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Principles of the Minimum Dissipated Power in Stationary Regime

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Abstract: - **The use of the power and energy functional in the analysis of the electric circuits makes it possible to appreciate the energetic equilibrium state attained in the circuit at a certain moment. In the present work, a power functional is built and its limit is determined. It is shown that the equilibrium state is one of a minimum energetic state.**

Key-Words: - electric circuits, Hilbert space, power functionals, minimum of functionals.

1 Introduction

Taking into consideration the functional and calculating their limits represents an important breakthrough in formulating and solving some problems related to the optimum.

The steady states in the mechanic, thermic, electric conservative systems generally represent limit states from an energetic point of view. For example [1], in the classical mechanics, Hamilton's principle of the minimum action states that the development in time of a system, from one steady state to another steady state, happens along a curve $\gamma : [a,b] \to R^n$, which extremates the functional

$$F_M = \int_a^b (T + U) dt \qquad (1)$$

called the integral of action, where T is the kinetic energy (the "life power") of the system and U is the mechanical effect of the power system which works on the system under consideration.

In the theory of the electrical circuits, the results obtained by Millar [2] and Stern [3] related to the cocontent function for nonlinear resistive and reciprocal network have a special theoretic importance due to their generality. C. A. Desoer and E. Kuh, [4] (pp. 770-772), had proved the same generally properties of the minimum dissipated power for the linear and resistive networks. As well, the Romanian professors V. Ionescu [5] and C.I. Mocanu [6] (pp. 350-353), had important contributions at the theoretical development of the electrical circuits minimax theorems. All these results are basically consequences of Maxwell's principles of minimum-heat [7] (pp. 407-408).

The electromagnetic field analysis admits a differential formulation and a variational equivalent formulation. The mathematical variational model presumes the establishment of a variational principle capable to supply the equations of the differential mathematical model through the stationarity condition of an adequate functional. The energetic functional of the non-stationary electromagnetic field associated to the domain D_c and the volume V, is:

$$F = \int_{D_c} \{(\int_0^{\bar{E}} \bar{D}d\bar{E} - \int_0^{\bar{B}} \bar{H}d\bar{B}) + (\bar{J}\bar{A} - \rho_v V)\}dxdydz \qquad (2)$$

where: $\bar{D}, \bar{E}, \bar{H}, \bar{B}$ are the vectors associated to the electric and magnetic field, $\bar{A}$ is the magnetic vector potential, V is the electric potential, $\bar{J}$ is the vector of conduction current density, and ρ_v is the volume density of electric charge. On demonstrate that the condition of minimum of functional (2) involve the fundamental equations, $rot\bar{H} = \bar{J} + \frac{\partial \bar{D}}{\partial t}, div\bar{D} = \rho_v$, of the electromagnetic field. Analogue, the energetic functional for electrokinetic field, where the state quantity is the electric potential V, is defined by:

$$F_1(V) = \int_{D_c} (\int_0^{\bar{J}} \bar{J}d\bar{E})dzxdydz + \int_{\Sigma_n} \bar{J}\bar{n}Vd\Sigma_n \qquad (3)$$

This paper presents three original contributions concerning the energetic functionals of the linear circuits. The first of them proposes an energetic functional for the linear circuits derived of the circuit equations properties. The second represents an other demonstration which proves that the stationary (equilibrium) electrokinetic state for the d.c. networks represents a minimum energetic state as far as the power dissipated by the circuit elements, and the third one proposes an original and general principle of the minimum dissipated active and reactive power for the a.c. linear circuits.

2. The Matrix Equations and the Minimum of Dissipated Power in D.C Circuits

Let's take the case of a reciprocal d.c. electric circuit, with l branches, in an equilibrium state. If G represents the graph of the circuit, then A is a tree and C its complementary tree. The branches voltages and the branches current verify the first and, respectively, the second theorem of Kirchhoff. The l-dimensional current and voltage matrix can be partitioned as follows

$$[I] = \begin{bmatrix} [I_c] \\ [I_r] \end{bmatrix} = \begin{bmatrix} [I_c] \\ -[\Lambda_{cr}][I_c] \end{bmatrix} \qquad (4)$$

$$[U] = \begin{bmatrix} [U_c] \\ [U_r] \end{bmatrix} = \begin{bmatrix} [\Lambda_{cr}][U_r] \\ [U_r] \end{bmatrix} \qquad (5)$$

where an adequate notation has been used to point out the sub matrix that refer to the current and voltage links, and the respective branches; $[\Lambda] = -\lfloor C_{B_c r} \rfloor$ is the essential incidence matrix. So all the voltage generators become an equivalent current generators, the matrix relation of the branches currents becomes:

$$[I] = [G][U] \qquad (6)$$

where [G] is the square matrix of the branch conductance which can be partitioned under the form of:

$$[G] = \begin{bmatrix} [G_{cc}][G_{cr}] \\ [G_{rc}][G_{rr}] \end{bmatrix} \qquad (7)$$

