_n X_n(g), \\ &\quad\ u_1 X_1(g) + \cdots + u_n X_n(g) \rangle \\ &= u_1^2 + \cdots + u_n^2 = 1. \end{aligned}$$

At the level of the cotangent bundle T^*G the variational problem can be tackled in a coordinate-free fashion, see [8]. However for the purpose of writing down the trajectories in the base manifold it is necessary to select coordinates. We shall consider coordinates given by pairs $g = (x, z) \in \mathbb{R}^n \times \mathfrak{so}_n$. In these coordinates the nonholonomic constraints

$$\dot{g}(t) = \sum_{i=1}^n u_i \, X_i(g)$$

yield the following expression for the vector fields

$$X_i = \frac{\partial}{\partial x_i} + \sum_{j \neq i} x_j \frac{\partial}{\partial z_{ij}},$$

furthermore, the skew-symmetric matrix z is subject to

$$\dot{z}_{ij} = \dot{x}_i \, x_j \, - \, \dot{x}_j \, x_i. \qquad (8)$$

The problem's Lagrangian $\mathcal{L}$ takes the form

$$\lambda_0 \, \frac{1}{2} \sum_{i=1}^n \dot{x}_i^2 \, + \, \sum_{i<j} \lambda_{ij} \, (\dot{z}_{ij} \, - \, \dot{x}_i \, x_j \, + \, \dot{x}_j \, x_i).$$

This variational problem is known in the literature as the *Gaveau Brockett problem*, see [5]. Two situations can occur. Either $\lambda_0 = 0$, which leads to the *singular* or *abnormal* case. The second situation corresponds to $\lambda_0 \neq 0$, the so called *normal* case. In this last case we can set $\lambda_0 = 1$, without loss of generality. In this problem, however, the second case contains the first, and therefore we will consider here only the normal extremals.

Let $\Lambda = (\lambda_{ij})$ the skew-symmetric matrix whose entries are the Lagrange multipliers. A direct calculation yields

Proposition 3.1 *The Euler-Lagrange equations for the $\mathcal{L}$ are written as follows*

$$\dot{u} \;=\; \Lambda \, u, \qquad (9)$$
$$\dot{\Lambda} \;=\; 0 \qquad (10)$$

The above equations lead us to the conclusion that the λ_{ij} are constants of motion and that

$$\frac{d}{dt} \, (\dot{x} - \Lambda x) = 0,$$

we obtain therefore that the initial velocity components $\dot{x} - \Lambda x = \dot{x}_0$ constitute a set of n constants of motion. The n initial conditions $u^T(0) = (u_1(0), \ldots, u_n(0))$, together with the $n(n-1)/2$ constants $\lambda_{ij},\ i < j$ provide a complete set of integrals of motion for the system, that guarantee the integrability of the system by quadratures, in conclusion the integral curves are given by

$$u(t) = \exp(t\Lambda)\, u(0).$$

Since Λ is a constant matrix, we can explicitly calculate the solutions by means of the classical Lagrange-Sylvester theorem. Recall that the rank of Λ is always an even number, that the nonzero eigenvalues are purely imaginary and appear in $\pm$ pairs, and that Λ can always be block-diagonalized. For simp[licity, for the remaining of the paper we shall take the following

> *Assumption.* For n even we shall assume that Λ is a nonsingular skew-symmetric $n \times n$ constant matrix having all its $n/2$ eigenvalues different For n odd we shall assume that Λ has only one eigenvalue equal to zero, and that the other non-zero eigenvalues are all different.

It should be mentioned that degenerated cases are important and can be studied by standard limit procedures. In any dimension σ shall denote the spectrum of Λ, and π shall denote its characteristic polynomial. Each $\mu_k \in \sigma$ determines its corresponding Hermitian projector

$$\pi_k(\Lambda) = \frac{1}{\pi'(\mu_k)} \prod_{\mu_j \neq \mu_k} (\Lambda - I\mu_j).$$

In the odd dimension, the projector $\pi_0(\Lambda)$ corresponding to the eigenvalue 0 writes as follows

$$\pi_0(\Lambda) = \frac{1}{\pi'(0)} \prod_{\mu_j \in (\sigma - \{0\})} (\Lambda - I\mu_j).$$

The Lagrange-Sylvester formula yields

$$\begin{aligned} \exp(t\Lambda) &= \sum_{\mu_k \in \sigma} e^{t\mu_k} \pi_k(\Lambda), \text{ and} \\ \exp(t\Lambda) &= \sum_{\mu_k \in (\sigma - \{0\})} e^{t\mu_k} \pi_k(\Lambda) + \pi_0(\Lambda), \end{aligned}$$

for n even and odd respectively.

We shall derive now explicit formulæ for the sub-Riemannian geodesics of the system in terms of the formulas for u and Λ of proposition 3.1.

