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Solving Systems of Nonlinear Ordinary Differential Equations using Fixed Point Theorems

NABIL MOUSSA
Chairman of Department of Mathematical Sciences
Ahlia University in Bahrain
P. O. Box 10878, Manama
BAHRAIN

http://www.aucegypt.edu/academic/maths/nabil.htm

Abstract: In this paper we construct periodical solutions for systems of nonlinear ordinary differential equations and study their stability using the fixed point theorems of Banach and Schauder showing the relation to the classical theorems of Picard-Lindeloef and Peano respectively.

Key-Words: Systems of Nonlinear Ordinary Differential Equations – Fixed Point Theorem of Banach – Fixed Point Theorem of Schauder – Periodical Solutions– Stability

1 Introduction

We construct periodical solutions for systems of nonlinear ordinary differential equations using the powerful fixed point theorems of Banach and Schauder. Also we study the stability of those solutions.

The methods of nonlinear functional analysis, especially the fixed point theorems, are in several cases generalization of classical analytical methods. Many of the results are qualitative and not quantitative but still very substantial to show global performance of the models to be studied (see [5]).

In section 2 we present the two fundamental fixed point principles of Banach and Schauder.

The proof of the fixed point theorem of Banach and its relation to Picard-Lindeloef theorem is given in section 3.

In section 4 we state the fixed point theorem of Schauder and show its relation to Brouwer fixed point theorem and Peano theorem.

Then we construct the periodical solutions of our problem and study their stability in section 5.

Finally the conclusion is then given in section 6.

2 Two Fundamental Fixed Point Principles

Several statements of nonlinear functional analysis emerge from one of the following two central fixed point theorems:

1- Fixed point theorem of Banach, which is a generalization of the method of successive approximation;

2- Fixed point theorem of Schauder, which is a generalization of the choice theorem on compactness.

We consider the initial value problem

$$x'(t) = f(t, x(t)) \quad , \quad x(t_0) = y_0 \quad (1)$$

If f is continuous, then (1) is equivalent to the integral equation

$$x(t) = y_0 + \int_{t_0}^{t} f(s, x(s))ds \qquad (2)$$

The theorem of Picard-Lindeloef proves the existence of a unique solution of (1) assuming f is Lipschitz continuous. If f is only continuous, the theorem of Peano proves the existence of a solution of (1) with no statement about uniqueness (see [6]).

To prove these classical theorems using functional analytical methods, we rewrite (2) as nonlinear operator equation

$$x = T(x) \quad , \quad x \in M \subset X \qquad (3),$$

in a suitable function space *X*.

Then we search for a solution of (3), that means we search for a fixed point of *T* on *M*.

3 Fixed Point Theorem of Banach

In this section we examine the possibility of solving the equation

$$x = T(x) \qquad , x \in M \qquad (4)$$

through the successive approximation

$$x_{n+1} = T(x_n) \qquad , x_0 \in M \qquad (5)$$

Banach Theorem:

Given

$$T : M \subseteq X \to M$$

with (X, d) a complete metric space, where *M* is closed and not empty.

Also the following inequality is assumed

$$d(T(x), T(\overline{x})) \leq K\, d(x, \overline{x})$$

$$for\ all\ x, \overline{x} \in M\ and\ 0 \leq K < 1 \quad (6)$$

Then we need to prove that (4) has exactly one solution and that the successive approximations $\{x_n\}$ converge to the solution x with the following error estimation

$$d(x_n, x) \leq K^n (1-K)^{-1} d(x_0, x_1) \quad (7)$$

Proof:

$\{x_n\}$ is a Cauchy sequence, since

$$d(x_n, x_{n+1}) = d(T(x_{n-1}, T(x_n)) \leq K\, d(x_{n-1}, x_n) \leq$$

$$\leq K^2\, d(x_{n-2}, x_{n-1}) \leq ... \leq K^n\, d(x_0, x_1)$$

and

$$d(x_n, x_{n+m}) \leq \sum_{r=0}^{m-1} d(x_{n+r}, x_{n+r+1}) \leq$$

$$\leq \sum_{r=0}^{m-1} K^{n+r}\, d(x_0, x_1) \leq K^n\, (1-K)^{-1}\, d(x_0, x_1)$$

Consequently $x_n \to x\ as\ n \to \infty\ and\ m \to \infty$ and we get (7).

