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On the Switching Effect of general predation on species living in different habitats with general harvesting

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Abstract:- The works of Bhatt, Khan, Jaju and Bhatt, Owen, Jaju, which deal with two prey species living in two different habitats with one predator specie, which is allowed to switch towards the most abundant prey specie, are extended by allowing the predators to attack the two prey species in two different ways and allowing harvesting of the prey species to be different for each specie and different from the predator-prey interactions. The stability of non-zero equilibrium states is examined and conditions for stability are obtained. Using the conversion rate of prey to predator as a bifurcation parameter, conditions for a bifurcation to occur are obtained. A Hopf bifurcation theorem is also presented. Six hypothetical systems are considered and corresponding bifurcation points determined. Graphic results for only one such system are displayed.

Key-Words: - Prey, Predator, Switching, Stability, Differential equations, Bifurcation Point.

1 Introduction

In predator-prey systems it is important to have a good functional representation of the interaction between the predators and preys and also if harvesting of preys, which is itself a predator-prey interaction is to take place, we need a representation of that interaction as well.

In order to obtain an interaction function sometimes it is good to try to get a general form of the function, together with any general conditions it should satisfy, then construct specific functions, using elementary functions, which satisfy the conditions. We shall use this as our basic guide for the interactions. As base models for the predator-prey systems we use the models of Bhatt, Khan, Jaju [1] and Bhatt, Owen, Jaju [2]. Ref.[1] considers systems of two preys living in two different habitats with one predator specie which is allowed to switch towards the most abundant prey specie. Ref.[2] extends [1] by including general predator-prey interactions where the predator interacts in the same way with both preys. That is, the predatory rates have the same functional form.

The object of the present work is to extend the above works in the sense that havesting of the preys is introduced whereby the two prey species can be harvested in different ways and the predators may also attack each prey specie in a different way. We do not apply any specific interaction functions to any actual systems, rather, we consider all interactions as general functions and determine the equilibrium states, their stability conditions and Hopf bifurcation points using the above works as our base models. A Hopf bifurcation theorem is also presented.

During our investigations we examined, numerically, the following hypothetical systems

System 1:

$$k_1\left(\frac{x_2}{x_1}\right) = \frac{1}{1+\left(\frac{x_2}{x_1}\right)^n}\,,\; h_1\left(\frac{x_2}{x_1}\right) = 1\,,$$

$$k_2\left(\frac{x_1}{x_2}\right) = \frac{1}{1+\left(\frac{x_1}{x_2}\right)^n}\,,\; h_2\left(\frac{x_1}{x_2}\right) = 1\,,$$

System 2:

$$k_1\left(\frac{x_2}{x_1}\right) = \frac{1}{\exp\left(\frac{x_2}{x_1}\right)^n}\,,\; h_1\left(\frac{x_2}{x_1}\right) = 1\,,$$

$$k_2\left(\frac{x_1}{x_2}\right) = \frac{1}{\exp\left(\frac{x_1}{x_2}\right)^n}\,,\; h_2\left(\frac{x_1}{x_2}\right) = 1\,,$$

System 3:

$$k_1\left(\frac{x_2}{x_1}\right) = h_1\left(\frac{x_2}{x_1}\right) = \frac{1}{1+\left(\frac{x_2}{x_1}\right)^n}\,,$$

$$k_2\left(\frac{x_1}{x_2}\right) = h_2\left(\frac{x_1}{x_2}\right) = \frac{1}{1+\left(\frac{x_1}{x_2}\right)^n}\,,$$

System 4:

$$k_1\left(\frac{x_2}{x_1}\right) = h_1\left(\frac{x_2}{x_1}\right) = \frac{1}{\exp\left(\frac{x_2}{x_1}\right)^n}\,,$$

$$k_2\left(\frac{x_1}{x_2}\right) = h_2\left(\frac{x_1}{x_2}\right) = \frac{1}{\exp\left(\frac{x_1}{x_2}\right)^n}\,,$$

