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Numerical Solution of the Hamilton-Jacobi-Bellman Equation for Stochastic Optimal Control Problems

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Abstract: This paper provides a numerical solution of the Hamilton-Jacobi-Bellman (HJB) equation for stochastic optimal control problems. The computation's difficulty is due to the nature of the HJB equation being a second-order partial differential equation which is coupled with an optimization. By using a successive approximation algorithm, the optimization gets separated from the boundary value problem. This makes the problem solveable by standard numerical methods. For a problem of portfolio optimization where no analytical solution is known, the numerical methods is applied and its usefulness demonstrated.

Key-Words: Stochastic Optimal Control, HJB Equations, Control of Financial Systems

1 Introduction

Stochastic optimal feedback control, concern of many different research activities, gains in importance to answer questions arising from various physical, biological, economic, and management systems. A necessary condition for an optimal solution of stochastic optimal control problems is the HJB equation, a second-order partial differential equation that is coupled with an optimization. Unfortunately, the HJB equation is difficult to solve analytically. Only for some special cases, with simple cost functionals and state equations, analytical solutions are known, e.g. the LQ regulator problem. In the following we provide a successive approximation algorithm for a numerical solution of the HJB equation to tackle problems with no known analytical solution.

The paper is organized as follows: In Section 2, we introduce the stochastic optimal control problem with all of the basic assumptions and state the HJB equation as a necessary condition for the value function. Section 3 reveals the successive approximation algorithm and Section 4 provides information on computational implementation and computational issues. Finally, an example of portfolio optimization demonstrates the usefulness of the method in Section 5.

2 Problem Formulation

Consider the n-dimensional stochastic process x which is governed by the given stochastic differential equation (SDE)

$$dx = f(t,x,u)dt + g(t,x,u)dZ, \tag{1}$$

where dZ denotes k-dimensional uncorrelated standard Brownian motion defined on a fixed, filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbf{P})$. The vector u denotes the control variables contained in some compact, convex set $U \subset \mathbb{R}^m$, the drift term $f(t,x,u)$ and the diffusion $g(t,x,u)$ are given functions

$$\begin{aligned} f:\ & [0,T]\times G\times U \longrightarrow \mathbb{R}^n, \\ g:\ & [0,T]\times G\times U \longrightarrow \mathbb{R}^{n\times k} \end{aligned}$$

for some open and bounded set $G \subset \mathbb{R}^n$.

The *value functional* of our problem starting at arbitrary time $t \in (0,T)$ and state $x \in G$ with respect to a fixed control law u is defined by

$$\mathcal{J}(t,x,u) = \mathbb{E}\left\{\int_t^\tau L(s,x,u)ds + K(\tau, x(\tau))\right\}, \tag{2}$$

*Now at Automatic Control Laboratory, Swiss Federal Institute of Technology, Zürich, Switzerland where $\mathbb{E}$ denotes the expectation operator and L, K are scalar functions:

$$\begin{array}{llll} L: & [0,T]\times G\times U & \longrightarrow & \mathbb{R}, \\ K: & [0,T]\times G & \longrightarrow & \mathbb{R}. \end{array}$$

The final time of our problem denoted by τ is the time when the solution $x(t)$ leaves the open set $Q = (0,T)\times G$:

$$\tau = \inf\left\{s \geq t \mid (s, x(s)) \notin Q\right\}.$$

Our aim is to find the admissible feedback control law u which maximizes the value of the functional $\mathcal{J}(t,x,u)$ leading to the *cost-to-go* (or *value*) function $J(t,x)$:

$$J(t,x) = \max_{u(t,x)\in U} \mathcal{J}(t,x,u).$$

We finally arrive at a definition for the stochastic optimal control problem:

$$\begin{aligned} & J(t,x) = \max_{u(t,x)\in U} \mathbb{E}\left\{\int_t^\tau L(s,x,u)ds + K(\tau, x(\tau))\right\} \\ & \quad \text{s.t.} \qquad\qquad\qquad\qquad\qquad\qquad\qquad (3) \\ & \quad dx = f(t,x,u)dt + g(t,x,u)dZ. \end{aligned}$$

In the following we will state the *Hamilton-Jacobi-Bellman* equation (or *dynamic programming* equation) as a necessary conditon for the cost-to-go function $J(t,x)$. For a detailed derivation, the reader is referred to [1, 2], or [3].

