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Control design for Timoshenko beams and easy test for stability of the closed loop system *

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Abstract: In this paper we discuss design problem of linear feedback controller for Timoshenko beams and test of stability of the closed loop systems. Herein we give some linear feedback control law for various boundary conditions, under which the closed loop systems are well-posed and asymptotically stable. At same time, we give test method for stability of the closed loop systems and then use the test method to determine exponential stability of the closed loop systems. Our main result is that if test system is a Riesz spectral system and decays exponentially, then the closed loop system also is exponentially stable. If the test system is either unstable or asymptotically sable but not exponentially stable, then the closed loop system is asymptotically stable.
Key-Words: Timoshenko beam; boundary and distributed feedback control; test of stability; Riesz basis

1 Introduction

Many mechanical systems, such as spacecraft and robot arm, can be modelled as Timoshenko beam equation (e.g., see, [1] and [2])

$$\begin{cases} \rho\ddot{w}(x,t) - K(w''(x,t) - \varphi'(x,t)) = 0, \\ I_\rho\ddot{\varphi}(x,t) - EI\varphi''(x,t) - K(w'(x,t) - \varphi(x,t)) = 0, \end{cases} \tag{1.1}$$

where t is time variable and $x \in (0,\ell)$ is spacial coordinate along beam in its equilibrium position and ℓ is length of beam. $w(x,t)$ is deflection of beam from its equilibrium line and $\varphi(x,t)$ is slope of the deflection curve when the shearing force is neglected, and ρ, K, I_ρ and EI are physical constants, for their precise meaning of them, see, Timoshenko's book [4].

For such a Timoshenko beam with appropriate boundary conditions, it describes dynamic behavior of corresponding mechanical system. Usually, Eqs.(1.1) has one of the following boundary conditions:

(B1) free–free:

$$\begin{cases} K(w'(0,t) - \varphi(0,t)) = EI\varphi'(0,t) = 0, \\ K(w'(\ell,t) - \varphi(\ell,t)) = EI\varphi'(\ell,t) = 0. \end{cases} \tag{1.2}$$

(B2) built in–free:

$$\begin{cases} w(0,t) = 0, \quad \varphi(0,t)) = 0, \\ K(w'(\ell,t) - \varphi(\ell,t)) = EI\varphi'(\ell,t) = 0. \end{cases} \tag{1.3}$$

(B3) hinged–hinged:

$$\begin{cases} w(0,t) = 0, \quad EI\varphi'(0,t) = 0, \\ w(\ell,t) = 0, \quad EI\varphi'(1,t) = 0. \end{cases} \tag{1.4}$$

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(B4) built in–built in:

$$\begin{cases} w(0,t) = \varphi(0,t) = 0, \\ w(\ell,t) = \varphi(\ell,t) = 0. \end{cases} \tag{1.5}$$

(B5) built in–hinged

$$\begin{cases} w(0,t) = 0, \quad \varphi(0,t) = 0, \\ w(\ell,t) = 0, \quad EI\varphi'(\ell,t) = 0. \end{cases} \tag{1.6}$$

For physical significance of these boundary conditions, we refer to a paper of Traill-Nash and Collar [3].

In engineering, the most important problem is suppression of beam vibration. Many engineers and mathematicians have designed various controllers to force the beam back to its equilibrium position. In recent years, design of feedback controllers for Timoshenko beam has attracted more attention and becomes an interesting research topic, for example, Kim and Renardy [5] for linear boundary feedback controllers, Feng et al [6] for nonlinear boundary feedback control law, and Shi et al [7] for distributed feedback controllers. Under these feedback control laws, analysis of stability of the closed loop system is a difficult and complicated task.

Although some nice results have been obtained for cantilever beam such as exponential stability (see Kim and Renardy [5]) and Riesz basis property of the closed loop system (see, Xu et al.[8],[9],[10]), and Shubov[11]), we have not a general test method for stability of the closed loop system. Can we give an effective way to check the effect of the control law? In this paper, we shall propose a test method for Timoshenko beam model (1.1).

