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Dynamics modeling of a non minimum phase flexible arm using the Causal Ordering Graph

Frédéric Colas, Jean-Yves Dieulot, Pierre-Jean Barre, Pierre Borne
Technological research team – ERT CEMODYNE
ENSAM, 8 bd Louis XIV, 59046 Lille cedex
FRANCE

http://www.lille.ensam.fr/cemodyne

Abstract: The dynamic behavior of a flexible arm fixed on a cart moving in one direction is studied in this paper. The arm is modeled by the Euler-Bernoulli beam theory including the effect of the beam's translation. The exact eigensolutions which verify the described geometric boundaries conditions are used and the Assumed Mode Method is applied to derive the equations of motion of the system. A proof of the non-minimum phase nature of such flexible structure is then determined and a Lumped Constants model is suggested to represent the behavior of the system. A link between the parameters of the two models is then given. The equation of motion of the Lumped Constants model is expressed in a Causal Ordering Graph representation and numerical simulations of the open-loop response of the system are carried out to show the influence of the system's parameters. This kind of model has been tested on an industrial robot.

Key-Words: Flexible mechanical structure, Non-minimum phase system, Causal Ordering Graph, Industrial Robot

1 Introduction

The vibration behavior of a flexible beam clamped on a moving cart has long been an important subject of investigation as it finds many practical applications such as the modeling of the dynamics of a Cartesian robotic arm. The major problem when modeling this system is that a mathematical model which is described by a partial differential equation is difficult to solve. Therefore, such a flexible structure is often approximated as a multi-mass system, or a lumped-parameter system [1], using for example the finite element method, which avoids these modeling difficulties as well as solving the governing partial differential equations. Such an approach provides a reasonably accurate representation of modal frequencies. For these reasons, the multi-mass-spring model is generally preferred to study flexible structures [2] and for control purposes. However, when the mass-spring systems are chained together, the resulting system is always minimum phase no matter where the actuators or sensors are located.

The dynamic motion of many controlled flexible structures can be characterized in terms of the poles and zeros of rational transfer functions. The poles of such transfer functions correspond to the modal frequencies of the system, but the zero values of the function are determined by the locations of actuators and sensors in the system [3]. A flexible beam always has non-minimum phase zeros when the sensors and actuators are non-collocated, which is quite important for future control design. This means that although the multi-mass-spring model can be used to model a flexible structure for a collocated case, it must be improved for representing a non-collocated case if the non-minimum phase behavior needs to be taken into account.

In this paper, the motion's equations for a flexible beam clamped on a moving cart are derived using the Euler-Bernoulli theory and the exact eigensolutions which verify these equations are determined since this system can be considered as a basic component of a flexible structure. The Assumed Mode Method is applied on the solutions of motion's equations. The non-minimum phase nature of such a flexible structure is discussed. A Lumped Constants Model for a non-minimum phase system is then derived in section 2.5. The Causal Ordering Graph representation [4] of this kind of model is determined in section 3. Numerical simulations are carried out in section 4 to show the influence of system parameters on the non-minimum phase behavior of the system. To test these models, experimental validations have been achieved on an industrial robot.

2 Dynamics Equations

2.1 Motion Equations for a flexible Arm

Fig. 1 shows the flexible structure considered. The elastic beam is assumed to follow flexure movement and is clamped on a moving cart. As illustrated in Fig. 1, this system can be considered as a model for a Cartesian axis robot. Under the Euler-Bernoulli assumptions, the equations of motion are obtained from the Hamilton's principle [5]:

$$M\,\ddot{y}(t)+\int_{l}^{0}\rho\left(\ddot{y}(t)+\frac{\partial^2 w(z,t)}{\partial t^2}\right)dz=U(t), \qquad (1)$$

$$EI\,\frac{\partial^4 w(z,t)}{\partial z^4}+\rho\left(\ddot{y}(t)+\frac{\partial^2 w(z,t)}{\partial t^2}\right)=0, \qquad (2)$$

where M is the mass of the cart, l is the length of the elastic beam, ρ is the mass per unit length of the elastic beam, EI is the flexural rigidity of the beam, y is the position of the cart and $w(z,t)$ is the beam's deflection. The boundary conditions are written below:

$$w(0,t)=\frac{\partial w(0,t)}{\partial t}=\frac{\partial^2 w(l,t)}{\partial t^2}=\frac{\partial^3 w(l,t)}{\partial t^3}=0. \qquad (3)$$