For the reciprocal circuits in stable electrokinetic state, the following power functional can be defined in the Hilbert space R^n

$$F : R^n \to R \qquad (8)$$

$$F \stackrel{\Delta}{=} \frac{1}{2}[U]^T[I] \qquad (9)$$

where the superscript T denotes transposition. By using the relations (4), (5), (6), (7) and (9), we can calculate

$$\begin{aligned} F([U_r]) &= \frac{1}{2}[U]^T[I] = \frac{1}{2}\left[[U_c]^T[U_r]^T\right] \\ &\begin{bmatrix} [G_{cc}][G_{cr}] \\ [G_{rc}][G_{rr}] \end{bmatrix}\begin{bmatrix} [U_c] \\ [U_r] \end{bmatrix} = \\ &= \frac{1}{2}\{[U_c]^T[G_{cc}][U_c] + [U_r]^T[G_{rr}][U_r]\} = \\ &= \frac{1}{2}\{[U_r]^T[\Lambda_{rc}][G_{cc}][\Lambda_{cr}][U_r] + \\ &+ [U_r]^T[G_{rr}][U_r]\} = \\ &= \frac{1}{2}[U_r]^T\{[\Lambda_{rc}][G_{cc}][\Lambda_{cr}] + [G_{rr}]\}[U_r] \end{aligned} \qquad (10)$$

where $[G_{cr}] = [G_{cr}] = 0$ for the reciprocal electric circuits.

In relation (10), we have obtained the expression of the functional under consideration, based only on the matrix of the branch voltages. To determine its extreme we apply the function of matrix properties. If $F'([U_r] = 0$, results

$$\begin{aligned} &[U_r]^T([\Lambda_{rc}][G_{cc}][\Lambda_{cr}] + [G_{rr}])[U_r] = 0 \\ &[\Lambda_{rc}][G_{cc}][\Lambda_{cr}][U_r] + [G_{rr}][U_r] = 0 \end{aligned} \qquad (11)$$

that is

$$[\Lambda_{rc}][I_c]+[I_r]=0 \qquad (12)$$

Consequently, the extreme point of the functional verifies the first theorem of Kirchhoff. To demonstrate that extreme point of the functional is minimum, we presumably take a different matrix of the branch voltages,

$$[\hat{U}_r]=[U_r]+\delta[U_r] \qquad (13)$$

which introduced in expression (8) leads to:

$$\begin{aligned}
F([\hat{U}_r]) &= \frac{1}{2}([U_r]+\delta[U_r])^T([\Lambda_{rc}][G_{cc}][\Lambda_{cr}]+ \\
&+[G_{rr}])([U_r]+\delta[U_r]) = F([U_r]) + \\
&+\delta[U_r]^T\{[\Lambda_{rc}][G_{cc}][\Lambda_{cr}]+[G_{rr}]\}[U_r]+ \\
&+\frac{1}{2}\delta^2[U_r]^T\{[\Lambda_{rc}][G_{cc}][\Lambda_{cr}]+[G_{rr}]\}[U_r]
\end{aligned} \qquad (14)$$

This demonstrates that the matrixes $[\Lambda_{rc}][G_{cc}][\Lambda_{cr}]$ and $[G_{rr}]$ are positively defined, [8], so $[\Lambda_{rc}][G_{cc}][\Lambda_{cr}]+[G_{cc}]>0$ and results:

$$F([\hat{U}_r]) > F([U_r]) \qquad (15)$$

this means that the functional has a minimum. According to definition (9), the functional $F([U_r])$ represents the power dissipated at the terminals of all the branches of a reciprocal circuit in stable electrokinetic state. Therefore, the results obtained (12) shows that the stable electrokinetic state is a minimal power state dissipated by the branches of the circuit.

