2005 WSEAS Int. Conf. on ENVIRONMENT, ECOSYSTEMS and DEVELOPMENT, Venice, Italy, November 2-4, 2005 (pp261-265)

A STATISTICAL ESTIMATION OF THE COEFFICIENT OF DISPERSION IN ONE-DIMENSIONAL HEAT CONDUCTION EQUATION AND ITS APPLICATION TO THE GROWTH OF YEAST POPULATION

AMRITASU SINHA DEVENDRA TRIPAHI
DEPARTMENT OF MATHEMATICS
DEPARTMENT OF MATHEMATICS
KIST
B.P. 3900 AVENUE DE L'ARMEE
KIGALI
IRT,ALLAHABAD
RWANDA, INDIA
http://www.geocities.amritasu.com

Abstract: **Dispersal of the species in the biological context is an important phenomenon to study. The basics of the theory of random dispersion of Biological Population took shape after the pioneering work of Skellam[7].His method involved applying the analytical expression for molecular diffusion directly To ecological problems, relating it to the interaction among and between species. In this work we shall be studying the one dimensional diffusion equation with constant coefficient of diffusivity as well as, time varying diffusivity. The main objective is to estimate the coefficient of dispersion in both the cases. For this a Statistical Estimation is being performed for the above coefficient in both cases by considering the growth of Yeast made by Carlson.**

Key-Words:- **Ecology, Diffusion Equation,Diffussivity,Coefficient of Dispersion, Growth of Yeast Population**

1. Introduction

Dispersal of the species in the biological context is an important phenomenon to study. The basics of the theory of random dispersion of Biological Population took shape after the pioneering work of Skellam[7].His method involved applying the analytical expression for molecular diffusion directly To ecological problems, relating it to the interaction among and between species. Accounting for differences in scale among ecological entities and processes Has been suggested as a way to understand the hierarchical complexity of Natural systems(O'Neill et al;[4]O'Neill[5],Salthe[6].Yet for all of its promise; examples of Mathematical or Stochastic development of the hierarchical approach are few Steele[8].Recently Timm and Okubo[9]observed (by considering an ecological model of Levin and Segal[] for prey-predator Planktonic species)that diffusive instability is less likely to occur in system with time varying diffusivity,than those of constant diffusivity. Their model Parameter estimates are based on the data in Wroblewski and O'brien[10]. In the present paper we shall be studying the one dimensional diffusion equation with constant coefficient of diffusivity as well as, time varying diffusivity. The main objective is to estimate the coefficient of dispersion in both the cases. For this a Statistical Estimation is being performed for the above coefficient in both cases by considering the growth of Yeast made by Carlson[1,2].

2. Problem Formulation

The one dimensional diffusion equation is,

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} \quad \textbf{(1)}$$

Where D is the coefficient of diffusivity. The above equation is being Extensively used in Biology and other related areas. Diffusion is the spreading Of particles ranging from molecules to bacteria, whose individual trajectories are regarded as random. An example to this situation is the dispersion of dye Particles that are released into a clear fluid from a needle. We are interested in the Statistical Estimation of the coefficient D in (1) called the coefficient of diffusivity.

3.Problem Solution

By a standard technique of separation of variables (1) is being solved to obtain,

$$u_n(x,t) = A_n \sin\left(\frac{n\pi x}{l}\right)\exp\left(-\frac{Dn\pi}{l}\right)^2 t \quad \textbf{(1).}$$

Where n =1,2,3,….