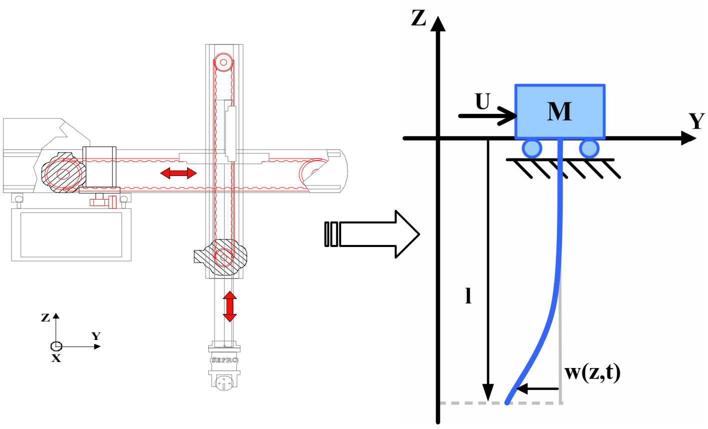

Fig. 1 – Flexible structure considered

2.2 Modal Analysis and Exact Eigensolutions

It is assumed that the position of cart $y(t)$ can be written as:

$$y(t)=\alpha(t)+\beta\,q(t)\ with\ w(z,t)=\Phi(z)q(t), \qquad (4)$$

where $\alpha(t)$ is the motion of the center of mass of the whole system without disturbances:

$$M_s\ddot{\alpha}(t)=U(t)\ with\ M_s=M+\rho|l|. \qquad (5)$$

β must therefore satisfy:

$$\beta=-\frac{\rho}{M}\int_{l}^{0}\Psi(z)\,dz\ with\ \Psi(z)\hat{=}\beta+\Phi(z). \qquad (6)$$

Substitution of (4) and (5) into (2) yields:

$$EI\Psi^{(4)}(z)q(t)+\rho\Psi(z)\ddot{q}(t)=-\rho\ddot{\alpha}(t). \qquad (7)$$

In a first time, the external forces should vanish, i.e. $\ddot{\alpha}(t)=0$ which allows to find the normal modes solutions. Thus, (7) can be decomposed into this two following equations:

$$\ddot{q}(t)+\omega^2\,q(t)=0, \qquad (8)$$

$$EI\Psi^{(4)}(z)-\omega^2\,\rho\Psi(z)=0. \qquad (9)$$

Using the boundary conditions, the mode shape's solution of (9) is:

$$\begin{aligned}\Psi(z)=&\frac{\beta}{2}\left[\cos(kz)+\cosh(kz)\right]\\&-\frac{\Psi''(0)}{2k^2}\left[\cos(kz)-\cosh(kz)\right]\\&-\frac{\Psi^{(3)}(0)}{2k^3}\left[\sin(kz)-\sinh(kz)\right],\end{aligned} \qquad (10)$$

with: $\Psi''(0)=\beta k^2\dfrac{\sin(kl)\sinh(kl)}{1+\cos(kl)\cosh(kl)}$, $k^4=\dfrac{\rho\omega^2}{EI}$

$$\Psi^{(3)}(0)=\beta k^3\frac{\cos(kl)\sinh(kl)+\cosh(kl)\sin(kl)}{1+\cos(kl)\cosh(kl)}.$$