System 5:

$$k_1\left(\frac{x_2}{x_1}\right) = \frac{1}{1+\left(\frac{x_2}{x_1}\right)^n}\,,$$

$$h_1\left(\frac{x_2}{x_1}\right) = \frac{1}{\exp\left(\frac{x_2}{x_1}\right)^n}\,,$$

$$k_2\left(\frac{x_1}{x_2}\right) = \frac{1}{1+\left(\frac{x_1}{x_2}\right)^n}\,,$$

$$h_2\left(\frac{x_1}{x_2}\right) = \frac{1}{\exp\left(\frac{x_1}{x_2}\right)^n}\,,$$

System 6:

$$k_1\left(\frac{x_2}{x_1}\right) = \frac{1}{\exp\left(\frac{x_2}{x_1}\right)^n}\,,$$

$$h_1\left(\frac{x_2}{x_1}\right) = \frac{1}{1+\left(\frac{x_2}{x_1}\right)^n}\,,$$

$$k_2\left(\frac{x_1}{x_2}\right) = \frac{1}{\exp\left(\frac{x_1}{x_2}\right)^n}\,,$$

$$h_2\left(\frac{x_1}{x_2}\right) = \frac{1}{1+\left(\frac{x_1}{x_2}\right)^n}\,,$$

where the functions $k_i, h_i,\ i = 1,2$ represent predatory and harvesting rates occurring in the differential equations (1).

We must mention that the parameters used in the numerical calculations in this work do not represent any real systems and are introduced for illustrative purposes only. The actual functions and parameter values were obtained by careful guessing.

We display graphically only the results for System 5 and mention that results for all of the systems examined support the present theory.

In Section 2 we set up the equations representing our model. In Section 3 we examine the stability of equilibrium states, while Section 4 contains the Hopf bifurcation analysis. Section 5 is devoted to the applications and in Section 6 we give the results.

2 Equations defining the model

The equations which we consider as defining the predator-prey model which allows the predator to switch towards the most abundant prey specie and which includes general predatory and harvesting rates are:

$$\begin{aligned}
\frac{dx_1}{dt} &= \alpha_1 x_1 - \epsilon_1 x_1 + \epsilon_2 p_{21} x_2 - \\
&\quad \beta_1 k_1 \left(\frac{x_2}{x_1}\right) x_1 y - x_1 h_1 \left(\frac{x_2}{x_1}\right) \delta_1 \\
\frac{dx_2}{dt} &= \alpha_2 x_2 - \epsilon_2 x_2 + \epsilon_1 p_{12} x_1 - \\
&\quad n\beta_2 k_2 \left(\frac{x_1}{x_2}\right) x_2 y - x_2 h_2 \left(\frac{x_1}{x_2}\right) \delta_2 \\
\frac{dy}{dt} &= \left[-\mu + c_1 \beta_1 k_1 \left(\frac{x_2}{x_1}\right) x_1 + \right. \\
&\quad \left. c_2 \beta_2 k_2 \left(\frac{x_1}{x_2}\right) x_2 \right] y \qquad (1)
\end{aligned}$$

where

x_i : represents the prey population in the two different habitats

δ_i : represents the harvesting rate of the prey population in the two different habitats

y : represents the abundance of predator species

β_i : the predator response rates towards the prey x_i

c_i : the rate of conversion of prey to predator

ϵ_i : inversion barrier strength in going out of the habitat

p_{ij} : the probability of successful transition from the i^{th} habitat to the j^{th} habitat

α_i : specific growth rate of the prey in the absence of predation

μ : per capita death rate of the predator and

$k_1\left(\frac{x_2}{x_1}\right)$, $h_1\left(\frac{x_2}{x_1}\right)$ and $k_2\left(\frac{x_1}{x_2}\right)$, $h_2\left(\frac{x_1}{x_2}\right)$ are general functions of $\left(\frac{x_2}{x_1}\right)$ and $\left(\frac{x_1}{x_2}\right)$ respectively, which must satisfy certain general conditions.