We shall refer this D as pure. If D is assumed to depend upon the time t

Then the solution of (1) is given by

$$u_n(x,t) = A_n \sin\left(\frac{n\pi x}{l}\right)\exp\left(-k\int D(t)dt\right) \quad \textbf{(2).}$$

In the present work we shall study the case when $D(t) = a + b\sin t$,where a and b are constants which are to be estimated. Substituting the value of D in equation (2) we obtain,

$$u_n(x,t) = A_n \sin\left(\frac{n\pi x}{l}\right)\exp(-at + b\cos t) \quad \textbf{(3).}$$

or,

$$u_n(x,t) = B_1' \exp(-at + b\cos t) \qquad \textbf{(4),}$$

where

$$B_1' = A_n \sin\left(\frac{n\pi x}{l}\right)$$

Taking logarithm of the equation (4) we get,

$$\log u_n(x,t) = \log B_1 - at + b\cos t$$

$$or$$

$$y(t) \;= B_1 + B_2 t + B_3 \cos t$$

$$where$$

$$y(t) = \log u_n(x,t)$$

$$B_1 = \log B_1'$$

$$B_2 = -a$$

$$B_3 = b$$

Case 1

The equation (2) can be written as,

$$u(t) = A_1' \exp\left(-A_2' t\right)$$
$$where \qquad (5)$$
$$A_1' = A_n \sin\left(\frac{n\pi x}{l}\right)$$

Upon considering

$$A_2' = \left(\frac{Dn\pi}{l}\right)^2,$$

We get
$\log u(t) = \log A_1' - A_2' t$ **,which upon further simplification results into;**

$$y(t) = A_1 + A_2 t$$
$$where$$
$$y(t) = \log u(t) \qquad (6).$$
$$A_1 = \log A_1'$$
$$A_2 = -A_2'$$

Case 2

Proceeding in the similar manner equation (6) can also be written as

$$\log u(t) = \log B_1' - at + b\cos t$$
$$where$$
$$B_1 = \log B_1' \qquad (7).$$
$$B_2 = -a$$
$$B_3 = b$$

We now estimate the constants in both the cases using the Method of Least Square

Suppose there are n-paired observation t and u(t).Then the normal equation in the first case are

$$\sum_{i=1}^{n} y_i = r_1 A_1 + A_2 \sum_{i=1}^{n} t_i$$

$$\sum_{i=1}^{n} y_i t_i = A_1 \sum_{i=1}^{n} t_i + A_2 \sum_{i=1}^{n} t_i^2 \qquad (8).$$

Similarly we can obtain the normal equations for Case 2, which are as follows

$$\sum_{i=1}^{n} y_i = r_1 B_1 + B_2 \sum_{i=1}^{n} t_i + B_3 \sum_{i=1}^{n} \cos t_i \qquad (9)$$
$$\sum_{i=1}^{n} y_i t_i = B_1 \sum_{i=1}^{n} t_i + B_2 \sum_{i=1}^{n} t_i^2 + B_3 \sum_{i=1}^{n} t_i \cos t_i$$
$$\sum_{i=1}^{n} y_i \cos t_i = B_1 \sum_{i=1}^{n} \cos t_i + B_2 \sum_{i=1}^{n} t_i \cos t_i + B_3 \sum_{i=1}^{n} \left(\cos t_i\right)^2$$

We have tried to obtain the Regression Model for the above two cases for the following set of data from Carlson[1,2] on the population of Yeast growth in Laboratory cultures

Table 1
Growth of Yeast Population

Hours (t)	Am.of Yeast (u(t))
0	9.6
1	18.3
2	29.0
3	47.2
4	71.1
5	119.1
6	174.6
7	257.3
8	350.7
9	441.0
10	513.3
11	559.7
12	594.8
13	629.4
14	640.8
15	651.1
16	655.9
17	659.6
18	661.8