To simplify, the mode shape can be rewritten as:

$$\Psi(z)=A(z)\beta. \qquad (11)$$

To determine the frequency equation (the equation which yields the modal frequencies), we use (11) into (6) and we obtain:

$$\beta=\frac{\rho}{M\,k^4}A^{(3)}(0)\,\beta. \qquad (12)$$

Since $\beta=0$ yields a trivial solution, the frequency equation is obtained as follow:

$$\begin{aligned}&1+\cos(kl)\cosh(kl)\\&+\frac{\rho}{kM}\left[\cos(kl)\sinh(kl)+\cosh(kl)\sin(kl)\right]=0.\end{aligned} \qquad (13)$$

For simplicity reason, β is given as:

$$\beta=2+2\cos(kl)\cosh(kl). \qquad (14)$$

The expression of the mode shape is then given as:

$$\Psi(z)=A(z)\left[2+2\cos(kl)\cosh(kl)\right]. \qquad (15)$$

For a given β_i associated with a modal pulsation ω_i and k_i solution of (13), the corresponding mode shape is defined as:

$$\Psi_i(z)=A_i(z)\left[2+2\cos(k_il)\cosh(k_il)\right]. \qquad (16)$$

The orthogonality condition of the mode shapes can be stated as follow [5]:

$$\int_l^0 \rho \Psi_i(z)\Phi_j(z)\,dz = 0 \;\; for\; i \neq j. \qquad (17)$$

2.3 Assumed Mode Method

It is assumed that the beam deflection and the position of the cart are given as:

$$w(z,t) = \sum_{i=1}^{\infty} \Phi_i(z) q_i(t), \qquad (18)$$

$$y(t) = \alpha(t) + \sum_{i=1}^{\infty} \beta_i q_i(t). \qquad (19)$$

Substituting (18) and (19) into (2), multiplying both sides of the resulting equation by $\Phi_j(z)$ and integrating this over the whole flexible beam yields:

$$\sum_{i=1}^{\infty}\left(\ddot{q}_i(t) + \omega_i^2 q_i(t)\right)\int_l^0 \rho \Psi_i(z)\Phi_j(z)\,dz = -\ddot{\alpha}(t)\int_l^0 \rho \Phi_j(z)\,dz. \qquad (20)$$

Substituting the orthogonality condition (17) into (20) leads to:

$$m_i \ddot{q}_i(t) + k_i q_i(t) = -\ddot{\alpha}(t)\,\mu_i \;\; for\; i = 1,\dots,\infty, \qquad (21)$$

with: $m_i = \int_l^0 \rho \Psi_i(z)\Phi_i(z)\,dz$, $k_i = m_i\,\omega_i^2$,

$$\mu_i = \int_l^0 \rho \Phi_i(z)\,dz.$$

For simulation purposes, assumed-mode solutions which consist to consider a finite number of eigensolutions are required. If the first n terms of (18) and (19) are considered, the dynamic behaviour of the system can reduce into a set of n+1 second-order ordinary differential equations:

$$\begin{pmatrix} M_s & 0 & \cdots & 0 \\ 0 & m_1 & \iddots & \vdots \\ \vdots & \iddots & \ddots & 0 \\ 0 & \cdots & 0 & m_n \end{pmatrix}\begin{pmatrix} \ddot{\alpha}(t) \\ \ddot{q}_1(t) \\ \vdots \\ \ddot{q}_n(t) \end{pmatrix} + \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & k_1 & \iddots & \vdots \\ \vdots & \iddots & \ddots & 0 \\ 0 & \cdots & 0 & k_n \end{pmatrix}\begin{pmatrix} \alpha(t) \\ q_1(t) \\ \vdots \\ q_n(t) \end{pmatrix} = \begin{pmatrix} M_s \\ \mu_1 \\ \vdots \\ \mu_n \end{pmatrix}\frac{U(t)}{M_s}. \qquad (22)$$

This set of equations can easily be implemented and simulated by Matlab/Simulink for an arbitrarily given forcing function $U(t)$ (See section 4).