All of the coefficients $\delta_i, \beta_i, c_i, \epsilon_i, p_{ij}, \alpha_i$, and μ are positive.

3 Stability of the equilibrium

Denoting the equilibrium values of the system in eqn.(1), when all species exist, by (X_1, X_2, Y) and letting $\bar{X} = \frac{X_1}{X_2}$ we have, on solving the equilibrium equations, i.e. eqns.(1) with the left-hand-sides equal to zero,

$$X_1 = \frac{\mu X_2}{c_1\beta_1 k_1\left(\frac{1}{\bar{X}}\right) + c_2\beta_2 k_2(\bar{X})},$$

$$X_2 = \frac{\mu}{c_1\beta_1 k_1\left(\frac{1}{\bar{X}}\right) + c_2\beta_2 k_2(\bar{X})},$$

$$\begin{aligned}
Y &= \frac{(\alpha_1 - \epsilon_1)\bar{X} + \epsilon_2 p_{21} - \delta_1 \bar{X} h_1\left(\frac{1}{\bar{X}}\right)}{\beta_1 \bar{X} k_1\left(\frac{1}{\bar{X}}\right)} \\
&= \frac{(\alpha_2 - \epsilon_2) + \epsilon_1 p_{12}\bar{X} - \delta_2 h_2(\bar{X})}{\beta_2 k_2(\bar{X})} \qquad (2)
\end{aligned}$$

The variable $\bar{X}$ satisfies the following equation, obtained by equating the two expressions for Y,

$$\begin{aligned}
&\left[\left\{\alpha_1 - \epsilon_1 - \delta_1 h_1\left(\frac{1}{\bar{X}}\right)\right\}\bar{X} + \epsilon_2 p_{21}\right] \times \\
&\quad \beta_2 k_2(\bar{X}) = \beta_1 \bar{X} k_1\left(\frac{1}{\bar{X}}\right) \times \\
&\quad (\alpha_2 - \epsilon_2 + \epsilon_1 p_{12}\bar{X} - \delta_2 h_2(\bar{X})) \qquad (3)
\end{aligned}$$

Now we observe that $\bar{X}$ is independent of c_1 and c_2 and hence also is Y from eqn.(2).

Further, since Y must be positive, then from eqn.(2) we must satisfy the following inequalities

$$\frac{\epsilon_2 - \alpha_2 + \delta_2 h_2(\bar{X})}{\epsilon_1 p_{12}} < \bar{X} < \frac{\epsilon_2 p_{21}}{\epsilon_1 - \alpha_1 + \delta_1 h_1\left(\frac{1}{\bar{X}}\right)} \qquad (4)$$

where $\beta_1, \beta_2, k_1, k_2$ are positive.

3.1 Assumptions and general stability conditions

We shall now examine the stability of the equilibrium point $E = (X_1, X_2, Y)$. We linearize the eqns.(1) by considering a small perturbation about the equilibrium point i.e. by substituting $x_1 = X_1 + u, x_2 = X_2 + v, y = Y + w$ and neglecting higher order terms in u, v and w. We now make the following assumptions:

Assumption 1: All predatory and harvesting functions, k_1, h_1, k_2, h_2, possess Taylor expansions,

Assumption 2: All predatory and harvesting functions, k_1, h_1, k_2, h_2 are decreasing functions,

Assumption 3: (**The Switching Assumption**)]

We point out here that this assumption enters only when we consider situations which must reflect the type of feeding mechanism by the predators(in the present case a switching mechanism). This assumption is placed in this section just so as to have all assumptions which are employed in the work in one place.

