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The application of wavelet transform for estimating the shape parameter of a Weibull pdf

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Abstract: - In this paper a method for the parameter estimation of a Weibull distribution is explained. It is based on the estimates of the cumulative probability function performed by a wavelet method. The estimates, obtained by the application of the proposed method, are compared with the results based on other literature estimators. In particular, it is shown that the estimates are stable also when samples show outliers.
Such a method could be utilised in mechanical applications (e.g. transmissions, rolling organs, kinematics couples effort subjected) or time series analysis.

Key-Words: - Wavelet analysis, numerical analysis, Weibull distribution, parameter estimation, mechanical lifetime.

1 Introduction

1.1 The Weibull distribution

The Weibull distribution is one of the most widely used lifetime distributions in reliability theory (e.g. see Fothergill (1990)) and in time series analysis (see Zuccolotto (2002)). It is a versatile distribution that can take on the characteristics of the other types of distributions, based on the value of a shape parameter. The three-parameter Weibull probability density function (p.d.f.) is given by

$$f(t) = \frac{\beta}{\eta}\left(\frac{t-\alpha}{\eta}\right)^{\beta-1} e^{-\left(\frac{t-\alpha}{\beta}\right)^{\beta}}, \tag{1}$$

where η is a scale parameter, β is the shape parameter or slope and α is a location parameter.

1.2 Estimation of the Weibull parameter

The estimates of the Weibull distribution parameters can be found graphically via probability plotting paper, or analytically, either using least squares or maximum likelihood.

Probability plotting for the Weibull distribution.
To better illustrate this procedure consider the example from Kececioglu (1994).

Weibull rank regression on X (RRX) or on Y (RRY).
It performs a rank regression on *X* or *Y* by fitting a straight line to a set of data points, in order to minimize both the horizontal or vertical deviations from the points.

Maximum likelihood for the Weibull distribution. \rm Maximum likelihood estimation works by developing a likelihood function on the available data and finding the values of the parameter estimates that maximize the likelihood function. Another method of finding the parameter estimates involves taking the partial derivatives of the likelihood function with respect to the parameters, setting the resulting equations equal to zero, and solving simultaneously to determine the values of the parameter estimates.

Three-parameter Weibull regression.
When the multivariate regression versus t_j points plotted on the Weibull probability paper do not fall on a satisfactory straight line, and the points on a curve, then a location parameter, α, might exist which may straighten out these points.
The goal in this case is to fit a curve, instead of a line, through the data points, using non-linear regression. The Gauss-Newton method can be used to solve for parameters, α, β and η, by performing a Taylor series expansion on $F(t_j, \alpha, \beta, \eta,)$. Then the non-linear model is approximated with linear terms and ordinary least squares are employed to estimate the parameters. This procedure is iterated until a satisfactory solution is reached.
The value of α is calculated by utilizing an optimized Nelder-Mead algorithm and adjusting the points by this value of α such that they fall on a straight line, and then plots both the adjusted and the original unadjusted points.
The aim of this study is to show a new method of calculating β and η parameter. It is based on the calculation of the cumulative probability distribution function (c.d.f.), performed by the wavelet analysis.

2 Mathematical background

Let $X_1, X_2, \ldots, X_N$ be be observed values of a random variable, whose density is given by the two dimensional Weibull p.d.f. ($\alpha = 0$)

$$f(t) = \frac{\beta}{\eta}\left(\frac{t}{\eta}\right)^{\beta-1} e^{-\left(\frac{t}{\beta}\right)^{\beta}}, \qquad (2)$$

whose c.d.f. is

$$F(t;\beta,\beta) = 1 - \exp\left[-\left(\frac{t}{\eta}\right)^{\beta}\right], \qquad (3)$$