2.4 Non-minimum phase structure

It is well known that a moving flexible beam is a non-minimum phase system [5, 6] but it was only proved for simple structures. In this section this feature is determined for a beam clamped on a moving cart. For the sake of simplicity, only the first deformation mode is considered so that n=1. First of all, the transfer function between the position of beam's end (y_2) and the input excitation (U) need to be determined. The position of beam's end is given by:

$$y_2(t) = \alpha(t) + \Psi_1(l) q_1(t). \qquad (23)$$

Substitution of (22) into (23) in the Laplace domain yields:

$$\frac{y_2(s)}{U(s)} = \frac{k_1\left(\frac{1}{\omega_1^2}\left(1 - \frac{\Psi_1(l)\mu_1}{m_1}\right)s^2 + 1\right)}{M_s s^2\left(m_1 s^2 + k_1\right)}. \qquad (24)$$

Then, the non-minimum phase nature of the system can be determined by the sign of the following criterion:

$$B(x) = 1 - \frac{\Psi_1(x)\mu_1}{m_1}, \qquad (25)$$

where $x = kl$.

More precisely, when $B(x)$ is negative, the system is non-minimum phase and when $B(x)$ is positive the system is minimum phase. Fig. 2 shows the evolution of the criterion $B(x)$ superimposed with the evolution of the frequency equation (13):

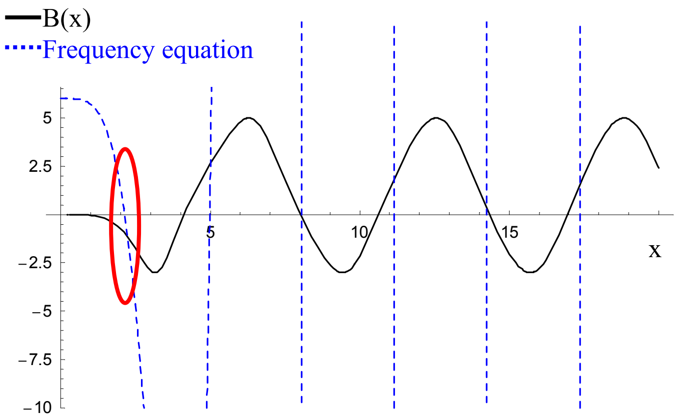

Fig. 2 – Evolution of B(x) (solid line) superimposed with the frequency equation (dashed line)

Although this is not the purpose of this paper, it can be shown that $B(x)$ is always negative for the first root of the frequency equation (i.e. for n=1). The structure is actually non-minimum phase.

2.5 Lumped Constants Model

The use of the Euler-Bernoulli theory for the modeling of flexible structures provides very accurate results but its practical application to the real world is very difficult because of the interactions between numerous components of a true mechanical system. For control purposes, such a flexible structure is often approximated as a lumped-parameters system. In this section, a Lumped constants model which is able to represent the dynamic behavior of a flexible structure is proposed and improves existing models. For simplicity reasons, the suggested model only considers the first deformation mode which is a good approximation because a lot of flexible structures have a predominant deformation mode. It can be extended easily to multi-mode models by chaining a number of elementary models together. By analogy with the previous model, y_1 represents the position of the cart and y_2 is for the position of the arm's end.

The suggested model (See Fig. 3) is not a classical two mass-spring model: the mass m_2 is not concentrated but is considered as a rod of length L. Equations of motion are given by (27).