The predatory rate functions satisfy the following conditions:

1. as $X_1 \to \infty$,
   $k_1\left(\frac{X_2}{X_1}\right) \to 1,$
   $k_2\left(\frac{X_1}{X_2}\right) \to 0$ and
2. as $X_2 \to \infty$,
   $k_1\left(\frac{X_2}{X_1}\right) \to 0,$
   $k_2\left(\frac{X_2}{X_1}\right) \to 1$

If we define $w_1,\ w_2,\ A,\ B,\ C$ as follows

$$w_1 = \beta_1 X_1 k_1\left(\frac{X_2}{X_1}\right),\ w_2 = \beta_2 X_2 k_2\left(\frac{X_1}{X_2}\right),$$

$$\begin{aligned}
A &= c_1\beta_1 k_1\left(\frac{X_2}{X_1}\right) + c_1\beta_1 \times \\
&\qquad \left\{-\frac{X_2}{X_1}k_1'\left(\frac{X_2}{X_1}\right)\right\} + c_2\beta_2 k_2'\left(\frac{X_1}{X_2}\right),\\
B &= c_1\beta_1 k_1'\left(\frac{X_2}{X_1}\right) + c_2\beta_2 \times \\
&\qquad \left\{-\frac{X_1}{X_2}k_2'\left(\frac{X_1}{X_2}\right)\right\} + c_2\beta_2 k_2\left(\frac{X_1}{X_2}\right),\\
C &= \epsilon_1 p_{12} - \beta_2 Y k_2'\left(\frac{X_1}{X_2}\right) - \delta_2 h_2'\left(\frac{X_1}{X_2}\right),\\
D &= \epsilon_2 p_{21} - \beta_1 Y k_1'\left(\frac{X_2}{X_1}\right) - \delta_1 h_1'\left(\frac{X_2}{X_1}\right)
\end{aligned} \qquad (5)$$

then the resulting characteristic equation is

$$\begin{vmatrix} -\frac{D}{\bar{X}} - \lambda & D & -w_1 \\ C & -C\bar{X} - \lambda & -w_2 \\ AY & BY & -\lambda \end{vmatrix} = 0. \qquad (6)$$

This equation can be written as

$$\lambda^3 + b_1\lambda^2 + b_2\lambda + b_3 = 0 \qquad (7)$$

where

$$\begin{aligned}
b_1 &= \bar{X}C + \frac{D}{\bar{X}},\\
b_2 &= Y(Aw_1 + Bw_2),\\
b_3 &= \frac{Y}{\bar{X}}(A\bar{X} + B)(Dw_2 + C\bar{X}w_1).
\end{aligned} \qquad (8)$$

In order to have stability of the equilibrium points the eigenvalue solutions, λ, must have negative real parts. Conditions for this to happen are provided by the Routh-Hurwitz criteria. These criteria imply that the eigenvalues will have negative real parts if and only if

$$b_1 > 0,$$

$$b_3 > 0,$$

$$b_1 b_2 - b_3 > 0.$$

From Assumption 2 we see that $C > 0,\ D > 0$ hence $b_1 > 0$. From eqn.(8) we see that b_3 is positive if $\bar{X}A + B > 0$ since $\bar{X}, Y, w_1, w_2, C, D$ are positive. Now replacing A, B by their definitions and collecting terms, we can show that

$$\begin{aligned}\bar{X}A + B = \\ &\frac{1}{X_2}\left[c_1\beta_1 k_1\left(\frac{X_2}{X_1}\right) + c_2\beta_2 k_2\left(\frac{X_1}{X_2}\right)\right] + \\ &c_1\beta_1\bar{X}\left[X_1 k_{1u}\left(\frac{X_2}{X_1}\right) + X_2 k_{1v}\left(\frac{X_2}{X_1}\right)\right] \\ &+c_2\beta_2\left[X_1 k_{2u}\left(\frac{X_1}{X_2}\right) + X_2 k_{2v}\left(\frac{X_1}{X_2}\right)\right].\end{aligned}$$