Consider all the combinations without repetition of pairs (X_i, X_j) ($i \in \{1, 2,..., N-1\}$ and $j \in \{i+j,...,N\}$). In this case, a preliminary estimate β_{ij} of the parameter β is given by, according to Thiel (see Birkes and Dodge (1993)):

$$\beta_{ij} = \frac{\ln\left[\ln\left(\frac{1}{1-\hat{F}(X_j)}\right)\right] - \ln\left[\ln\left(\frac{1}{1-F(X_i)}\right)\right]}{\ln X_j - \ln X_i} \qquad (4)$$

where $\hat{F}(X_i)$ in an estimator of $F(X_i)$. Formally, β_{ij} is obtained by writing twice the equation (3) in the linear form, both for $x = X_i$ and $x = X_j$,

$$\ln x = \ln \eta + \frac{1}{\beta}\ln\left[\ln\left(\frac{1}{1-F}\right)\right] \qquad (3')$$

and by solving for β. Following Fothergill (1990), an estimator of F_i is, for any $i \in \{1, 2,...,N\}$,

$$F = \frac{i-0.3}{N+0.4}.$$

Now, imagine to set the $M := N(N-1)/2$ values of β_{ij} in ascending order. Therefore, the point estimator $\hat{\beta}$ of β is defined as the median value of this so obtained ordered set. By Thiel's method it is also possible to calculate an estimate $\hat{\eta}$ of the parameter η. In fact, one can calculate N drafted estimate η_i; ($i \in \{1, 2,...,N\}$), of the parameter η, by solving the equation (3') with respect to η:

$$\eta_i = \frac{X_i}{\left[-\ln\left(1-\hat{F}(X_i)\right)\right]^{\frac{1}{\beta}}}.$$

By sorting these N values in ascending order, one can take their median value as an estimate $\hat{\eta}$ of η.

3 Proposed method

3.1 Overview on Wavelet methodology on function approximation

Let $X_1, X_2, \ldots, X_N$ be independent identically distributed random variables whose density is given by the two dimensional Weibull p.d.f. (2), where β and η are unknown. Recall that a wavelet estimator of f may be given by (e. g. Härdle et al. (1988) and Ogden (1997))

$$\hat{f} = \sum_k \hat{a}_{j_0k}\varphi_{j_0k}(x) + \sum_{j=j_0}^{j_1}\sum_k \hat{b}_{jk}\psi_{jk}(x), \qquad (5)$$

where

$$\psi_{jk}(x) = 2^{j/2}\psi(2^j x - k),$$

$$\varphi_{jk}(x) = 2^{j/2}\varphi\left(2^j x - k\right), \quad k \in \mathbf{Z},$$

$$\hat{a}_{j_0k} = \frac{1}{n}\sum_{i=1}^{n}\varphi_{j_0k}(X_i) \quad \text{and} \quad \hat{b}_{jk} = \frac{1}{n}\sum_{i=1}^{n}\psi_{jk}(X_i).$$

Moreover, $\psi(x)$ is a function (mother wavelet) whose first h; ($h \in \mathbf{N}$) moments are zero. φ is a function (father wavelet) orthonormal to $\psi(x)$, according to the L^2 norm. It is easy to show that if $\psi(x)$ is a mother wavelet, then also $\psi_{ik}(x)$ is a mother wavelet. Moreover, in this case, the systems of functions $\{\{\varphi_{j_0k}\}, \{\psi_{jk}\}, k \in \mathbf{Z}, j = 0, 1, 2,\ldots\}$ is an orthonormal system in $L^2(\mathbf{R})$.

The estimator (5) is based on Parseval Theorem. According to this result, any $h \in L^2(\mathbf{R})$ can be represented as a convergent series

$$h(x) = \sum_k a_{j_0k}\varphi_{j_0k}(x) + \sum_{j=j_0}^{j_1}\sum_k b_{jk}\psi_{jk}(x), \qquad (6)$$

where

$$a_{j_0k} = \int_{-\infty}^{+\infty} h(x)\cdot\frac{1}{\sqrt{2^{j_0}}}\psi\left(\frac{x-k}{2^{j_0}}\right)dx$$

and

$$b_{jk} = \int_{-\infty}^{+\infty} h(x)\cdot\frac{1}{\sqrt{2^{j}}}\varphi\left(\frac{x-k}{2^{j}}\right)dx .$$