By introducting the differential operator

$$\mathcal{A}(t,x,u) = \frac{1}{2}\sum_{i,j=1}^{n} \sigma_{ij}\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i},$$

where the symmetric matrix $\sigma = (\sigma_{ij})$ is defined by $\sigma(t,x,u) = g(t,x,u)g^T(t,x,u)$, the HJB equation can be written as follows [1]:

$$\begin{aligned} & J_t + \max_{u\in U}\left\{L(t,x,u) + \mathcal{A}(t,x,u)J\right\} = 0, \\ & (t,x) \in Q, \qquad\qquad\qquad\qquad\qquad\qquad (4) \end{aligned}$$

with the boundary data

$$J(t,x) = K(t,x), \qquad (t,x) \in \partial^* Q, \qquad (5)$$

where $\partial^* Q$ denotes a closed subset of the boundary ∂Q such that $(\tau, x(\tau)) \in \partial^* Q$ with probability 1:

$$\partial^* Q = ([0,T]\times\partial G) \cup (\{T\}\times G).$$

[1]For simplicity, we will use index notation to express partial derivatives and often suppress function arguments, e.g. $J_t = \frac{\partial J(t,x)}{\partial t}$.

The HJB equation (4) is a scalar linear second-order PDE which is coupled with an optimization over u. This makes solving the problem so difficult (apart from computational issues arising from problem sizes in higher dimensions). Note that the coefficients of (4) are dependent on t, x, and u.

3 Numerical HJB Solutions

As already mentioned in the previous section, analytical solutions of the HJB equation are only known for some special cases with simple state equations and cost functional. In this section we will reveal a numerical approach for solving the HJB equation enabling us to investigate a broader class of optimal control problems.

3.1 Successive Approximation of the HJB Equation

Solving the PDE and optimization problem at once would lead to unaffordable computational costs. Chang and Krishna propose a successive approximation algorithm which will be used in the following [4].

Lemma 1 (without proof) *Let J^u be the solution of the boundary value problem corresponding to the arbitrary but fixed control law $u \in U$:*

$$J^u_t + L(t,x,u) + \mathcal{A}(t,x,u)J^u = 0, \quad (t,x)\in Q, \qquad (6)$$

with boundary data

$$J^u(t,x) = K(t,x), \qquad (t,x) \in \partial^* Q. \qquad (7)$$

Then

$$J^u(t,x) = \mathcal{J}(t,x,u), \qquad (8)$$

where $\mathcal{J}(t,x,u)$ denotes the value functional defined in (2). For the proof refer to section V§7 in [3, pp. 129–130].

We are now going to reveal the successive approximation algorithm. To begin with, assume J^k to be the solution of (6)–(7) corresponding to the arbitrary but fixed control law $u^k \in U$

$$J^k_t + L(t,x,u^k) + \mathcal{A}(t,x,u^k)J^k = 0, \quad (t,x)\in Q, \qquad (9)$$

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$$J^k(t,x) = K(t,x), \qquad (t,x) \in \partial^* Q. \tag{10}$$

Then the sequence of control laws is given by

$$u^{k+1} = \arg\max_{u\in U} \left\{ L(t,x,u) + \mathcal{A}(t,x,u) J^k \right\}. \tag{11}$$

Note that because of (9) and (11)

$$\begin{aligned} &L(t,x,u^{k+1}) + \mathcal{A}(t,x,u^{k+1})J^k \\ &\quad \geq L(t,x,u^k) + \mathcal{A}(t,x,u^k)J^k. \end{aligned} \tag{12}$$

Now let J^{k+1} be the solution of the boundary value problem (6)–(7) corresponding to the new control law u^{k+1}:

$$J_t^{k+1} + L(t,x,u^{k+1}) + \mathcal{A}(t,x,u^{k+1})J^{k+1} = 0, \quad (t,x) \in Q, \tag{13}$$

$$J^{k+1}(t,x) = K(t,x), \; (t,x) \in \partial^* Q. \tag{14}$$

Lemma 2 *Let the sequences of control laws u^k and their affiliated value functionals J^k be defined as above. Then the sequence J^k satisfies*

$$J^{k+1} \geq J^k. \tag{15}$$

Proof 1 *Define $W = J^{k+1} - J^k$ on Q and compute $W_t + \mathcal{A}(t,x,u^{k+1})W$:*

$$\begin{aligned} W_t + \mathcal{A}(t,x,u^{k+1})W = &J_t^{k+1} + \mathcal{A}(t,x,u^{k+1})J^{k+1} \\ &- J_t^k - \mathcal{A}(t,x,u^{k+1})J^k. \end{aligned} \tag{16}$$