Using the above data the regression models are

$u(t) = 28.46663 \exp(0.2238915\, t)$ **(10)**

for Case 1,and

$u(t) = 28.77304 \exp(0.2227680\, t - 0.079207 \cos t)$ **(11).**

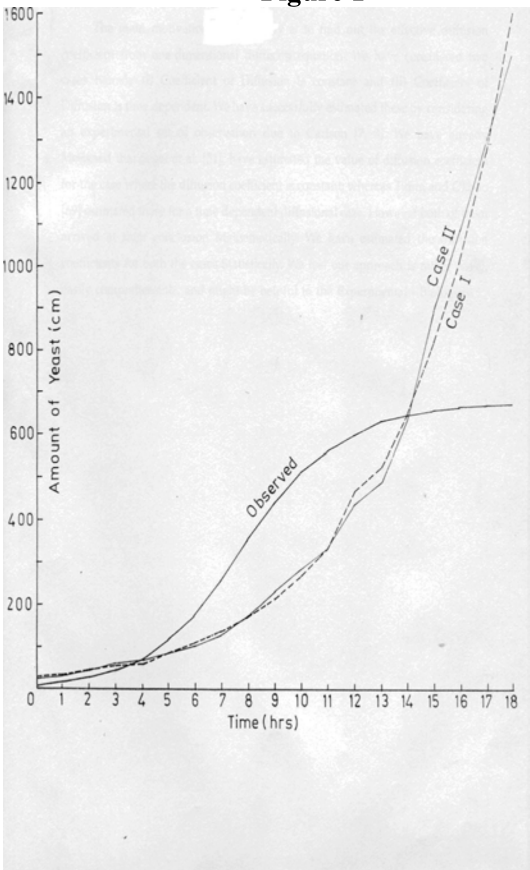
These are estimated and shown for the above set of data in the following table.

Table 2
Growth of Yeast population with its estimated value

Hours(t)	Amount of Yeast	Estimate for	
	observed	Case1 (Constant D)	Case2 D as a function of time
0	9.6	28.47	26.58
1	18.3	35.61	34.45
2	29	44.55	46.43
3	47.2	55.72	60.71
4	71.1	69.71	73.87
5	119.1	87.2	85.7
6	174.6	109.08	101.49
7	257.3	136.45	128.91
8	350.7	170.69	172.97
9	441	213.53	229.64
10	513.3	267.11	285.31
11	559.7	334.13	333.46
12	594.8	417.98	389.87
13	629.4	522.86	484.71
14	640.8	654.07	643.78
15	651.1	818.2	863.62
16	655.9	1023.51	1096.17
17	659.6	1280.35	1297.62
18	661.8	1601.63	1505.62

The attached Figure 1 shows the three cases.

Figure 1



Conclusion

The main motivation of this study is to estimate the coefficient of diffusion in both the cases Statistically. However Timm and Okubo[9] have estimated these Mathematically. We feel that our approach is rather novel, easily comprehensible and might be helpful to Experimental Biologists.

REFERENCES

[1].Carlson,T
Uber Gescwindigkeit and Grosse Der Hefevermechrung in Wurze, Biochemie Zeistschrift **57 (1913), 313-34.**

[2]Carlson,T
Genetics of natural population XV,Rate of diffusion of a Mutant gene through a Population of Drosophila Pseudo obscura, **Genetics 32,303-324.**

[3].Levin,S.A., Segal, L.E.
Hypothesis for origin of plank--tonic patchiness. Nature.**259 (1978).**

[4].O'Neill, R.V.Deanglis, D.A.Waide,J.B.and Allen,T.F.H.
A hierachical concept of Ecosystems.
Princeton.Univ. Press. Princeton,N.J.(1986).

[5].O'Neill,R.V.
Perspectives in hierarchy and Scale in"Perspectives in Ecological theory."
(J.Roughgarden,R.M.May and S.A. Levine Eds.) Princeton univ.Press, Princeton,N.J. (1989),pp.140-156.

[6].Salthe,S.N.
"*Evolving hierarchical Systems.*"
Columbia Univ.Press., New York,(1995).

[7]. Skellam,J.G.
Random dispersal in Theoritical Populations.
Biometrica.38,(1951),196-218.

[8].Steele,J.H.
Scale and coupling in ecological system in "Perspectives in ecological theory."
(J.Roughgarden et al.Eds.), (1989),pp.177-188, PrincetonUniv.Press, Princeton N.J.

[9].Timm,U and Okubo.A
Diffusion-driven instability In a predator-prey system With time-varying diffusi--vities.
J.Math.Biol.30,(1992), 307-320.

[10].Wroblewski,J.S.; O'Brien,J.J.
A spatial model of phyto--planklon patchiness.
Mar.Biol.35(1976), 161-175.