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Sampling Theorem for Multidimensional, Multiband Signals

ALDO DE SABATA
Faculty of Electronics and Telecommunications
"Politehnica" University of Timisoara
Bd. V. Parvan nr. 2, 300223 Timisoara
ROMANIA

Abstract: - We approach the problem of sub-Nyquist sampling of multidimensional, multiband signals, with known spectral support that does not necessarily tile $\Re^N$ under translation. Undersampling is achieved by adapting the Papoulis' multichannel sampling scheme. Minimum (Landau) sampling density is obtained in some special conditions.

Key-Words: - multichannel sampling scheme, multidimensional signals, sampling theorem

1 Introduction

Sampling strategies and procedures for one-dimensional, multiband signals have been treated by various authors [1], [2], [3], etc with the purpose of taking advantage of the spectral support structure for reducing the sampling rate beneath the Nyquist rate, as close as possible to the Landau rate. For this, Papoulis' multichannel sampling scheme (MSS) [4] provides an appropriate framework. Typical applications are in communications, radar, and measuring techniques.

Sampling theorems for multidimensional signals are also extensively covered in literature, for the case of signals having the spectral support included in an *N*-dimensional parallelepiped. The case of a rectangular lattice [5] is extended to general lattices [6], by using the MSS, and recently further extended to periodic/nonperiodic hybrids [7]. Such theorems are used in Fourier imaging applications (sensor array imaging, synthetic aperture radar, magnetic resonance imaging), space-time sampling etc.

In the present paper we consider complex valued, finite energy, continuous, *N*-dimensional signals, whose spectral support is a union of finite cells (also parallelepipeds) in the shape of (6) below. We thus generalize to multidimensional MSS the one dimensional multicoset sampling treated in [2].

2 Sampling theorem

Consider an *N*-dimensional parallelepiped X that contains the spectral support of the signal. Without loss of generality, we can assume that a vertex coincide with the origin of $\Re^N$. X is tiled by the translations of a cell X_0, of (column) vectors $\mathbf{v}_i$, $i=1..N$. The disjoint cells generated by translation are

$$C_{\mathbf{l}} = C_{\mathbf{0}} \oplus \mathbf{Vl}, \quad \mathbf{V} = [\mathbf{v}_1\ \mathbf{v}_2 .. \mathbf{v}_N], \qquad (1)$$

where $\oplus$ denotes translation, and $\mathbf{l}$ is a vector with integer components from a set

$$L = \{\mathbf{l} = [l_1\ l_2 ... l_N]^T \mid l_i = 0..(L_i - 1), i = 1..N\} \qquad (2)$$

for some integers L_i, i=1..N. We have

$$C = \bigcup_{\mathbf{l} \in L} C_{\mathbf{l}}. \qquad (3)$$

A two dimensional version of this arrangement is represented in Fig. 1.

The cell X_0 is divided in parallelepiped subcells $\Gamma_{\mathbf{m}}$, not necessarily of equal dimensions (Fig. 1). The vector index $\mathbf{m}$ is taken from a set

$$M = \{[m_1\ m_2 ... m_N]^T \mid m_i = 0..M_i - 1, i = 1..N\}, \qquad (4)$$

with M_i: integers.

Parts of the signal's spectral support are contained by translations of the subcells $\Gamma_{\mathbf{m}}$, which are contained in X. We denote by $\Sigma_{\mathbf{m}}$ the part of the spectral support that is contained by translations of $\Gamma_{\mathbf{m}}$:

$$\Sigma_{\mathbf{m}} = \bigcup_{\mathbf{p}\in\Pi_{\mathbf{m}}} \left(\Gamma_{\mathbf{m}} \oplus \mathbf{Vp}\right), \qquad (5)$$

where $\Pi_{\mathbf{m}}$ is a set of integer vectors, and let $P_{\mathbf{m}} = \mathrm{card}(\Pi_{\mathbf{m}})$.
The complete spectral support is contained by

$$\Sigma = \bigcup_{\mathbf{m}\in M} \Sigma_{\mathbf{m}}, \qquad (6)$$

but some of the sets that appear in the right hand side can be empty.
There exist translations of the subcells that are disjoint of Σ. Their union is

$$\Sigma_{\mathbf{m}}^{c} = \bigcup_{\mathbf{q}\in Q_{\mathbf{m}}} \left(\Gamma_{\mathbf{m}} + \mathbf{Vq}\right), \qquad (7)$$

where $\mathrm{Q}_{\mathbf{m}} = \Lambda \setminus \Pi_{\mathbf{m}}$. Let $\mathrm{card}(\mathrm{Q}_{\mathbf{m}}) = Q_{\mathbf{m}}$.
In the one-dimensional case, the parallelepiped, cells, and subcells are intervals. The way of selecting these intervals for a given spectral support is discussed in [2]. For our case, we will skip this discussion, for brevity, and we will simply assume that the spectral support is given by (6).
Let $P = \max_{\mathbf{m}\in M} \Pi_{\mathbf{m}}$. We consider the MSS represented in Fig. 2. The filters $H_i(\mathbf{f}), i = 1..P$ are known, while the filters $Y_i(\mathbf{f})$ must be determined such that $X_0(\mathbf{f}) = X(\mathbf{f})$, if it is possible. The input-output relation is