Adding and subtracting $L(t,x,u^{k+1})$ on the right-hand side and using (13) and (12) yields

$$\begin{aligned} &W_t + \mathcal{A}(t,x,u^{k+1})W = \\ &= \underbrace{J_t^{k+1} + \mathcal{A}(t,x,u^{k+1})J^{k+1} + L(t,x,u^{k+1})}_{=0} \\ &\quad - \Big[J_t^k + \underbrace{\mathcal{A}(t,x,u^{k+1})J^k + L(t,x,u^{k+1})}_{\geq \mathcal{A}(t,x,u^k)J^k + L(t,x,u^k)} \Big]. \end{aligned}$$

Thus,

$$\begin{aligned} &W_t + \mathcal{A}(t,x,u^{k+1})W \\ &\quad \leq - \underbrace{\Big[J_t^k + \mathcal{A}(t,x,u^k)J^k + L(t,x,u^k) \Big]}_{=0}. \end{aligned}$$

The squared bracket term on the right-hand side of the previous equation is equal to zero due to (9). Hence,

$$W_t + \mathcal{A}(t,x,u^{k+1})W \leq 0. \tag{17}$$

Next, we compute the expectation of $W(\tau, x(\tau))$ by integration of its total derivative according to Itô's rule subject to (1) and the last control law u^{k+1}:

$$\begin{aligned} &\mathbb{E}\{W(\tau, x(\tau))\} = W(t,x) \\ &+ \mathbb{E}\left\{ \int_t^\tau \Big[W_t(t,s) + \mathcal{A}(t,s,u^{k+1})W(t,s) \Big] ds \right\}. \end{aligned}$$

Knowing that by definition $W(\tau, x(\tau)) = J^{k+1}(\tau, x(\tau)) - J^k(\tau, x(\tau))$ vanishes on $\partial^* Q$ because of (10) and (14) and considering (17) we obtain

$$\begin{aligned} &W(t,x) = \underbrace{\mathbb{E}\{W(\tau, x(\tau))\}}_{=0} \\ &- \mathbb{E}\left\{ \int_t^\tau \underbrace{\Big[W_t(t,s) + \mathcal{A}(t,s,u^{k+1})W(t,s) \Big]}_{\leq 0} ds \right\} \geq 0. \end{aligned}$$

Since $W(t,x) = J^{k+1} - J^k$, it follows directly that

$$J^{k+1} \geq J^k.$$

Theorem 1 *Let the sequences of control laws u^k and their corresponding value functionals J^k be defined as above. Then they converge to the optimal feedback control law $u(t,x)$ and the value function $J(t,x)$ of our optimal control problem (3), i.e.:*

$$\lim_{k\to\infty} u^k(t,x) = u(t,x) \text{ and } \lim_{k\to\infty} J^k(t,x) = J(t,x).$$

Proof 2 *Due to Theorem 15 in [5, p. 80] (see also Theorem VI.6.1 in [3, pp. 208–209]) J^k, $J^k_{x_i}$ converge uniformly on Q to J^*, $J^*_{x_i}$ and J^k_t, $J^k_{x_i x_j}$ weakly to J^*_t, $J^*_{x_i x_j}$ (for necessary properties of f, g, L and K refer to [3, p. 167]). Consider now the limit of (13) for $k \to \infty$:*

$$\lim_{k\to\infty} \Big[J_t^{k+1} + L(t,x,u^{k+1}) + \mathcal{A}(t,x,u^{k+1})J^{k+1} \Big] = 0.$$

Since the limits of J^k and its derivatives do exist, we may change iteration indices:

$$0 = \lim_{k\to\infty} \Big[J_t^k + L(t,x,u^{k+1}) + \mathcal{A}(t,x,u^{k+1})J^k \Big].$$