As T is continuous (from (6)), applying (5), we obtain that

$$x = T(x)\ as\ n \to \infty\ .$$

To prove the uniqueness we apply

$x = T(x)\ and\ \overline{x} = T(\overline{x})$ to get

$$d(x, \overline{x}) = d(T(x), T(\overline{x})) \leq K\, d(x, \overline{x}) \quad ; 0 \leq K < 1$$

That means $x = \overline{x}$.

We can state thar Banach fixed point theorem is a generalization of Picard-Lindeloef theorem (see [5]).

4 Fixed Point Theorem of Schauder

Before stating Schauder fixed point theorem, we state first Brouwer fixed point theorem:

Brouwer Theorem:

Given

$$f : M \subset R^n \to M \; continuous \, ; n \in N$$

with

M compact, convex and not empty.

Then f has a fixed point.

Now we use an approximation process to generalize the fixed point theorem of Brouwer in an infinite dimensional Banach space to get the following fixed point theorem of Schauder.

Schauder Theorem:

Given

$$T : M \subset X \to M \; compact \, ;$$

$$X \; Banach\, space$$

with

M closed, bounded, convex and not empty,

Then T has a fixed point.

Here we can also state that Schauder fixed point theorem is a generalization of Peano theorem (see [5]).

5 Construction of Solutions

We consider the following system of ordinary differential equations (see [1]).

$$x'(t) = A\, x(t) + f\,(t, x(t))$$

with

$$x(t) = (\xi_1(t), \xi_2(t), ..., \xi_n(t)) \in R^n \, ;$$
$$f = (f_1, f_2, ..., f_n) \, ;$$
$$f\,(t+\omega, y) = f\,(t, y) \; for\, all \; (t, y) \in R^{n+1} \, ;$$
$$A = (a_{ij}) \; independent\, of\; time \qquad (8)$$

Now we use the fixed point theorems to construct $\omega -$periodical solutions of (8) and study their stability.

Assuming

$$f : R\, x\, R^n \to R \; continuous$$

with

$$\frac{\| f\,(t, y)\|}{\|y\|} \to 0 \quad for \;\; \|y\| \to \infty$$

uniformly with respect to all $t \ge t_0$ (9),

and

$$\| f\,(t, y) - f\,(t, \bar{y})\| \le L(z, r) \|y - \bar{y}\|$$

for all

$$t \ge t_0 \,, z \in R \,, r > 0 \, ;$$

$$y, \bar{y} : \|y - z\|, \|\bar{y} - z\| \le r \qquad (10)\, ,$$

If there are numbers a,b > 0 such that for all $t \ge 0$, we have

$$\|\exp tA\| \le a \, \exp\,(-bt) \qquad (11)\, ,$$

we can prove that (8) has $\omega -$periodical solutions.

If for $c \in (0,1)$ *and all* $y \in R^n, t \ge t_0$, we have

$$\frac{a^2 \|y\| \| f\,(t, y)\|}{2b} < c \qquad (12)\, ,$$

then we get

$$\|x(t; y_0, t_0)\| \le K_1 \|y_0\| \exp(-K_2\,(t - t_0)$$

with

$$K_1, K_2 > 0 \qquad (13)\, ,$$

which implies the stability of the solutions

of (8) (see [2]).

Example:

Given is the system

$$x_1' = x_1 + x_2 \sin t$$
$$x_2' = x_2 - x_1 \cos t \qquad (14),$$

with

$$f = (x_2 \sin t, - x_1 \cos t);$$
$$f : R \, x \, R^2 \to R \; continuous;$$
$$f(t + 2\pi, y) = f(t, y),$$

where

$$A = I;$$
$$t_0 = 0$$

For a suitable choice of $a, b, c, r, z, K_1 \; and \; K_2$ we can show that (14) has $2\pi -$periodical solutions and that these solutions are stable.

6 Conclusion

The powerful fixed point theorems of Banach and Schauder enable us to solve numerous nonlinear problems of mathematical physics like waves of liquids, quantum field theory, nonlinear oscillations and deflection of plates (see [3] and [4]).

The case of constructing periodical solutions of systems of nonlinear ordinary differential equation as well as studying the stability of the solutions has been successfully solved in a qualitative approach.

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