We shall now assume that $(x, z) \in \mathbb{R}^n \times \mathfrak{so}_n$ is a geodesic arc defined in certain interval $[0, T]$, (with x a column vector), and with initial conditions $(x(0), z(0)) = (0, \mathbf{0})$. We shall assume also that (x, z) is the projection of a normal extremal (u, Λ) with Λ a constant skew-symmetric matrix satisfying the assumption written above, and we shall denote $u(0) = u_0$. We have that

$$\begin{aligned} x &= \sum_{\mu_k \in \sigma} \frac{1}{\mu_k} \left(e^{\mu_k t} - 1\right) \pi_k(\Lambda) u_0, \text{ and} \\ x &= \sum_{\mu_k \in (\sigma - \{0\})} \frac{1}{\mu_k} \left(e^{\mu_k t} - 1\right) \pi_k(\Lambda) u_0 \\ &\quad + t\pi_0(\Lambda) u_0, \end{aligned}$$

for n even and odd respectively.

3.1 Low dimensional cases

We now specialize the results to the cases $n = 2$ and $n = 3$. In both cases we assume, as before, that (x, z) is a geodesic arc, with initial point $(0, \mathbf{0})$.

(2,3)-Case. The Heisenberg algebra. This case corresponds to $n = 2$, and leads to the three dimensional Lie algebra given by X_1, X_2 and the nonzero bracket:

$$[X_1, X_2] = X_{12}.$$

It has been widely studied since the pioneering paper by R. Brockett [10]. In this case, $(u, \Lambda) \in \mathbb{R}^2 \times \mathfrak{so}_2$, and $\lambda_{12} \neq 0$. In consequence

$$\begin{aligned} x(t) &= \frac{1}{\lambda_{12}} \Big(\sin(\lambda_{12} t) I \\ &\quad - \frac{(\cos(\lambda_{12} t) - 1)}{\lambda_{12}} \Lambda \Big) u_0. \end{aligned}$$

(3,6)-Case. This case corresponds to $n = 3$, it has been studied, to some extent, by W. Liu and H.Sussmann [11], O. Myasnichenko [12] and others. It consists of the six dimensional nilpotent Lie algebra given by X_1, X_2, X_3 and the nonzero brackets

$$\begin{aligned} [X_1, X_2] = X_{12},\ [X_1, X_3] &= X_{13}, \\ [X_2, X_3] &= X_{23}. \end{aligned}$$

In this case, $(u, \Lambda) \in \mathbb{R}^3 \times \mathfrak{so}_3$, and the matrix $\Lambda \in \mathfrak{so}_3$ has eigenvalues $\{i\lambda, -i\lambda, 0\}$ with

$$\begin{aligned} \lambda &= \sqrt{\lambda_{12}^2 + \lambda_{23}^2 + \lambda_{31}^2} \\ &= \sqrt{(-\mathrm{tr}(\Lambda^2)/2)}. \end{aligned} \tag{11}$$

Thus Λ satisfies $\Lambda(\Lambda^2+\lambda^2 I)=0$, and

$$\begin{aligned} x(t) &= \Big(tI - (\cos(\lambda t)-1)\frac{\Lambda}{\lambda^2} \\ &+ (\lambda t - \sin(\lambda t))\frac{\Lambda^2}{\lambda^3}\Big) u_0. \end{aligned}$$

4 Sub-Riemannian geometry associated to the structure (G,Δ)

Let $[0,T]$ be a sufficiently small interval, and let $t\mapsto (g(t),u(t))$ be a trajectory of the Gaveau-Brockett system, the geodesics emanating from $g(0)$, are given by $t\mapsto Exp(t,u(0))$.

Let $\rho>0$, the *wave front* of radius ρ emanating from $g(0)$, is defined as

$$W_\rho = Exp(\rho,\cdot),$$

the *sub-Riemannian sphere* S_ρ of radius ρ and center $g(0)$ is the set of points on G with sub-Riemannian distance to $g(0)$ exactly ρ, clearly $S_\rho\subset W_\rho$.

4.1 The exponential mapping

Let us consider the initial condition $g(0)$ as the identity element of G, and the co-vectors $(u(0),\Lambda(0))$. The exponential mapping can then be written as follows

$$\begin{aligned} \mathbb{R}^{n(n+1)/2} &\longrightarrow \mathbb{R}^{n(n+1)/2}, \\ (u(0),\Lambda) &\longmapsto (x(u(0),\Lambda), z_{ij}(u(0),\Lambda)). \end{aligned}$$

Since $u_1^2+\cdots+u_n^2=R^2$ is constant, we can write

$$u_{2i-1}(0)=R_i\cos\phi_i, \quad \text{and} \quad u_{2i}(0)=R_i\sin\phi_i,$$

for $i=1,\dots,\lfloor n/2\rfloor$, clearly $u_n(0)=R_{\lfloor n/2\rfloor+1}$ for n odd. But then

$$\sum_{i=1}^{\lfloor n/2\rfloor+1} R_i^2 = R^2,$$

and $R_{\lfloor n/2\rfloor+1}=0$ for n even. Thus, the momenta u_j can be parametrized in terms of spherical coordinates for the R_i.