Our idea is to take dominant part of the system (including boundary conditions) as a test system. More precisely saying, if the system is controlled by distributed feedback controllers, i.e.,

$$\begin{cases} \rho\ddot{w}(x,t) - K(w''(x,t) - \varphi'(x,t)) \\ \qquad -a(x)\dot{w}(x,t) = 0, \quad x \in (0,\ell), t > 0, \\ I_\rho\ddot{\varphi}(x,t) - EI\varphi''(x,t) - K(w'(x,t) - \varphi(x,t)) \\ \qquad -b(x)\dot{\varphi}(x,t) = 0, \quad x \in (0,\ell), t > 0, \end{cases} \tag{1.7}$$

then we take the following system as a test system

$$\begin{cases} \rho\ddot{w}(x,t) - Kw''(x,t) + a(x)\dot{w}(x,t) = 0, \\ \qquad x \in (0,\ell), t > 0, \\ I_\rho\ddot{\varphi}(x,t) - EI\varphi''(x,t) + b(x)\dot{\varphi}(x,t) = 0, \\ \qquad x \in (0,\ell), t > 0. \end{cases} \tag{1.8}$$

If Tomoshenko beam is controlled by boundary feedback control, then we take test system as

$$\begin{cases} \rho\ddot{w}(x,t) - Kw''(x,t) = 0, \quad 0 < x < \ell, t > 0 \\ I_\rho\ddot{\varphi}(x,t) - EI\varphi''(x,t) = 0, \; 0 < x < \ell, t > 0. \end{cases} \tag{1.9}$$

In above both cases we can judge stability of test system by simplify boundary conditions. We can prove that frequencies of test system are asymptotic values of frequencies of the closed loop system, and hence the test system and the closed loop system have same exponential stability.

Our main result is the following.

Theorem 1.1 *Let Timoshenko beam be attached linear feedback controllers and the closed loop system be dissipative and asymptotically stable. If test system is a Riesz spectral system, then the closed loop system and test system have same exponential stability.*

Usually verification of exponential stability of the closed loop system is very complicated, and checking of stability of test system is relative easy, advantage of this result is that we can assert stability of the closed loop system from test system.

The contents of this paper are organized as follows. In next section, we shall give some design of linear feedback control laws for Timoshenko beam. In section 3, we shall give test systems for various closed loop systems and assert stability of the closed loop systems using theorem 1.1.

2 Design of feedback control law for Timoshenko beam

In this section we shall attach some feedback controllers for Timoshenko beam. Herein the feedback controllers are classified two types: pointwise controls (including boundary control) and distributed controls. We shall show that under these designs of feedback control law, energy of the closed loop systems decays. Hereafter we always assume system is given by (1.1).

2.1 Pointwise(boundary) feedback control

For boundary condition (B1), we apply controls $u(t)$ and $v(t)$ to one end and take feedback control law as

$$\begin{cases} K(w'(0,t)-\varphi(0,t)) = EI\varphi'(0,t) = 0, \\ K(w'(\ell,t)-\varphi(\ell,t)) = u(t) = -\alpha\dot{w}(\ell,t) - w(\ell,t) \\ EI\varphi'(\ell,t) = v(t) = -\beta\dot{\varphi}(\ell,t) - \varphi(\ell,t). \end{cases} \tag{2.1}$$

For boundary condition (B2), we apply controls $u(t)$ and $v(t)$ to free end and adopt feedback control law as

$$\begin{cases} w(0,t) = 0, \quad \varphi(0,t) = 0, \\ K(w'(\ell,t)-\varphi(\ell,t)) = u(t) = -\alpha\dot{w}(\ell,t), \\ EI\varphi'(\ell,t) = v(t) = -\beta\dot{\varphi}(\ell,t). \end{cases} \tag{2.2}$$

For boundary condition (B3), we take control law as

$$\begin{cases} w(0,t) = 0, \quad EI\varphi'(0,t) = 0, \\ w(\ell,t) = 0, \\ EI\varphi'(1,t) = v(t) = -\beta\dot{\varphi}(\ell,t) - \varphi(\ell,t). \end{cases} \tag{2.3}$$