Consequently, we get the following principle (*1st Principle of Minimum dissipated Power – PMP*): *the minimum of the dissipated power by the branches of linear and resistive circuit in stationary regime (d.c.) is satisfied by the solutions in the currents and voltages of the circuit, and these are the currents and voltages which verify the 1st and 2nd theorem of Kirchhoff.*

3. Hilbert Space Techniques for Determining the Minimum of the Active and Reactive Dissipated Power Functionals for Linear A.C. circuits

We can demonstrate a similar principle for the cvasistationary regime (a.c.) of linear electric circuit.

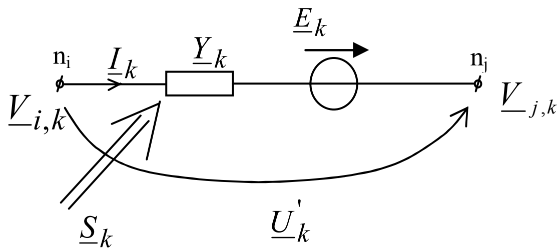

Fig. 1. A.c. circuit branch

By using the symbolical method, the voltage at every branch (fig.1) of the circuit is equal to:

$$\underline{U}_k+\underline{E}_k=\frac{\underline{I}_k}{\underline{Y}_k} \qquad (16)$$

If we note: $\underline{V}_{i,k}=x_{i,k}+jy_{i,k}$; $\underline{V}_{j,k}=x_{j,k}+jy_{j,k}$; $\underline{Y}_k=G_k-jB_k$; $\underline{E}_k=a_{E,k}+jb_{E,k}$,

where G_k, B_k, $a_{E,k}$ and $b_{E,k}$ are constants, then the complex conjugated current of branch k can be expressed [9]:

$$\begin{aligned}
\underline{I}_k^* &= \underline{Y}_k^*(\underline{V}_{i,k}-\underline{V}_{j,k}+\underline{E}_k)^* = \\
&[G_k(x_{i,k}-x_{j,k}+a_{E,k})+ \\
&+B_k(y_{i,k}-y_{j,k}+b_{E,k})]+ \\
&+j[B_k(x_{i,k}-x_{j,k}+a_{E,k})- \\
&-G_k(y_{i,k}-y_{j,k}+b_{E,k})]
\end{aligned} \qquad (17)$$

The complex power dissipated by the admittances of all the L branches of the circuit is:

$$\begin{aligned}
&\sum_{k=1}^{L}\underline{S}_k = \sum_{k=1}^{L}\underline{U}_k\underline{I}_k^* = \\
&= \sum_{k=1}^{L}(G_k+jB_k)[(x_{i,k}-x_{j,k}+a_{E,k})^2 + \\
&+(y_{i,k}-y_{j,k}+b_{E,k})^2]
\end{aligned} \qquad (18)$$

The real and imaginary parts of the complex power can be defined as the functionals in Hilbert space [10]

$$F_R \equiv \frac{1}{2}\mathrm{Re}[\underline{S}] : R^{2N} \to R$$

$$F_R(x_1,x_2,...,x_N,y_1,y_2,...,y_N) \equiv \frac{1}{2}\mathrm{Re}[\underline{S}] =$$

$$\frac{1}{2}\sum_{k=1}^{L} G_k[(x_{i,k}-x_{j,k}+a_{E,k})^2+(y_{i,k}-y_{j,k}+b_{E,k})^2]$$

(19)

$$F_I \equiv \frac{1}{2}\mathrm{Im}[\underline{S}] : R^{2N} \to R$$

$$F_I(x_1,x_2,...,x_N,y_1,y_2,...,y_N) = \frac{1}{2}\mathrm{Im}[\underline{S}] =$$

$$\frac{1}{2}\sum_{k=1}^{L} B_k[(x_{i,k}-x_{j,k}+a_{E,k})^2+(y_{i,k}-y_{j,k}+b_{E,k})^2]$$

and they are quite obviously a function class C^2 in R^{2N} , and are positively defined, [9], i.e. for all the pair
$(x_i, y_i), i = 1,..., N$, then:
$F_R(x_1, x_2,..., x_N, y_1, y_2,..., y_N)\rangle 0$
and $F_I(x_1, x_2,..., x_N, y_1, y_2,..., y_N)\rangle 0$.