$$X_0(\mathbf{f}) = V_0 \sum_{\mathbf{n}} X(\mathbf{f} - \mathbf{Vn}) \sum_{k=1}^{P} \hat{H}_k(\mathbf{f} - \mathbf{Vn}) Y_k(\mathbf{f}), \qquad (8)$$

where

$$\hat{H}_k(\mathbf{f}) = \begin{cases} H(\mathbf{f}), \mathbf{f} \in \mathrm{supp}(X(\mathbf{f})) \\ 0, \text{otherwise} \end{cases}, \qquad (9)$$

and V_0 is the volume of X_0. The summation index $\mathbf{n}$ runs over Λ^N. The condition $X_0(\mathbf{f}) = X(\mathbf{f})$ is true if

$$\sum_{k=1}^{P} \hat{H}_k(\mathbf{f} - \mathbf{Vn}) Y_k(\mathbf{f}) = \frac{1}{V_0}\delta_{\mathbf{n}}^{0}, \quad \mathbf{f} \in \mathrm{C}. \qquad (10)$$

We can rewrite (10) by using only translations of the fundamental subcell:

$$\sum_{k=1}^{P} \hat{H}_k(\mathbf{f} + \mathbf{Vr}) Y_k(\mathbf{f} + \mathbf{Vs}) = \frac{1}{V_0}\delta_{\mathbf{r}}^{\mathbf{s}}, \quad \mathbf{f} \in \mathrm{C}_0, \quad \mathbf{r},\mathbf{s} \in \Lambda. \qquad (11)$$

We split now the system (11) in a maximum of $\mathrm{card}(M)$ systems, one for each subcell, by taking into account (9), as follows

$$\begin{aligned} &\sum_{k=1}^{P} \hat{H}_k(\mathbf{f} + \mathbf{Vp}) Y_k(\mathbf{f} + \mathbf{Vr}) = \frac{1}{V_0}\delta_{\mathbf{p}}^{\mathbf{r}} \\ &\sum_{k=1}^{P} \hat{H}_k(\mathbf{f} + \mathbf{Vp}) Y_k(\mathbf{f} + \mathbf{Vq}) = 0 \\ &\mathbf{f} \in \Gamma_{\mathbf{m}}, \quad \mathbf{p} \in \Pi_{\mathbf{m}}, \quad \mathbf{q} \in \mathrm{Q}_{\mathbf{m}}. \end{aligned} \qquad (12)$$

We define the following matrices:

$$\begin{aligned} &\mathbf{H}_{\mathbf{m}} = \| a_{\mathbf{p}k} \| = \| \hat{H}_k(\mathbf{f} + \mathbf{Vp}) \|, \\ &\mathbf{Y}_{\mathbf{m}} = \| b_{k\mathbf{p}} \| = \| Y_k(\mathbf{f} + \mathbf{Vp}) \|, \\ &\mathbf{Y}_{\mathbf{m}}^{c} = \| c_{k\mathbf{q}} \| = \| Y_k(\mathbf{f} + \mathbf{Vq}) \|, \\ &\mathbf{p} \in \Pi_{\mathbf{m}}, \mathbf{q} \in \mathrm{Q}_{\mathbf{m}}, k = 1..P, \mathbf{f} \in \Gamma_{\mathbf{m}}. \end{aligned} \qquad (13)$$

For every vector index, the lexicographic order is taken. The dimensions of these matrices are $P_{\mathbf{m}} \times P$, $P \times P_{\mathbf{m}}$, and $P \times Q_{\mathbf{m}}$ respectively. The first line in (12) becomes:

$$\mathbf{H}_{\mathbf{m}} \mathbf{Y}_{\mathbf{m}} = \frac{1}{V_0} \mathbf{I}_{P_{\mathbf{m}}}. \qquad (14)$$

If $\mathbf{H}_{\mathbf{m}}$ has full row rank, then

$$\mathbf{Y}_{\mathbf{m}} = V_0^{-1} \mathbf{H}_{\mathbf{m}}^{-1}, \qquad (15)$$

where $\mathbf{H}_{\mathbf{m}}^{-1}$ is a left inverse for $\mathbf{H}_{\mathbf{m}}$. Then $\mathbf{Y}_{\mathbf{m}}^{c}$ can be any matrix such that

$$\mathbf{H}_{\mathbf{m}}\mathbf{Y}_{\mathbf{m}}^{c} = \mathbf{0}_{P_{\mathbf{m}},Q_{\mathbf{m}}}. \quad (16)$$

By taking the inverse Fourier transform of (8) with $X_0(\mathbf{f}) = X(\mathbf{f})$, the following sampling expansion can be obtained after some straightforward calculations:

$$x(\mathbf{t}) = \sum_{k=1}^{P}\sum_{\mathbf{n}} g_k((\mathbf{V}^{-1})^T\mathbf{n})y_k(\mathbf{t}-(\mathbf{V}^{-1})^T\mathbf{n}), \quad (17)$$

where $x(\mathbf{t})$, $g_k(\mathbf{t})$, and $y_k(\mathbf{t})$ are the inverse Fourier transforms of $X(\mathbf{f})$, $G_k(\mathbf{f})$, and $Y_k(\mathbf{f})$ defined in Fig. 2.