For boundary condition (B4), we apply controls at a middle point ξ and adopt pointwise feedback control law as

$$\begin{cases} w(\xi^-,t) = w(\xi^+,t), \quad \varphi(\xi^-,t) = \varphi(\xi^+,t), \\ w'(\xi^-,t) - w'(\xi^+,t) = -\alpha\dot{w}(\xi,t) \\ \varphi'(\xi^-,t) - \varphi'(\xi^+,t) = -\beta\dot{\varphi}(\xi,t). \end{cases} \tag{2.4}$$

For boundary condition (B5), we take feedback control law as

$$\begin{cases} w(0,t) = 0, \quad \varphi(0,t) = 0, \\ w(\ell,t) = u(t) = -\alpha K\int_0^t (w'(\ell,s) - \varphi(\ell,s))ds, \\ EI\varphi'(\ell,t) = v(t) = -\beta\dot{\varphi}(\ell,t). \end{cases} \tag{2.5}$$

In the above, the constants α and β are positive feedback gain.

2.2 Distributed feedback controls

For any one of boundary conditions we always take distributed feedback controls as

$$\begin{cases} \rho\ddot{w}(x,t) - K(w''(x,t) - \varphi'(x,t)) \\ \qquad +a(x)\dot{w}(x,t) = 0, \\ I_\rho\ddot{\varphi}(x,t) - EI\varphi''(x,t) - K(w'(x,t) - \varphi(x,t)) \\ \qquad +b(x)\dot{\varphi}(x,t) = 0, \end{cases} \tag{2.6}$$

where $a(x), b(x)$ are positive continuous functions on interval $[c,d] \subset [0,\ell]$, and there is $\alpha > 0$ such that $\max\{a(x)\} \geq \alpha$, $\max\{b(x)\} \geq \alpha$

2.3 Energy of closed loop systems

The energy of Timoshenko beam system is given by

$$\begin{aligned} E(t) = &\tfrac{1}{2}\int_0^\ell [K|w'(x,t) - \varphi(x,t)|^2 + EI|\varphi'(x,t)|^2]dx \\ &+ \tfrac{1}{2}\int_0^\ell [\rho|\dot{w}(x,t)|^2 + I_\rho|\dot{\varphi}(x,t)|^2]dx. \end{aligned} \tag{2.7}$$

If we adopt pointwise or boundary feedback controls, then

$$\begin{aligned} \tfrac{dE(t)}{dt} = &\dot{w}(x,t)K(w'(x,t) - \varphi(x,t))|_{x=0}^{x=\xi^-} \\ &+\dot{\varphi}(x,t)EI\varphi'(x,t)|_{x=0}^{x=\xi^-} \\ &+\dot{w}(x,t)K(w'(x,t) - \varphi(x,t))|_{x=\xi^+}^{x=\ell} \\ &+\dot{\varphi}(x,t)EI\varphi'(x,t)|_{x=\xi^+}^{x=\ell}. \end{aligned} \tag{2.8}$$

Under these feedback controls we have

$$\frac{dE(t)}{dt} = \begin{cases} -\alpha|\dot{w}(\ell,t)|^2 - \beta|\dot{\varphi}(\ell,t)|^2 - \varphi(\ell,t)\dot{\varphi}(\ell,t) - w(\ell,t)\dot{w}(\ell,t), & \text{if (2.1) holds} \\ -\alpha|\dot{w}(\ell,t)|^2 - \beta|\dot{\varphi}(\ell,t)|^2, & \text{if (2.2) holds} \\ -\beta|\dot{\varphi}(\ell,t)|^2 - \varphi(\ell,t)\dot{\varphi}(\ell,t), & \text{if (2.3) holds} \\ -\alpha|\dot{w}(\xi,t)|^2 - \beta|\dot{\varphi}(\xi,t)|^2 & \text{if (2.4) holds} \\ -\alpha^{-1}|K(w'(\ell,t) - \varphi(\ell,t))|^2 - \beta|\dot{\varphi}(\xi,t)|^2 & \text{if (2.5) holds} \end{cases} \tag{2.9}$$

For the distributed feedback controls we have

$$\frac{dE(t)}{dt} = \begin{array}{l} -\int_0^\ell a(x)|\dot{w}(x,t)|^2 dx \\ -\int_0^\ell b(x)|\dot{\varphi}(x,t)|^2 dx. \end{array} \quad (2.10)$$

From above we see easily that energy of the closed loop system decays, but we do not know whether energy of the systems decays exponentially.