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By using (11) *and substituting* $\lim_{k\to\infty} J^k$, $\lim_{k\to\infty} J_t^k$ *with their limits* J^* *resp.* J_t^*, *we obtain*

$$\begin{aligned} 0 &= \lim_{k\to\infty}\left[J_t^k + \max_{u\in U}\left\{L(t,x,u) + \mathcal{A}(t,x,u)J^k\right\}\right] \\ &= J_t^* + \max_{u\in U}\left\{L(t,x,u) + \mathcal{A}(t,x,u)J^*\right\}, \end{aligned}$$

where $(t,x) \in Q$. *Since* $J^*(t,x) = K(t,x)$ *on* $\partial^* Q$, $J^*(t,x)$ *solves* (4) *and* (5) *and hence is the solution of the optimal control problem* (3). *Consequently,*

$$u^*(t,x) = \arg\max_{u\in U}\left\{L(t,x,u) + \mathcal{A}(t,x,u)J^*\right\}$$

is the optimal feedback control law.

Before we continue, let us recapitulate the successive approximation algorithm:

1. $k = 0$; choose an arbitrary initial control law $u^0 \in U$.

2. Solve the boundary value problem for the fixed control law u^k, i.e., $J^k(t,x)$ solves

$$\begin{aligned} &J_t^k + L(t,x,u^k) + \mathcal{A}(t,x,u^k)J^k = 0, \\ &(t,x) \in Q, \\ &J^k(t,x) = K(t,x),\ (t,x) \in \partial^* Q. \end{aligned}$$

3. Compute the succeeding control law u^{k+1}, i.e., solve the optimization problem:

$$u^{k+1} = \arg\max_{u\in U}\left\{L(t,x,u) + \mathcal{A}(t,x,u)J^k\right\}.$$

4. $k = k+1$; back to step 1.

To sum up, the algorithm is an iterative approach which decouples the optimization from the boundary value problem and thus avoids the need of doing the whole work at once. In other words, the difficult problem of finding a numerical solution of the HJB equation (4) has been seperated into two easier ones which are solvable by standard numerical means: 1. Solving boundary value problem (9)–(10). 2. Optimization of the nonlinear function with possible controller constraints (11).

4 Computational Implementation

In this section, we will point out a possible way of solving the PDEs and optimization problems arising from the successive approximation algorithm introduced in the previous section.

4.1 Numerical Solution of the HJB-PDE

Boundary value problem (9)–(10) is a scalar second-order PDE with nonlinear coefficients and hence can be tackled by standard methods for linear parabolic PDEs. With the mixed derivatives left out of consideration, (9) has the structure of the heat equation with advection and source terms. However, in contrast to the heat problem, the HJB-PDE has a terminal condition $J^k(T,x) = K(T,x)$ rather than an initial condition and therefore has to be integrated backwards in time. With the simple substitution $\bar{t} = T - t$ the problem is converted into a PDE which can be integrated forward in time (i.e. $\bar{t}$ runs from 0 to T):

$$\begin{aligned} &J_{\bar{t}}^k - \left[L(\bar{t},x,u^k) + \mathcal{A}(\bar{t},x,u^k)J^k\right] = 0,\ (\bar{t},x) \in Q, \\ &J^k(\bar{t},x) = K(\bar{t},x),\quad (\bar{t},x) \in \partial^* Q. \end{aligned}$$

Since available standard codes were not able to handle mixed derivatives and coefficients dependent on t, x and u, we developed our own solvers. Although many different methods for solving PDEs have been developed so far, there does not exist one which is best-suited for all types of applications. We use finite difference schemes as they are both well-suited for simple (rectangular) shaped domains Q and rather easy to implement. For a good introduction into the topic of finite difference schemes, the reader is advised to read [6]. Our solver employs an implicit scheme and uses upwind differences for the first order derivatives for stability reasons. Second order and mixed derivatives are approximated by central space differences. Although not required by (9) we extended the code to handle an additional linear term of J^k and Neumann boundary conditions on ∂G. This enables us to get better results for some financial applications which can be significantly simplified by a special transformation adding the linear term (for details see Section 5).

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4.2 Optimization

According to Section 3, we are facing the following nonlinear optimization problem to compute the succeeding control law:

$$u^{k+1} = \arg\max_{u \in U} \left\{ L(t,x,u) + \mathcal{A}(t,x,u) J^k \right\}.$$

Since we approximate $J^k(t,x)$ on a finite grid, the optimization must be solved for every grid point. This can be accomplished by standard optimization tools. For problems with simple functions, it may be possible to obtain an analytical solution for the optimal control law. This is to be prefered in return of less computational time. For our example of portfolio optimization provided in Section 5, we are able to obtain an explicit expression for the optimal control law.