In this work, it has been performed the choice

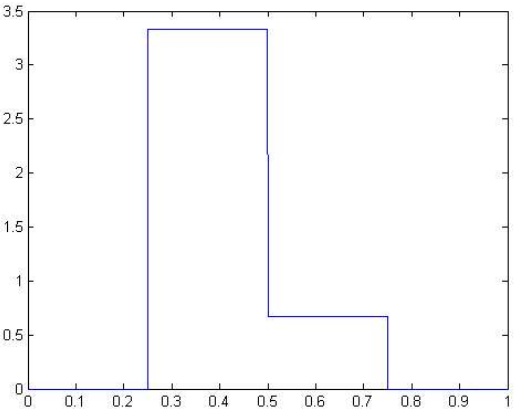
$$\varphi(x) = \begin{cases} 1, x \in (0,1] \\ 0, \ x \notin (0,1] \end{cases} \quad \text{and} \quad \psi(x) = \begin{cases} -1, x \in \left[0, \frac{1}{2}\right] \\ 1, \ x \in \left(\frac{1}{2}, 1\right] \\ 0, x \notin [0,1] \end{cases}.$$

They are called Haar system. These curves have been chosen because the order N, of the observations, can be relatively small.

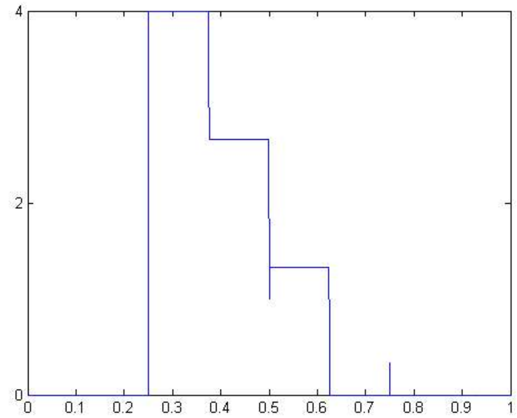
Recall that, many authors (e. g. Härdle et al. (1988)) propose to use $j_0 = 0$ and $j_1 = \log_2 N$. Note that $-1 \le k \le 2^{j_1}\max_i X_i + 1$.

3.2 Proposed methodology on Weibull p.d.f. approximation

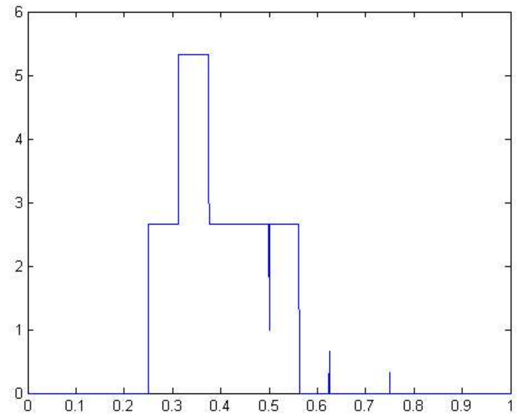
It is simulated N observed values, $x = (x_1, x_2,..., x_N)$. Therefore, it is calculated the normalised vector



$j_1 = 1$



$j_1 = 2$



$j_1 = 3$

Fig.1. *Estimate of F by wavelet method. The parameters values (not normalized) used here are x* = (23 28 33 37 41 48) .

Sample	j_1	$\hat{\beta}$	$\hat{\eta}$
16 34 53 75 93 120	1	1.4059	83.6991
	2	1.4059	83.6991
	3	1.6220	76.3844
	4	1.2998	73.4655
23 28 33 37 41 48	1	4.2048	38.7560
	2	3.8240	37.8986
	3	5.0524	38.2889
	4	5.0496	38.1055

Table 1: *Comparison of estimates of* β *obtained by the proposed method, for various levels of* j_1. *It has been set* $j_0 = 0$.

$$\left(X_1, X_2, \ldots, X_N\right) = \frac{(x_1, x_2 .. x_N)}{\mathrm{N}(x)},$$

where $\mathrm{N}(x) = \left(\sum_{i=1}^{N} x_1^{\,2}\right)^{\frac{1}{2}}$.