3 Test of stability for the closed loop system

In this section we shall give assertion for stability of the closed loop system using theorem 1.1. For different system we shall give distinct test system.

3.1 Cases of pointwise and boundary feedback controls

For these cases we always take (1.9) as test system. More precisely, for closed loop system with boundary condition (2.1), we take test system as

$$\begin{cases} \rho\ddot{w}(x,t) - Kw''(x,t) = 0, \quad 0 < x < \ell, t > 0 \\ I_\rho\ddot{\varphi}(x,t) - EI\varphi''(x,t) = 0, \ 0 < x < \ell, t > 0, \\ w'(0,t) = 0, \quad \varphi'(0,t) = 0, \\ Kw'(\ell,t) = -\alpha\dot{w}(\ell,t) - w(\ell,t), \\ EI\varphi'(\ell,t) = -\beta\dot{\varphi}(\ell,t) - \varphi(\ell,t). \end{cases} \quad (3.1)$$

It is easy to prove that test system (3.1) is a Riesz spectral system when $\alpha \neq \sqrt{\rho/K}$ and $\beta \neq \sqrt{I_\rho/EI}$, whose frequencies satisfy $Re\lambda_{j,n} < 0, j = 1, 2$, and their asymptotic values are given by

$$\lambda_{1,n} = \begin{cases} \frac{1}{2\ell}\ln\left|\frac{\alpha-\sqrt{\rho/K}}{\alpha+\sqrt{\rho/K}}\right| + \frac{in\pi}{\ell} + O(\frac{1}{n}), \\ \quad \text{if } \alpha > \sqrt{\rho/K}, \\ \frac{1}{2\ell}\ln\left|\frac{\alpha-\sqrt{\rho/K}}{\alpha+\sqrt{\rho/K}}\right| + \frac{i(2n+1)\pi}{2\ell} + O(\frac{1}{n}), \\ \quad \text{if } \alpha < \sqrt{\rho/K}. \end{cases} \quad (3.2)$$

and

$$\lambda_{2,n} = \begin{cases} \frac{1}{2\ell}\ln\left|\frac{\beta-\sqrt{I_\rho/EI}}{\beta+\sqrt{I_\rho/EI}}\right| + \frac{in\pi}{\ell} + O(\frac{1}{n}), \\ \quad \text{if } \beta > \sqrt{I_\rho/EI}, \\ \frac{1}{2\ell}\ln\left|\frac{\beta-\sqrt{I_\rho/EI}}{\beta+\sqrt{I_\rho/EI}}\right| + \frac{i(2n+1)\pi}{2\ell} + O(\frac{1}{n}), \\ \quad \text{if } \beta < \sqrt{I_\rho/EI}. \end{cases} \quad (3.3)$$

We assert from theorem 1.1 that this system is exponentially stable.

For closed loop system with boundary condition (2.2), we can take test system as

$$\begin{cases} \rho\ddot{w}(x,t) - Kw''(x,t) = 0, \quad 0 < x < \ell, t > 0 \\ I_\rho\ddot{\varphi}(x,t) - EI\varphi''(x,t) = 0, \ 0 < x < \ell, t > 0, \\ w(0,t) = 0, \quad \varphi(0,t) = 0, \\ Kw'(\ell,t) = -\alpha\dot{w}(\ell,t), \\ EI\varphi'(\ell,t) = -\beta\dot{\varphi}(\ell,t). \end{cases} \quad (3.4)$$