4.3 Numerical Issues

Since the number of unknown grid points at which we approximate $J^k(t,x)$ grows by an order of magnitude with dimension (*Bellman's curse of dimensionality*) and grid resolution we have to face issues of

- memory limitations and
- computation time and accuracy.

The PDE solvers outlined in Section 4.1 require the solution of large systems of linear equations. The coefficient matrix of these linear systems is banded and therefore strongly encourages the use of sparse matrix techniques to save memory. Furthermore, applying indirect solution methods for linear systems such as *successive overrelaxation* provides higher accuracy and memory efficiency than direct methods (for details see [7]).

MATLAB's memory requirement for storing the coefficient matrix corresponding to the implicit scheme described in Section 4.1 is outlined examplarily in [8]. While we need only 2.4 MB for a two dimensional grid of 150 points in each space coordinate, the matrix will allocate approximately 732 MB in the three dimensional case. Considering the fact that todays 32-bit architectures limit the virtual memory for variable storage to 1.5 GB (0.5 GB are needed by MATLAB), it is obvious that our solvers are restricted to rather coarse grids and low dimensions showing that Bellman's curse of dimensionality can't be overcome by the successive approximation. However, enhanced numerical methods such as alternating direction implicit (ADI) methods or domain decompositon algorithms could contribute to tackle bigger problems. For further information on these topics the reader is referred to [6] and [9].

5 Case Study

The case study presents a portfolio optimization problem in continuous-time. In the late 1960's Robert Merton showed showed the connection between stochastic optimal control and portfolio optimization under uncertainty [10, Chapter 4 and 5].

5.1 Portfolio Optimization Problem

We consider a portfolio optimization problem where an investor has the choice of investing in the stock market or to put his money in a bank account. We model the stock market as geometric Brownian motion with time-varying and stochastic mean returns and time-varying and stochastic diffusion term (volatility). Mathematically, the stock market model and the bank account model are given by

$$\frac{dS(t)}{S(t)} = [Fx(t) + f]dt + \sqrt{v(t)}dZ_1, \quad (18)$$

$$\frac{dB(t)}{B(t)} = r dt\,, \quad (19)$$

where $S(t)$ is the stock market index, $B(t)$ is the value of the bank account, r is the risk-free interest rate, $x(t)$ is a factor that directly affects the mean return, and $v(t)$ is the square of the volatility. The dynamics for the factor and the volatility are modelled as

$$dx(t) = (a_1 + A_1 x(t))dt + \nu dZ_2, \quad (20)$$

$$dv(t) = (a_2 + A_2 v(t))dt + \sigma\sqrt{v(t)}dZ_3, \quad (21)$$

where a_1, A_1, a_2, A_2, ν, and $\sigma \in \mathbb{R}$ are parameters describing the models. The volatility model is described in detail in [11], the factor model in [12]. Furthermore, we assume that all three Brownian motions are correlated: $dZ_1 dZ_2 = \rho_{12} dt$, $dZ_1 dZ_3 = \rho_{13} dt$, and $dZ_2 dZ_3 = \rho_{23} dt$. The portfolio dynamics (wealth equation) for this investment universe is given by

$$\begin{aligned} dW(t) &= W(t)(r+u(t)(Fx(t)+f-r))dt \\ &+ W(t)u(t)\sqrt{v(t)}dZ_1\,, \end{aligned} \tag{22}$$

where $W(t) \in \mathbb{R}$ describes the value of the portfolio and $u(t)$ is the fraction of wealth invested in the stock market. For the derivation of (22), the reader is referred to [10, Chapter 5]. The objective of the investor is to maximize the expectation of the power utility of his wealth at a finite fixed time horizon T: $\max \mathbb{E}\{\frac{1}{\gamma}W^\gamma(T)\}$. Thus the portfolio optimization problem is