Consequently, the minimum points of the real and imaginary component of the complex power functionals, [10], are the solutions of the system which contains 4N equations:

$$\frac{\partial F_R}{\partial x_1}=0,\ \frac{\partial F_R}{\partial y_1}=0,...,\ \frac{\partial F_R}{\partial x_N}=0, \frac{\partial F_R}{\partial y_N}=0,$$
$$\frac{\partial F_I}{\partial x_1}=0,\ \frac{\partial F_I}{\partial y_1}=0,...,\ \frac{\partial F_I}{\partial x_N}=0, \frac{\partial F_I}{\partial y_N}=0 \quad (20)$$

If we calculate the algebrical sum of the solutions, with one of them multiplied with (-1) or ±j , we obtain the expressions:

$$\sum_{l_k \in n_1} \underline{I}_k^* = 0,\ \sum_{l_k \in n_2} \underline{I}_k^* = 0,...,\ \sum_{l_k \in n_N} \underline{I}_k^* = 0 \quad (21)$$

which are identical with the Kirchhoff's equations for currents (1st Kirchhoff theorem), expressed in all the N nodes of the circuit.

Consequently, the following principle can be issued *(2nd Principle of Minimum Active and Reactive dissipated Power –PMARP)*: *the minimum of the active and reactive dissipated power by the branches of a linear circuit in a cvasistationary regime (a.c.) is satisfied by the solutions in currents and voltages of the circuit, and these are the currents and voltages that verify the 1st and 2nd theorem of Kirchhoff.*

4. Examples

4.1. Determination of the minimum power functional for a d.c. circuit

We consider the d.c. circuit shown in figure 2, where $R_1 = 1\Omega, R_2 = 2\Omega, R_3 = 3\Omega, E_1 = 4V$ .Because $U_r = V_2 - V_1$, the power functional dissipated by the

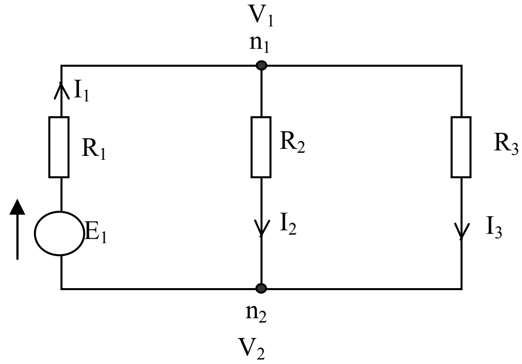

Fig. 2. D.c. circuit with three branches

branches of the circuit, (9), are calculated depending on potentials V_1, V_2 :

$$F = \frac{1}{2}\{G_1(V_2 - V_1 + E_1)^2 + G_2(V_1 - V_2)^2 +$$
$$+ G_3(V_1 - V_2)^2\}$$

The minimum of the power functional are the solutions of the system

$$\frac{\partial F}{\partial V_1} = -G_1(V_2 - V_1 + E_1) + G_2(V_1 - V_2) +$$
$$+ G_3(V_1 - V_2) = -I_1 + I_2 + I_3 = \sum_{l_k \in n_1} I_k = 0$$

$$\frac{\partial F}{\partial V_2} = G_1(V_2 - V_1 + E_1) - G_2(V_1 - V_2) -$$
$$- G_3(V_1 - V_2) = I_1 - I_2 - I_3 = \sum_{l_k \in n_2} I_k = 0$$

which represent the 1st theorem of Kirchhoff expressed in node 1 and 2. We calculate using PSPICE and MATHCAD the variation of the dissipated powers P_1, P_2, P_3 depending by the currents I_1, I_2, I_3, shown respectively in figures 3, 4 and 5. Using Thèvenin's theorem, we obtain the expressions:
$P_i = (U_{i,0} - R_{e,i}I_i)I_i, i = 2,3$, where $U_{i,0}, R_{e,i}$, are the open voltage and the equivalent resistance of the circuit at the terminals of branch 2, respectively 3.

It is remarked that the dissipated power in each branch is a minimum value compared with the maximum value of the power, which is obtained for the current value of short-circuit divided by 2 ($\frac{I_{sc}}{2}$),

$P_1 \approx 3{,}305W \langle P_{1\max} = 3{,}33W; P_2 \approx 2{,}38W \langle P_{2\max} = 3W;$
$P_3 \approx 1{,}586W \langle P_{3\max} \approx 2{,}67W.$

As well, we observe in figures 3, 4, and 5, the existence of two current branch values which correspond with the value of dissipated power for each branch

$I_1^{'} = 1.818A, I_1^{''} = 1.515A; I_2^{'} = 1.09A, I_2^{''} = 2.909A;$
$I_3^{'} = 0.727A, I_3^{''} = 3.272A.$

Only one of these values verifies the Kirchhof's theorems ($I_1^{'}, I_2^{'}, I_3^{'}$).