It is easy to prove that test system (3.4) is a Riesz spectral system as $\alpha \neq \sqrt{\rho/K}$ and $\beta \neq \sqrt{I_\rho/EI}$, whose frequencies are given by

$$\lambda_{1,n} = \begin{cases} \frac{1}{2\ell}\ln\left|\frac{\alpha-\sqrt{\rho/K}}{\alpha+\sqrt{\rho/K}}\right| + \frac{in\pi}{\ell}, \\ \quad \text{if } \alpha > \sqrt{\rho/K}, \\ \frac{1}{2\ell}\ln\left|\frac{\alpha-\sqrt{\rho/K}}{\alpha+\sqrt{\rho/K}}\right| + \frac{i(2n+1)\pi}{2\ell}, \\ \quad \text{if } \alpha < \sqrt{\rho/K}. \end{cases} \quad (3.5)$$

and

$$\lambda_{2,n} = \begin{cases} \frac{1}{2\ell}\ln\left|\frac{\beta-\sqrt{I_\rho/EI}}{\beta+\sqrt{I_\rho/EI}}\right| + \frac{in\pi}{\ell}, \\ \quad if \beta > \sqrt{I_\rho/EI}, \\ \frac{1}{2\ell}\ln\left|\frac{\beta-\sqrt{I_\rho/EI}}{\beta+\sqrt{I_\rho/EI}}\right| + \frac{i(2n+1)\pi}{2\ell}, \\ \quad if \beta < \sqrt{I_\rho/EI}. \end{cases} \quad (3.6)$$

Again we deduce from theorem1.1 that the system decays exponentially.

Similarly, for closed loop system with boundary condition (2.3), we take test system as

$$\begin{cases} \rho\ddot{w}(x,t) - Kw''(x,t) = 0, \quad 0 < x < \ell, t > 0 \\ I_\rho\ddot{\varphi}(x,t) - EI\varphi''(x,t) = 0, \ 0 < x < \ell, t > 0, \\ w(0,t) = 0, \quad EI\varphi'(0,t) = 0, \\ w(\ell,t) = 0, \\ EI\varphi'(\ell,t) = -\beta\dot{\varphi}(\ell,t) - \varphi(\ell,t). \end{cases} \quad (3.7)$$

It is easy to prove that test system (3.7) is a Riesz spectral system as $\beta \neq \sqrt{I_\rho/EI}$. However test system is unstable, so the closed loop system is not exponentially stable.

Using above approach we can check stability of closed loop system with boundary condition (2.4) or (2.5) by test system.

3.2 Case of distributed feedback controls

For closed loop system with distributed feedback controls (2.6), we can take (1.8) associated with any one of five-type boundary conditions as test system. In order to simplify the calculation, we can take the dominant part of boundary conditions, for example, for (B2) we can take test boundary conditions as

$$\begin{cases} w(0,t) = \varphi(0,t) = 0, \\ Kw'(\ell,t) = EI\varphi'(\ell,t) = 0. \end{cases} \tag{3.8}$$

Thus corresponding test system is

$$\begin{cases} \rho\ddot{w}(x,t) - Kw''(x,t) + a(x)\dot{w}(x,t) = 0, \\ \quad x \in (0,\ell), t > 0, \\ I_\rho\ddot{\varphi}(x,t) - EI\varphi''(x,t) + b(x)\dot{\varphi}(x,t) = 0, \\ \quad x \in (0,\ell), t > 0, \\ w(0,t) = \varphi(0,t) = 0, \\ Kw'(\ell,t) = EI\varphi'(\ell,t) = 0. \end{cases} \tag{3.9}$$

The stability of this test system is equivalent to stability of wave equation

$$\begin{cases} \xi^2\ddot{w}(x,t) - w''(x,t) + \widetilde{a}(x)\dot{w}(x,t) = 0, \\ \quad x \in (0,\ell), t > 0, \xi > 0, \\ w(0,t) = 0, \quad w'(\ell,t) = 0. \end{cases} \tag{3.10}$$