Finally, for any $i \in \{1, 2, \ldots, N\}$, it is estimated $\hat{F}(x)$ by numerical quadrature of $\hat{f}$, by putting

$$\int_0^{X_i} \hat{f}(x)dx =: \hat{F}(X_i).$$

The calculation of β_{ij} and $\hat{\beta}$ is obtained by applying Thiel's method. In Fig.1 and in Tables 1-2 are shown the results deriving from the proposed procedure.
In particular, Fig.1 illustrates the estimate of a p.d.f. function, based on Haar Wavelets, given $N = 6$ sample, for various values of the parameter j_1. In Table 1, estimates of the parameters, are compared for various values of the parameter j_1. In Table 2, estimates of β are compared with other methods: the Least Square Regression (LS) and The Maximum Likelihood (ML). Observe that, by following Härdle et al. (1998), it has been chosen $j_1 = 0$. Recall that Härdle et al. (1998) propose to set $j_1 = \log_2 N$ and Ogden (1997) suggests to set $j_1 \approx \log_2 N - \log_2(N)$. Therefore, in Table 2 the comparisons are made by choosing

$$j_1 = Int\left[\log_2\left(\frac{1}{median(x_i - x_{i-1})}\right) + - \ln\left(\log_2\left(\frac{1}{median(x_i - x_{i-1})}\right)\right)\right], \tag{7}$$

Sample	method	$\hat{\beta}$	$\hat{\eta}$
16, 34, 53, 75, 93, 120	Proposed	1.4059	83.6991
	ML	1.9327	73.5272
	LS	1.4397	76.1096
16, 34, 53, 75, 93, 190*	Proposed	1.3233	88.7073
	ML	1.4319	85.0637
	LS	1.2492	86.6415
23 28 33 37 41 48	Proposed	3.8240	37.8986
	ML	4.7772	38.2666
	LS	4.0023	38.5410
*8 28 33 37 41 48	Proposed	3.5909	38.0022
	ML	2.9117	36.2540
	LS	1.7827	37.7669
23 28 33 37 41 66*	Proposed	3.3598	38.2537
	ML	2.8747	42.6796
	LS	3.0936	42.1628
29.32 35.37 39.28 42.40 45.16 47.75 50.35 53.11 56.35 61	Proposed	4.7393	50.5728
	ML	5.7628	54.3803
	LS	4.9851	50.0056
29.32 35.37 39.28 42.40 45.16 47.75 50.35 53.11 56.35 79*	Proposed	4.7393	50.5660
	ML	3.7234	60.5231
	LS	4.2440	52.2645
14 35.37 39.28 42.40 45.16 47.75 50.35 53.11 56.35 82	Proposed	4.7269	50.3929
	ML	3.0484	61.4257
	LS	2.6172	52.6014

Table 1: *Comparison of* $\hat{\beta}$ *and* $\hat{\eta}$ *with other estimators, also in presence of outliers* (*indicated with* *). *It has been set* $j_0 = 0$ *and* j_1 *is given by* (*7*).

where $x_i = \dfrac{X_i}{\sqrt{\sum_{i=1}^{N} X_i^2}}$.

4 Conclusions

It has been proposed a method to calculate the parameters of a Weibull distribution. The proposed method belongs to the family of non-parametric methodologies. It is based on the (non-parametric) determination of the values of the cumulative distribution, given the assigned sample. The estimates, for the shape and scale parameters, have been compared with the ones derived by applying other methods: LS and ML. The parameter estimates, performed by the proposed methodology, appear to be very consistent and stable, even in the presence of outliers in the data. For this reason, the proposed methodology can be indicated for analysing the real working conditions of mechanical equipment.

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