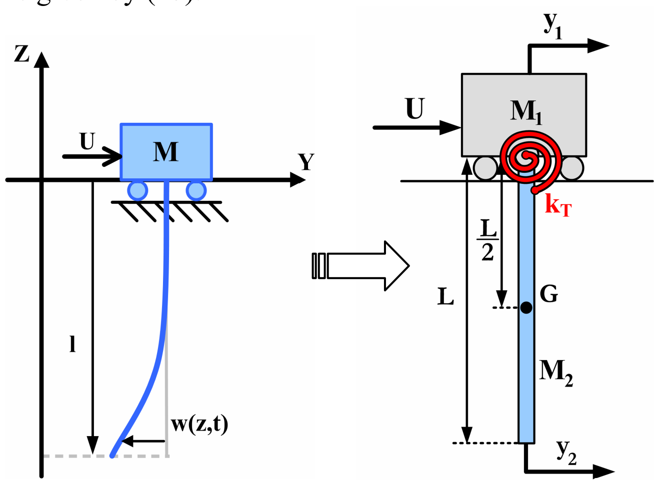

Fig. 3 – Lumped Constant Model for a non-minimum phase system

$$\begin{cases} M_1\ddot{y}_1 = U + F_{2/1},\ M_2\ddot{y}_G = -F_{2/1}, \\ \left(\dfrac{M_2L^2}{12}\right)\ddot{\theta} = F_{2/1}\dfrac{L}{2} - k_T\theta, \\ \theta = \dfrac{y_G - y_1}{L/2}, \\ y_2 = 2y_G - y_1. \end{cases} \quad (26)$$

Transfers functions of y_1 and y_2 with respect to U are given by:

$$\frac{y_1(s)}{U(s)} = \frac{12k_T + 4L^2M_2s^2}{s^2\left(12k_T(M_1+M_2) + L^2\left(4M_1M_2 + M_2^2\right)s^2\right)}, \quad (27)$$

$$\frac{y_2(s)}{U(s)} = \frac{12k_T - 2L^2M_2s^2}{s^2\left(12k_T(M_1+M_2) + L^2\left(4M_1M_2 + M_2^2\right)s^2\right)}. \quad (28)$$

Equation (28) is a proof of the non-minimum phase nature of the arm's end because the transfer function between y_2 and the input excitation U always has an instable zero.

The different parameters of this model can be linked with the parameters of (24) by estimating the two transfer functions (28) and (24). More precisely, parameters of the Lumped constants model are given as a function of the parameters of the flexible structure according to:

$$\begin{cases} k_T = \dfrac{k_1M_s\left(2\mu_1\Psi_1(l) - 3m_1\right)}{9\left(m_1 - \mu_1\Psi_1(l)\right)^2}L^2, \\ M_1 = \dfrac{M_s\left(\mu_1\Psi_1(l) - 3m_1\right)}{3m_1 - 3\mu_1\Psi_1(l)}, \\ M_2 = \dfrac{M_s\left(-4\mu_1\Psi_1(l) + 6m_1\right)}{3m_1 - 3\mu_1\Psi_1(l)}. \end{cases} \quad (29)$$

3 COG Model synthesis of the structure

3.1 Introduction

The Causal Ordering Graph formalism was used to represent the introduced Lumped Constants model. It is build up with several graphical processors, which are attached to different objects located in the studied process. The evolution of these objects is characterized by a transformation relation between influencing quantities and influenced quantities. This relation is induced by the principle of causality governing the energetic relation of an object or group of objects. To sum up, the output of a processor only depends on present or past values of the inputs. Such a formulation expresses the causality in integral form and there are many significant electrical and mechanical examples which illustrate this concept. Since the flux in a self is an integral function of the voltage; by analogy the kinetic moment of a rigid mass is the integral function of the applied efforts.

In general, the expression of the transformation relations by means of the state equations is the best warranty against physical misinterpretation. To simplify the presentation, two complementary definitions of the integral causality are only retained: (a) If an object accumulates information, causality is internal: the output is necessarily a function of the energy state, the relation then oriented is known as causal. Time and the initial state are implicit inputs and are not represented. (b) If an object does not accumulate information, causality is external. The output is an instantaneous function of the input, the relation, which is not oriented, is then known as rigid. Fig. 4 gives the selected symbolism to differentiate the two kinds of processors.

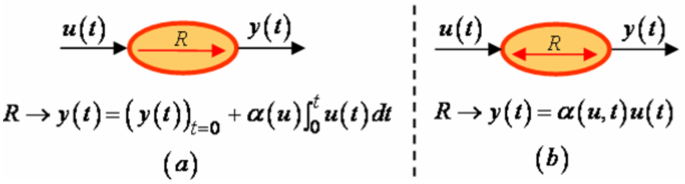

Fig. 4 – COG symbolisms: (a) causal relation, (b) rigid relation.