Generally speaking, for the branches with resistances, these current values represent the roots of a second degree equation: $I^{','' } = \frac{U_0}{2R_e}(1 \pm \frac{R_e - R}{R_e + R}).$ *The graphic positions of these two values of current compared with the value* $\frac{I_{sc}}{2}$ *are depending of the resistance value: if* $R \langle R_e$, $I^{'} \rangle \frac{I_{sc}}{2}$, *and if* $R \rangle R_e$, $I^{''} \langle \frac{I_{sc}}{2}$, *which is presented in the graphics of figures 4 respectively 5. .*

In conclusion, the currents of circuits are distributed so that they verify 1st Kirchhoff theorem and the powers dissipated by the circuit are minimum.

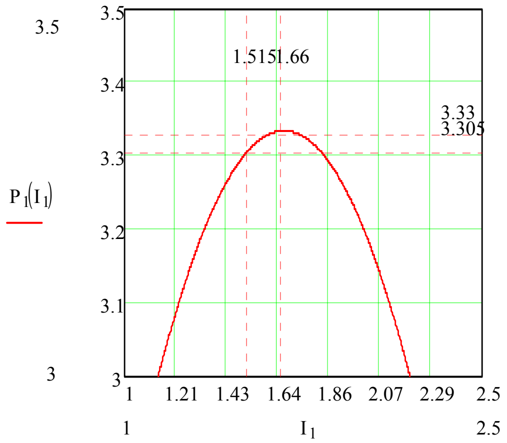

Fig. 3. The variation of P_1

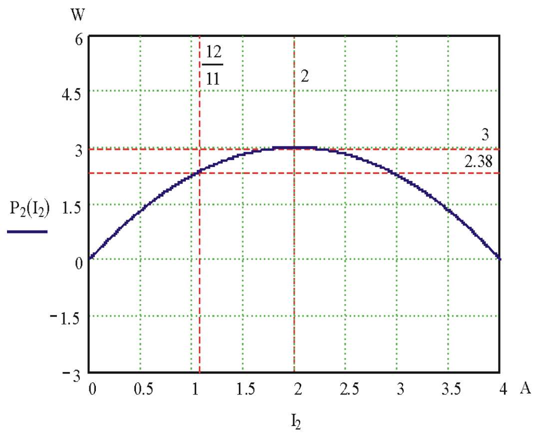

Fig. 4. The variation of P_2

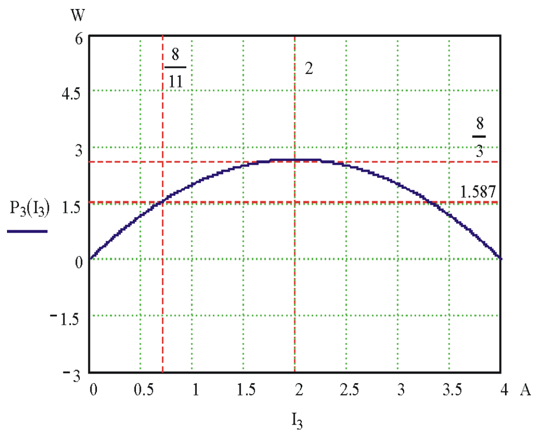

Fig. 5. The variation of P_3

4.2.Determination of the minimum power functional for a.c. circuits

We consider the a.c. circuit shown in figure 6. The expressions of the complex potentials, of the complex source and the expression of the complex conjugated currents (15), of the circuit are:

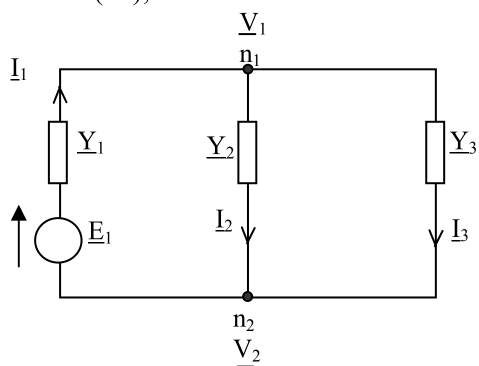

Fig. 6. A.c. circuit with three branches

$\underline{V}_1 = x_1 + jy_1; \underline{V}_2 = x_2 + jy_2; \underline{E}_1 = a + jb;$