$$\begin{aligned} &\max_{u\in[-1,1]} \mathbb{E}\Big\{\frac{1}{\gamma}W^\gamma(T)\Big\} \\ s.t.& \\ dW(t) &= W(t)(r+u(t)(Fx(t)+f-r))dt \\ &+ W(t)u(t)\sqrt{v(t)}dZ_1 \\ dx(t) &= (a_1+A_1x(t))dt+\nu dZ_2 \\ dv(t) &= (a_2+A_2v(t))dt+\sigma\sqrt{v(t)}dZ_3, \end{aligned} \tag{23}$$

where $\gamma < 1$ is coefficient of risk aversion and $u(t) \in [-1,1]$. For this problem of portfolio optimization no analytical solution is known and thus, we solve the problem by the proposed numerical method. We make the assumption that both of the processes $x(t)$ and $v(t)$ are measurable and we have both of the time series to estimate the model parameters. The HJB equation for portfolio problem (23) (suppressing t in all functions for compactness) is

$$\begin{aligned} J_t + \max_{u\in[-1,1]} &\Big\{W(r+u(Fx+f-r))J_W \\ &+ (a_1+A_1x)J_x \\ &+ (a_2+A_2v)J_v + \frac{1}{2}\Big(W^2u^2vJ_{WW} + v\sigma^2J_{vv} \\ &+ \nu^2J_{xx}\Big) + Wu\sqrt{v}\rho_{12}\nu J_{Wx} + Wu\rho_{13}v\sigma J_{Wv} \\ &+ \sqrt{v}\sigma\rho_{23}\nu J_{xv}\Big\} = 0\,, \end{aligned} \tag{24}$$

with terminal condition $J(T,\cdot) = \frac{1}{\gamma}W^\gamma(T)$. For this type of HJB we make the Ansatz $J = \frac{1}{\gamma}W^\gamma(t)H(t,x,v)$ in order to simplify the problem. Putting the Ansatz into (24) yields

$$\begin{aligned} H_t + \max_{u\in[-1,1]} &\Big\{\gamma\Big(r+u(Fx+f-r) \\ &+ \frac{1}{2}u^2v(\gamma-1)\Big)H \\ &+ (a_1+A_1x+\gamma u\sqrt{v}\rho_{12}\nu)H_x + \frac{1}{2}\nu^2H_{xx} \\ &+ \frac{1}{2}v\sigma^2H_{vv} + (a_2+A_2v+\gamma uv\rho_{13}\sigma)H_v \\ &+ \sqrt{v}\sigma\rho_{23}\nu H_{xv}\Big\} = 0, \end{aligned} \tag{25}$$

with terminal condition $H(T,x,v) = 1$. The HJB equation (25) is a problem with two state variables $x(t)$ and $v(t)$. The optimal control law is given by

$$\begin{aligned} u^* &= \frac{1}{(1-\gamma)v}\Big((Fx+f-r) + \sigma\rho_{13}v\frac{H_v}{H} \\ &+ \nu\sqrt{v}\rho_{12}\frac{H_x}{H}\Big) \end{aligned} \tag{26}$$

and note that $u(t) \in [-1,1]$. In order to compute u^* we need to solve the PDE for H and calculated the two derivatives. If u^* violates the constraint, the value of u^* is set to the limits of the constraint.

5.2 Simulation with Historical Data

The portfolio optimization problem (5.1) is applied to US data. We use the S&P 500 index as stock market index, the volatility index (VIX) as measurable time series for the volatility, and the difference between the E/P ratio of the S&P 500 and the 10 year Treasury Bond interest rate as factor that explains the expected mean returns. As short-term interest rate, we use the 1-month Treasury Notes. All five time series start on 1/1/1992 and end on 1/1/2004 and were obtained from Thomson DATASTREAM. We use the data from 1/1/1992 to 31/12/1997 to estimate the model parameters. The data from 1/1/1998 to 1/1/2004 is then used to evaluate the performance of our optimal portfolio controller.