We obtained the following

Theorem. A finite energy, complex valued, continuous signal, defined on $\Re^N$, which has a spectral support in the shape of (6), sampled with the MSS of Fig. 2, can be interpolated from its samples taken at points $(\mathbf{V}^{-1})^T\mathbf{n}$ according to (17), if the matrix $\mathbf{H}_{\mathbf{m}}$ defined in (18) has full row rank for each $\mathbf{m}$ for which the set $\Pi_{\mathbf{m}}$ is not empty.

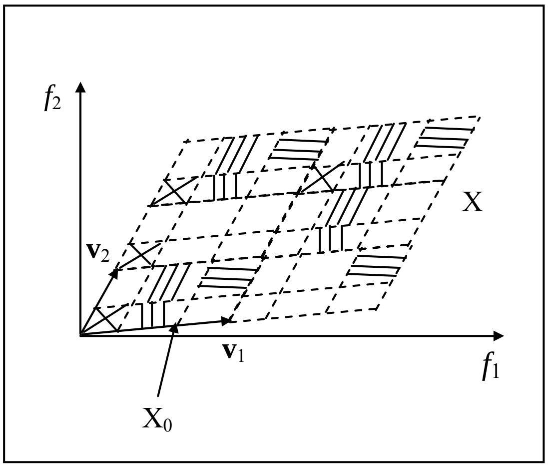

Fig. 1. Spectral support in two dimensions. The marked subcells belong to Σ. Subcells marked with the same pattern correspond to a given $\Pi_{\mathbf{m}}$.

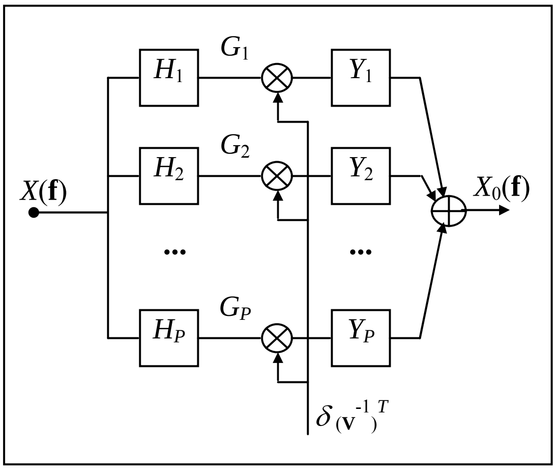

Fig. 2. Multichannel sampling scheme

3 Discussion

The Nyquist rate, i.e. the smallest uniform sampling rate that guarantees no aliasing, is, in our case, the volume of X, $V(C)$. This can be much larger than the Landau (minimum) sampling rate, equal to the Lebesgue measure of the spectral support. We used a sampling rate equal to $PV(C_0)$. If all subcells where completely occupied, a minimum sampling rate would have been $\sum_{\mathbf{m}\in M} P_{\mathbf{m}}V(C_{\mathbf{m}})$. The ratio of the minimum sampling rate to the actual one gives the sampling efficiency. For $P_{\mathbf{m}} = P$, a unit sampling efficiency is obtained, and then the solution (15) is unique; this case is treated in [6]. In general $P_{\mathbf{m}} < P$, and several solutions are possible, all providing equivalent sampling expansions (17). The efficiency is subunitary, but some evidence in the one-dimensional, multicoset sampling suggests that a particular solution might be chosen to minimize the sensitivity to errors [2]. In the present approach, every direction in X has been undersampled, and each side of the cell X_0 was parallel to a side of the parallelepiped X. This has been achieved by choosing the sampling pattern in the time domain as $\left(V^{-1}\right)^T\mathbf{n}$. A more general, and sometimes more useful pattern can be chosen [6,7], an issue we will address elsewhere.

4 Conclusion

We presented a sampling theorem for multiband, multidimensional signals in terms of the multichannel sampling scheme introduced by Papoulis, and consisting of two continuous time filter banks, one for analysis and one for reconstruction. The frequency responses of the analysis filters must satisfy a certain independence condition in order for the reconstruction to be possible. We obtained a generalized sampling expansion in terms of the sampled outputs of the analysis filters.

We generalized in this way the one-dimensional multicoset sampling. The sampling density is below the Nyquist rate and generally above the Landau rate. Sampling at a rate higher than the minimum allowable one provides flexibility in choosing the reconstruction filters, so that a minimization of sensitivity to errors might be possible, similar to the cited one-dimensional case.

We undersampled each direction in the signal's spectral support. Other sampling patterns can be considered, a problem left for future work.

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