Using the "Y-equilibrium equation" and performing the differentiations implied on the right hand side, we get

$$\bar{X}A + B = \frac{\mu}{X_2}$$

which tells us that $b_3 > 0$.
Thus, with the Assumptions 1 and 2, we have stability of the general equilibrium point, if and only if,

$$\begin{aligned}b_1 b_2 - b_3 = \\ &\left\{\beta_2 k_2\left(\frac{X_1}{X_2}\right) - \beta_1 k_1\left(\frac{X_2}{X_1}\right)\right\} \times \\ &(BCX_1 - ADX_2) > 0 \qquad (9)\end{aligned}$$

where A, B, C, D are defined in (5).

4 Hopf Bifurcation

Following the analysis of [1] and [2] we obtain the following theorem:

Theorem 1 *Let the equilibrium $E = (X_1, X_2, Y)$ exist and let $A > 0, B > 0$, then if $\bar{c}_1$ is a positive root of the equation $b_1 b_2 = b_3$, we have a Hopf bifurcation, [3], as c_1 passes through $\bar{c}_1$ provided $\bar{X} \neq \dfrac{w_1}{w_2}$.*

If we perform a similar analysis with c_2 (the rate of capture of the prey in the second habitat to the predator) as the variable parameter, we shall get a similar result.

5 Applications

As mentioned in the Introduction, we examined six hypothetical systems the results for which support the present theory. However, we display graphically the results for only System 5. This is the system given by eqn.(1) with

$$k_1\left(\frac{x_2}{x_1}\right) = \frac{1}{1 + \left(\frac{x_2}{x_1}\right)^n},$$

$$h_1\left(\frac{x_2}{x_1}\right) = \frac{1}{\exp\left(\frac{x_2}{x_1}\right)^n},$$

$$k_2\left(\frac{x_1}{x_2}\right) = k_1\left(\frac{x_1}{x_2}\right), h_2\left(\frac{x_1}{x_2}\right) = h_1\left(\frac{x_1}{x_2}\right)$$

In light of Theorem 1 we can write the following Theorem:

Theorem 2 *Let the equilibrium $E = (X_1, X_2, Y)$ exist and $A > 0,\ B > 0$, and $\bar{c}_1$ be a positive root of the equation $b_1 b_2 = b_3$, then a Hopf bifurcation occurs when c_1 passes through $\bar{c}_1$ provided $\bar{X}^n \neq \dfrac{\beta_2}{\beta_1}$.*

6 Results

Both Predatory rates Multiplicative and both Harvesting rates Exponential

In our work we examined several parameter sets but we display only one since we do not apply our results to any real system. They all supported the theory. The parameter set used is as follows:
$\mu = 0.01,\ \alpha_1 = 0.015,\ \alpha_2 = 0.025,\ \beta_1 = 0.01,\ \beta_2 = 0.02,\ p_{12} = 0.3,\ p_{21} = 0.2,\ \epsilon_1 = 0.02,\ \epsilon_2 = 0.03,\ \delta_1 = 0.0001,\ \delta_2 = 0.0005,$ with $c_1 = 0.1,\ c_2 = 0.1,\ n = 1$(stable) and $c_1 = 0.1,\ c_2 = 0.4,\ n = 2$(unstable). The behavior of the populations are graphically displayed in Figs.(1) and (2). These behaviors support the theory as do all other sets examined.

It is interesting to observe that in the cases we examined, when there is an instability and the populations of the preys have a ”minimum” i.e. decrease, the population of the predator shows a ”maximum” i.e. it appears that some of the predators give ”birth”.

As mentioned in the Introduction we do not apply our work to any real system, we only consider hypothetical cases, however, we feel that our work is of interest in the sense that we have used arbitrary predatory and harvesting rate functions, hence it will compliment other existing works.