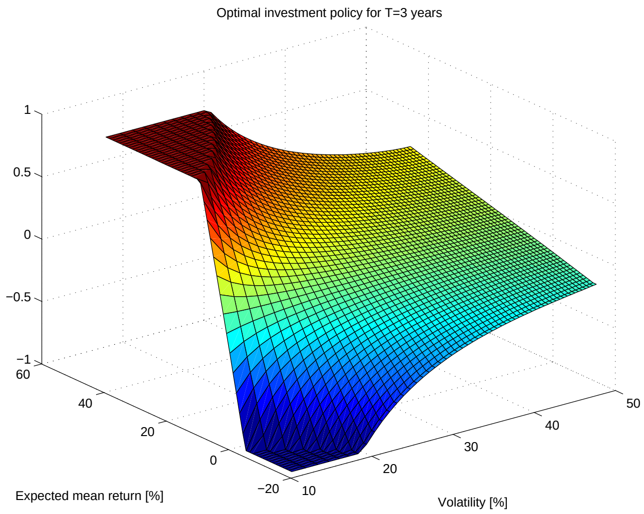
The controller is numerically computed based on the estimated parameters till 31/12/1997. The time horizon is 1/1/2004 and the HJB equation is solved with a monthly time step and $\gamma = -5$. The parameters of model are again estimated based on the data from 1/1/1992 to 31/12/2000 in order to calculate the controller based on more recent data. As similar procedure is used in [13]

Table 1: Parameter estimates of the portfolio model

	31/12/1997	31/12/2000
a_1	0.13	0.12
A_1	-4.73	-2.86
a_2	-0.013	-0.015
A_2	-0.85	-0.83
ν	0.013	0.013
σ	0.287	0.367
ρ_{12}	0.20	0.05
ρ_{13}	-0.62	-0.73
ρ_{23}	-0.16	-0.03
F	8.96	10.48
f	0.32	0.35
r	0.044	0.046

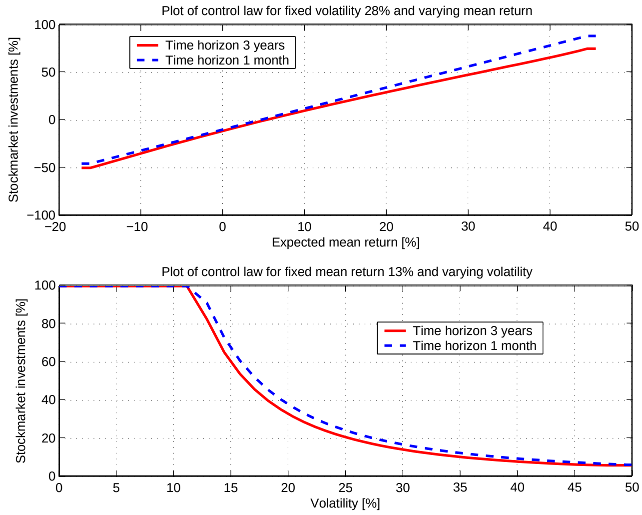
and [12]. The performance of the portfolio optimization therefore is always tested on the out-of-sample data. The parameters are estimated using the discrete-time equivalent of the models and using a pseudo maximum likelihood procedure for (21) and the exact maximum likelihood procedure for (20) and (18). For details see [14]. The parameter estimates are given in Table 1.

Figure 1: Optimal investment policy



In Fig. 1, the optimal investment policy as function of $x(t)$ and $v(t)$ is shown. The controller is the solution of (25). It is computed with the numerical procedure as outlined in Section 3. In order to interpret the resulting controller, we fix the

Figure 2: Optimal investment policy



mean return and plot $u(t)$ as function of volatility and we fix the volatility and plot $u(t)$ as function of the mean return, as shown in Fig. 2. The optimal investment policy is almost a linear function of the expected returns and resembles a hyperbola with respect to the volatility. The investment in the risky stock market decreases when the volatility for each given expected return increases. The effect of the time horizon can be seen in Fig. 2, where the optimal policy for time horizon 1 month and three years are shown.

Figure 3: Results of the historical simulation with US data from 1998 to 2004

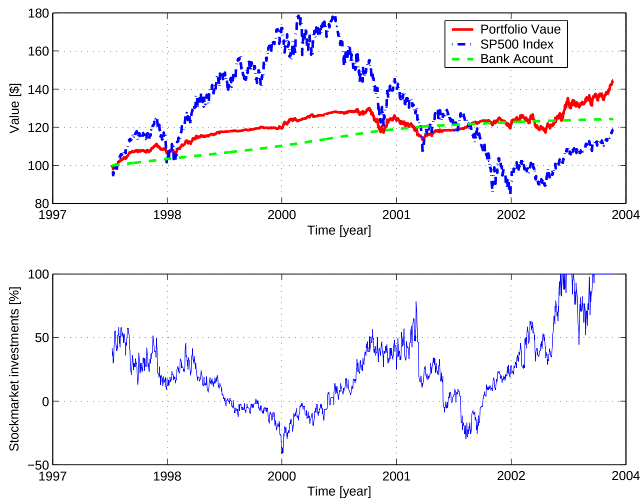


Table 2: Statistics of the historical simulation

	Portf.	S&P 500	Bank ac.
return (%)	5.98	2.8	3.54
volatility (%)	6.8	21.9	-
Sharpe ratio	0.36	-0.03	-