$\underline{I}_1^* = (G_1 + jB_1)[(x_2 - x_1 + a) - j(y_2 - y_1 + b)];$

$\underline{I}_2^* = (G_2 + jB_2)[(x_1 - x_2) - j(y_1 - y_2)];$

$\underline{I}_3^* = (G_3 + jB_3)[(x_1 - x_2) - j(y_1 - y_2)].$

The total complex dissipated power of the admittances of the circuit is (16):

$$\underline{S} = \sum_{k=1,2,3} \underline{S}_k = (G_1 + jB_1)[(x_2 - x_1 + a)^2 + (y_2 - y_1 + b)^2] +$$

$$+ (G_2 + jB_2)[(x_1 - x_2)^2 + (y_1 - y_2)^2] +$$

$$+ (G_3 + jB_3)[(x_1 - x_2)^2 + (y_1 - y_2)^2$$

If the variables are x_1 and y_1, which are the real and imaginary parts of the potential $\underline{V}_1$, the minimum of the functionals (17) is the solution of the system:

$$\frac{1}{2}\frac{\partial \mathrm{Re}[\underline{S}]}{\partial x_1} = -G_1(x_2 - x_1 + a) + G_2(x_1 - x_2) + G_3(x_1 - x_2) = 0$$

$$\frac{1}{2}\frac{\partial \mathrm{Re}[\underline{S}]}{\partial y_1} = -G_1(y_2 - y_1 + b) + G_2(y_1 - y_2) + G_3(y_1 - y_2) = 0$$

$$\frac{1}{2}\frac{\partial \mathrm{Im}[\underline{S}]}{\partial x_1} = -B_1(x_2 - x_1 + a) + B_2(x_1 - x_2) + B_3(x_1 - x_2) = 0$$

$$\frac{1}{2}\frac{\partial \mathrm{Im}[\underline{S}]}{\partial y_1} = -B_1(y_2 - y_1 + b) + G_2(y_1 - y_2) + G_3(y_1 - y_2) = 0.$$

If we sum the first equation of system with the third equation multiplied with $(-j)$, and if we sum the second equation multiplied with $(-j)$ with the forth equation, we obtain the 1st theorem of Kirchhoff expressed in node 1:

$$-\underline{I}_1^* + \underline{I}_2^* + \underline{I}_3^* = \sum_{l_k \in n_1} \underline{I}_k^* = 0$$

We obtain a similar result if the variables of the system will be x_2 and y_2, which are the real and imaginary parts of the potential $\underline{V}_2$:

$$\frac{1}{2}\frac{\partial \mathrm{Re}[\underline{S}]}{\partial x_1} = -G_1(x_2 - x_1 + a) + G_2(x_1 - x_2) + G_3(x_1 - x_2) = 0$$

$$\frac{1}{2}\frac{\partial \mathrm{Re}[\underline{S}]}{\partial y_1} = -G_1(y_2 - y_1 + b) + G_2(y_1 - y_2) + G_3(y_1 - y_2) = 0$$

$$\frac{1}{2}\frac{\partial \mathrm{Im}[\underline{S}]}{\partial x_1} = -B_1(x_2 - x_1 + a) + B_2(x_1 - x_2) + B_3(x_1 - x_2) = 0$$

$$\frac{1}{2}\frac{\partial \mathrm{Im}[\underline{S}]}{\partial y_1} = -B_1(y_2 - y_1 + b) + G_2(y_1 - y_2) + G_3(y_1 - y_2) = 0$$

and the solution of the system verifies the equation:

$$\underline{I}_1^* - \underline{I}_2^* - \underline{I}_3^* = \sum_{l_k \in n_2} \underline{I}_k^* = 0$$

which represent the 1st theorem of Kirchhoff expressed in node 2.

5.Conclusions

To determine the extreme of the power functional in case of the linear circuits is a problem of utmost importance, with quite useful didactic, theoretical and practical applications.

Using the variational principles it has been established that the solutions of the linear electric d.c. and a.c. circuit represent a minimum of the dissipated power in the stationary regime of the circuit.

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