3.2 COG representation of the suggested Lumped Constants Model

Using the previous rules, the COG representation of the Lumped constants model represented in Fig. 3 is illustrated in Fig. 5.

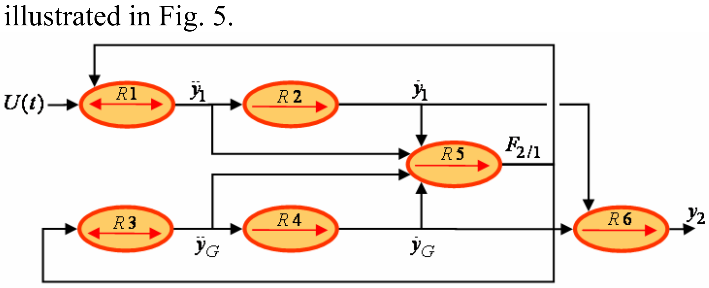

Fig. 5 – COG representation of the Lumped constants model previously proposed

The different relations are given as follow:

$$R1 \rightarrow M_1\ddot{y}_1 = U(t) + F_{2/1},\ R2 \rightarrow \frac{d\dot{y}_1}{dt} = \ddot{y}_1,$$

$$R3 \rightarrow M_2\ddot{y}_G = -F_{2/1},\ R4 \rightarrow \frac{d\dot{y}_G}{dt} = \ddot{y}_G,$$

$$R5 \rightarrow F_{2/1}\frac{L}{2} = (\dot{y}_1 - \dot{y}_G)_{t=0} + \frac{2K}{L}\int_0^t(\dot{y}_1 - \dot{y}_G)dt - \frac{M_2L}{24}\ddot{y}_1 - \ddot{y}_G,$$

$$R6 \rightarrow \frac{dy_2}{dt} = 2\dot{y}_G - \dot{y}_1.$$

4 Numerical Simulations

In this section, Numerical Simulations are carried out to obtain the open-loop response of the system represented in Fig. 1 under arbitrary excitation and to show the influence of some parameters on the nature of the system. For each simulation, the input excitation U is given by:

$$\begin{cases} U(t) = 50N \text{ if } 0 \le t < 0.5s, \\ U(t) = -50N \text{ if } 0.5s \le t < 1s, \\ U(t) = 0 \text{ otherwise.} \end{cases} \quad (30)$$

4.1 Very flexible structure

In this part, the parameters of the simulated flexible structure are listed as follow:

$$E = 0.74\times10^9\,N/m^2,$$
$$I = 300\times10^{-8}\,m^4,$$
$$M = 10\,kg,$$
$$\rho = 1.25\,kg/m,$$
$$l = 4\,m.$$

The length of the rod is excessively high to allow a very flexible behavior and is made with aluminum ($E = 0.74\times10^9\,N/m^2$).

Fig. 6 shows the evolution of the position of the arm's end for these parameters.

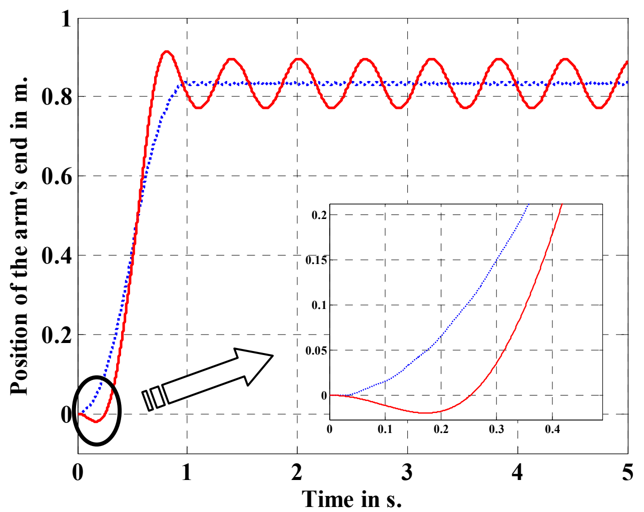

Fig. 6 – Evolution of the position of the arm's end for a very flexible structure (in red full line) and for a less flexible structure (in blue dashed line)

The non-minimum phase nature can be seen in this figure: the arm's end begins by moving back before moving forward. It is a well known characteristic of a non-minimum phase system.