The result of the historical simulation is given in Fig. 3 where the portfolio value, the S&P 500 index, the bank account, and the investment in the stock market are shown. In Table 2 summarizes the statistics of the simulation. The portfolio outperforms both stock market and bank account. The investments in the stock market vary from -30% to 100%. The risk aversion used in this simulation is fairly high. Therefore the portfolio exhibits a much lower volatility than the stock market. The portfolio manages to have higher return than both assets and possesses a markedly higher Sharpe ratio than the stock market. The simulation shows that an investor could have exploited the partial predictability of the returns as well as the information on risk which is implied in the volatility index VIX.

6 Conclusion

In this paper we derive a numerical method to solve the HJB equation. The idea of the successive approximation method is explained and its convergence is proven. To our knowledge, this is the first convergence proof of the successive approximation algorithm. The algorithm and its numerical implementation are discussed and applied in a numerical case study. The case study describes a portfolio optimization problem in continuous-time where the risky assets are modeled with time-varying and stochastic drift and diffusion. The problem is explained and the optimization is numerically solved by applying the proposed method. The portfolio optimization is used in a real-world case study with U.S. asset market data. The benefit of the portfolio optimization is shown in an six year out-of-sample test, where the portfolio beats the S&P 500 index measured in returns and risk-adjusted returns.

References

[1] J. Yong and X. Y. Zhou, *Stochastic Controls, Hamilton Systems and HJB Equations.* Springer Verlag New-York, 1999.

[2] B. Hanzon, H. Schumacher, and M. Vellekoop, "Finance for control engineers," Tutorial Workshop at the European Control Conference 2001, Porto, Portugal.

[3] W. H. Fleming and R. W. Rishel, *Deterministic and Stochastic Optimal Control.* Springer-Verlag, 1975.

[4] M. H. Chang and K. Krishna, "A successive approximation algorithm for stochastic control problems," *Applied Mathematics and Computation*, vol. 18, no. 2, pp. 155–165, 1986.

[5] A. Friedman, *Partial Differential Equations of Parabolic Type.* Englewood Cliffs, N.J.: Prentice-Hall, 1964.

[6] J. C. Strikwerda, *Finite difference schemes and partial differential equations.* Pacific Grove, California: Wadsworth & Brooks/Cole Advanced Books & Software, 1989.

[7] W. H. Press, S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery, *Numerical Recipes in C++, The Art of Scientific Computing*, 2nd ed. Campbridge University Press, 2002.

[8] H. Peyrl, "Numerical solution of the Hamilton-Jacobi-Bellman equation for technical and financial applications of stochastic optimal control," Master's thesis, Swiss Federal Insitute of Technology, 2003, http://control.ee.ethz.ch/research/ publications/publications.msql?id=1774, accessed Februar 2, 2004.

[9] M. Sun, "Domain decompostion algorithms for solving Hamilton-Jacobi-Bellman equations," *Numerical Functional Analysis and Optimization*, vol. 14(1+2), 1993.

[10] R. C. Merton, *Continuous-time finance.* Oxford: Backwell Publishers, 1992.

[11] D. Psychoyios, G. Skiadopoulos, and P. Alexakis, "A review of stochastic volatility

processes: Properties and implications," *The Journal of Risk Finance*, pp. 43–59, Spring 2003.

[12] T. Bielecki, S. Pliska, and M. Sherris, "Risk sensivitve asset allocation," *Journal of Economic Dynamics & Control*, vol. 24, pp. 1145–1177, 2000.

[13] M. Brennan, E. Schwartz, and R. Lagnado, "Strategic asset allocation," *Journal of Economic Dynamics & Control*, vol. 21, pp. 1377–1403, 1997.

[14] F. Herzog, "Maximum likelihood parameter estimation for stochastic differential equations," to be published.