Corresponding boundary eigenvalue problem is

$$\begin{cases} \lambda^2\xi^2 w(x) - w''(x) + \widetilde{a}(x)\lambda w(x) = 0, \\ \quad x \in (0,\ell), \\ w(0) = 0, \quad w'(\ell) = 0. \end{cases} \tag{3.11}$$

Let $u(x) = w'(x) + \lambda\xi w(x)$,$v(x) = w'(x) - \lambda\xi w(x)$ and $W(x) = (u(x), v(x))^\tau$. Then we have

$$\frac{dW(x)}{dx} = [\lambda M + M_0(x)]W(x)$$

where

$$M = \begin{bmatrix} \xi & 0 \\ 0 & -\xi \end{bmatrix}$$

and

$$M_0(x) = \begin{bmatrix} \frac{\widetilde{a}(x)}{2\xi} & -\frac{\widetilde{a}(x)}{2\xi} \\ \frac{\widetilde{a}(x)}{2\xi} & -\frac{\widetilde{a}(x)}{2\xi} \end{bmatrix},$$

corresponding boundary conditions can be rewritten as

$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} W(0) + \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} W(\ell) = 0. \tag{3.12}$$

According to asymptotic expansion theorem of fundamental matrix(cf. R. Mennicken and M. Möller's book[13, pp83, Theorem 2.8.2]), we have

$$W(x) = [A_0(x) + \lambda^{-1}A_1(x,\lambda)]e^{\lambda M x}\eta, \quad \eta \in \mathbb{C}^2, \tag{3.13}$$

where

$$A_0(x) = \begin{bmatrix} e^{\frac{1}{2\xi}\int_0^x \widetilde{a}(s)ds} & 0 \\ 0 & e^{-\frac{1}{2\xi}\int_0^x \widetilde{a}(s)ds} \end{bmatrix},$$

$$e^{\lambda M x} = \begin{bmatrix} e^{\lambda\xi x} & 0 \\ 0 & e^{-\lambda\xi x} \end{bmatrix},$$

and $A_1(x,\lambda)$ is uniformly bounded on $[0,\ell]$ and $|\lambda| > \gamma > 0$ and $A_1(0,\lambda) = 0$. Substituting (3.13) into (3.12) leads to

$$\begin{bmatrix} 1 & -1 \\ e^{\frac{1}{2\xi}\int_0^\ell \widetilde{a}(s)ds + \lambda\xi\ell} + [0]_0 & e^{-\frac{1}{2\xi}\int_0^\ell \widetilde{a}(s)ds - \lambda\xi\ell} + [0]_0 \end{bmatrix} \eta = 0, \tag{3.14}$$

where $[a]_0$ means $[a]_0 = a + \lambda^{-1}a_1(\lambda)$ and $a_1(\lambda)$ is bounded. Thus asymptotic values of eigenvalues of wave equation are determined via

$$e^{2\lambda\xi\ell} = -e^{-\frac{1}{\xi}\int_0^\ell \widetilde{a}(s)ds},$$

they are given by

$$\lambda_n = -\frac{1}{2\xi^2\ell}\int_0^\ell \widetilde{a}(s)ds + \frac{(2n+1)\pi i}{2\xi\ell}, \quad n \in \mathbb{Z}.$$

Therefore asymptotic values of eigenvalues of test system (3.9) are given by

$$\lambda_{1,n} = -\frac{1}{2\rho\ell}\int_0^\ell a(s)ds + \frac{(2n+1)\pi i}{2\sqrt{\rho/K}\ell}, \quad n \in \mathbb{Z}.$$

$$\lambda_{2,n} = -\frac{1}{2I_\rho\ell}\int_0^\ell b(s)ds + \frac{(2n+1)\pi i}{2\sqrt{I_\rho/EI}\ell}, \quad n \in \mathbb{Z}.$$

Using a result in [12], we can prove that test system is a Riesz spectral system. Therefore test system is exponentially stable, so is system (2.6) with boundary (B2).

Remark 3.1 *In this paper we always require test system being a Riesz spectral system. If the test system is not a Riesz system, then eigenvalues of test system need not to be asymptotic values of eigenvalues of the closed loop system. So we must keep this in mind.*

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