4.2 Less flexible structure

In this part, the length of the rod is reduce to one meter and ρ is multiply by four to have the same overall mass than the previous structure. The new parameters are thus given by:

$$E = 0.74\times10^9\,N/m^2,$$
$$I = 300\times10^{-8}\,m^4,$$

$$M = 10\,kg,$$
$$\rho = 5\,kg/m,$$
$$l = 1\,m.$$

Fig. 6 shows the evolution of the position of the arm's end for these new parameters. The structure is more rigid and the non-minimum phase behavior is insignificant.

5 Experimental validation

Fig. 7 shows an industrial robot devoted to injection molding machine. On this real flexible structure, the system is not very flexible so that the non-minimum phase nature of the system can be neglected.

Fig. 7 – Overview of an industrial robot (stroke [mm]: X-1000 Y-400 Z-800, maximum speed: 120m/min, maximum acceleration: $4 m.s^{-2}$).

For this kind of not very flexible structure, a classical two mass-spring model for which the stiffness of the spring depends on the position of the load mass in the working space (See Fig. 8) allows to represent the first deformation modes [1]. The COG representation of this Lumped Constants model is represented in Fig. 8.

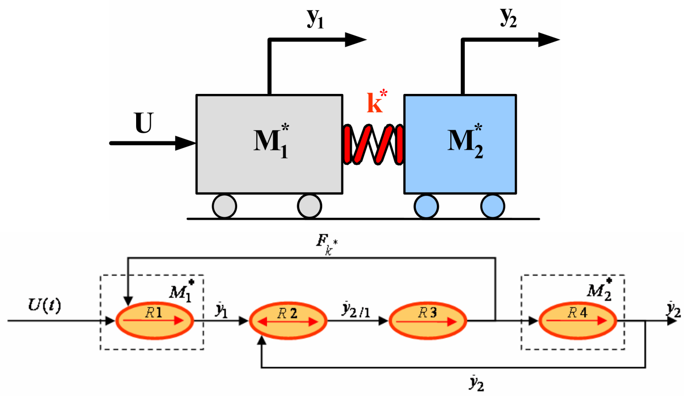

Fig. 8 – Lumped Constant Model with its COG representation for an industrial robot

The different relations of this COG are given as follow:

$$R1 \to M_1^* \frac{d\dot{y}_1}{dt} = U(t) + F_{k^*}, R2 \to \dot{y}_{2/1} = \dot{y}_2 - \dot{y}_1,$$

$$R3 \to \frac{dF_{k^*}}{dt} = k^*.y_{2/1}, R4 \to M_2^* \frac{d\dot{y}_2}{dt} = -F_{k^*}.$$

Modal analysis can be carried experimentally using an impulse response obtained when exciting the end-effector with an impact hammer [7]. The signal are recorded using a FFT analyser and data response can be analysed to obtain the modal parameters M_1^*, M_2^* and k^*.

6 Conclusion

In this paper, an Euler-Bernoulli beam fixed on a moving cart was considered and the equations of motion which describe the global motion as well as the vibrations motion derived. The exact and assumed-mode solutions were obtained and the non-minimum nature of such flexible was shown. A Lumped Constants model which is able to represent this nature was derived and a link between the two model's parameters was given. The COG formalism was used to represent the suggested Lumped Constants model and simulations were carried out to show the influence of some parameters on the non-minimum phase nature of the flexible system. Finally, this kind of model was applied on an industrial robot.

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