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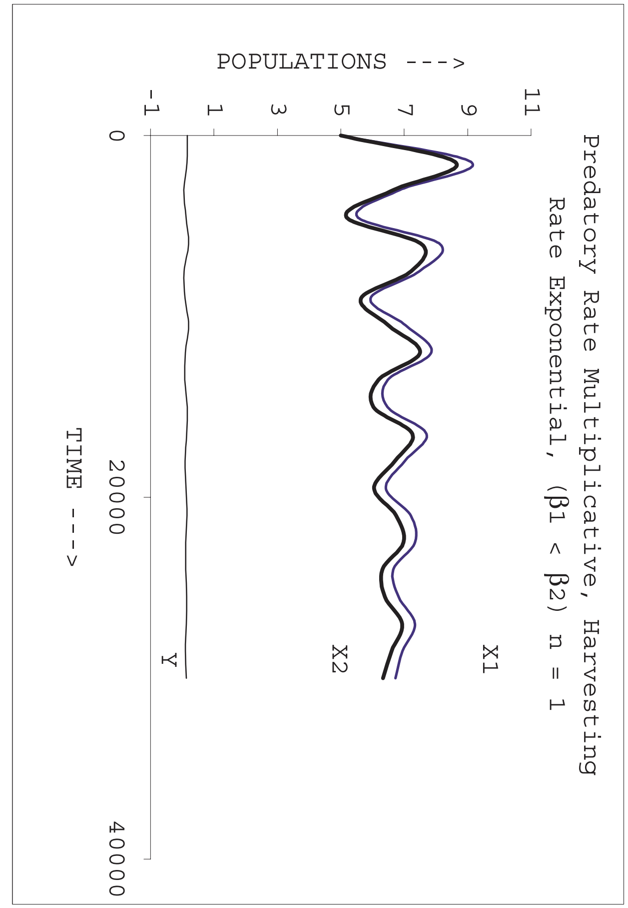

Fig. 1: Population with predatory rate multiplicative,harvesting rate exponential

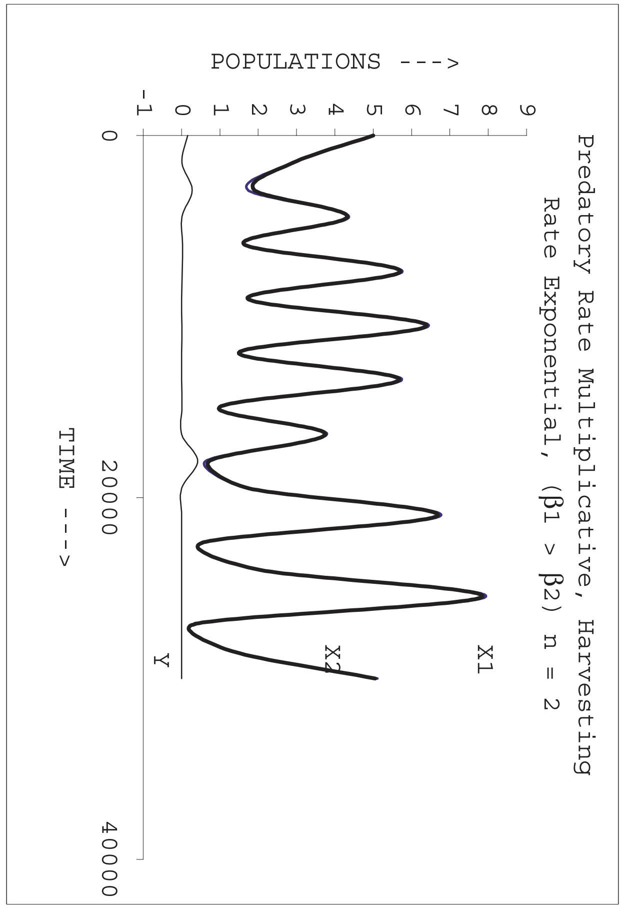

Fig. 2: Population with predatory rate multiplicative,harvesting rate exponential