The parametrization of the momenta λ_{ij} in the general case is by far more complicated, to our knowledge there is no a general procedure.

(3,6)-Case. As we mentioned above R_1 and R_2 can be parametrized by spherical coordinates as follows $R_2=R\cos\theta$ and $R_1=R\sin\theta$, therefore

$$\begin{aligned} u_1 &= R\sin\theta\cos\phi = R_1\cos\phi, \\ u_2 &= R\sin\theta\sin\phi = R_1\sin\phi, \\ u_3 &= R\cos\theta = R_2 = R_{\lfloor\frac{3}{2}\rfloor+1\neq 0}. \end{aligned}$$

with $\theta\in[0,\pi]$ and $\phi\in[0,2\pi]$.

For the parametrization of the momenta λ_{ij}, let us recall that the Lie algebra $\mathfrak{so}_3$ is generated by the following 3×3 skew-symmetric matrices

$$\mu_1=-e_2\wedge e_3,\ \mu_2=e_1\wedge e_3,\ \mu_3=e_1\wedge e_2.$$

By using the basis $\{\mu_1,\mu_2,\mu_3\}$ we have the mapping

$$\begin{aligned} \varphi:\mathfrak{so}_3 &\longrightarrow \mathbb{R}^3, \\ x_1\mu_1+x_2\mu_2+x_3\mu_3 &\longmapsto (x_1,x_2,x_3) \end{aligned}$$

which is a linear isomorphism and clearly satisfy

$$\varphi[A,B]=\varphi(A)\times\varphi(B),$$

here $\times$ denotes the standard cross product in $\mathbb{R}^3$. Hence φ is a Lie algebra isomorphism. Furthermore, for all $R\in\mathrm{SO}_3$ we have $\phi(RAR^T)=R\phi(A)$.

Equation (11) is equivalent to $\lambda^2=\|\phi(\Lambda)\|^2$, and we take spherical coordinates

$$\begin{aligned} \lambda_{12} &= -\lambda\cos\alpha, \\ \lambda_{13} &= \lambda\sin\alpha\cos\beta, \\ \lambda_{23} &= -\lambda\sin\alpha\sin\beta, \end{aligned}$$

with $\alpha\in[0,\pi]$ and $\beta\in[0,2\pi]$. The parametrization of the exponential mapping

$$(R,\theta,\phi,\lambda,\alpha,\beta)\longmapsto(x,z),$$

with $x=(x_1,x_2,x_3), z=(z_{12},z_{13},z_{23})$, is now complete in terms of the geometric invariants of the problem.

Observe that we can always find $R\in\mathrm{SO}_3$ such that $R\Lambda R^T$ is equal to $\lambda\mu_1,\lambda\mu_2$, or $\lambda\mu_3$. In this sense we can distinguish *privileged* directions given by vectors $(\lambda,0,0)$, $(0,\lambda,0)$ or $(0,0,\lambda)$, depending on a selected rotation axis associated with R.

4.2 Small radii spheres and the wave fronts

The wave front is defined as the set of end points of geodesics of fixed length, which we take equal to 1. The unit sphere is the set of points of geodesics at unit sub-Riemannian distance from the origin. Examples are given in this subsection for $t = 1$, in arc length units. Cross-sections of the unit sphere and the wave front are shown in figures 1 and 2, for the case $(3, 6)$ taking $\lambda_{23} = \lambda_{31} = 0$. The surfaces are parametrized by $\lambda = \lambda_{12}$ and the initial conditions $\dot{x}_1(0) = 0,\ \dot{x}_2(0) = \cos\alpha,\ \dot{x}_3(0) = \sin\alpha$.

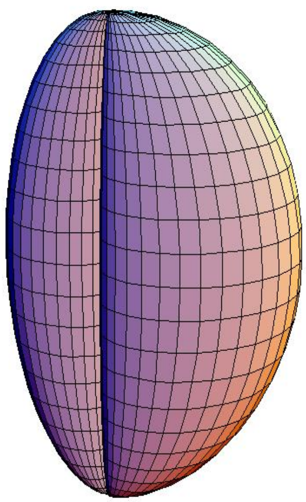

Figure 1. Section of the unit sphere.

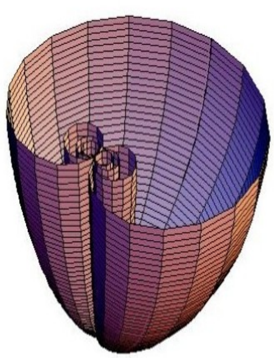

Figure 2. Section of the wave front.

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