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Optimal control of a non-isothermal solidification model

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Abstract

In this paper, we consider optimal control problems which are related to phase-field model. The goal of this study is to reach a given target set, by a phase trajectory, of a model describing the process of non-isothermal solidification, under action of disturbances. Firstly optimal control problems are formulated. Afterwards the existence and the uniqueness of the solution of the perturbed problem is proved along with stability under mild assumptions. An optimal solution is proven to exist and, finally, we give necessary optimality conditions.

Key words. Phase-field model, non-isothermal solidification, optimal control, necessary conditions of optimality, data assimilation

AMS subject classification. 49J20, 49J50, 35K55, 49K35, 35B45

1 Introduction

The aim of this contribution is the study of optimal control problems related to non-isothermal solidification, using a phase-field model. It involves temperature U and a phase-field variable ϕ which varies sharply but smoothly, between 0, in the solid phase and 1, in the liquid phase, over a thin layer which separates the two phases. The time evolution of (ϕ, U) is governed by the following system:

$$\left\{\begin{array}{l} \dfrac{\partial\phi}{\partial t} - \nu\Delta\phi = F_1(x,t,\phi) \\ \qquad +(U-U_a)F_2(x,t,\phi) \quad in \;\; \mathcal{Q}, \\ \dfrac{\partial U}{\partial t} + \alpha\dfrac{\partial\phi}{\partial t} - \mu\Delta U = f \;\; in \;\; \mathcal{Q}, \\ \text{subjected to the boundary conditions} \\ \dfrac{\partial\phi}{\partial n} = \dfrac{\partial U}{\partial n} = 0 \;\; in \;\; \Sigma = \partial\Omega\times(0,T), \\ \text{and the initial conditions} \\ \phi(0)=\phi_0, U(0)=U_0 \;\; in \;\; \Omega, \end{array}\right. \tag{1}$$

where $\mathcal{Q} = \Omega \times (0,T)$, Ω is an open bounded domain in $\mathbb{R}^m$, $m \leq 3$, with a smooth C^∞ boundary $\partial\Omega$, n is the unit normal to $\partial\Omega$, α, μ, ν are given positive constants and U_a, the temperature at melting points, belongs to the space

$$L^\infty_a(\mathcal{Q}) := \{v \in L^2(\mathcal{Q}) \mid a_1 \leq v \leq a_2 \;\; a.e. \; in \; \mathcal{Q}\},$$

where a_1 and a_2 are given positive constants.

Various problems associated with the phase-field models have been studied over the last years (see e.g. [5, 6, 7, 10, 13, 4] and the references therein). For the optimal control problems associated with the phase-field models, we can mention [9] in which the phase transitions of pure materials due to thermal effects are analyzed, [2] in which the author considers a robust control problem for a phase-field model of isothermal solidification of a binary alloy. Here, we consider an optimal control problem of a more general phase-field model of non-isothermal solidification, in order to take into account the influence of noise on data. Actually, during solidification of metals, the solid/liquid interface becomes unstable with respect to small perturbations caused by the introduction of fluctuation terms. In this paper we consider that a control, represented by f, is applied to the system and that the temperature U_a is not accurately known. Therefore, using the classical output least square formulation we may consider that U_a also plays the role of a control variable. On the other hand we will assume that the initial conditions (U_0, ϕ_0) are not accurately known (data assimilation).

1.1 Assumptions and notations

We denote by $V = H^1(\Omega)$ and V' the dual of V. We denote by $<,>_{V',V}$ the duality product between V' and V. As usual, for any non negative integer m, we define the space $H^m(0,T,L^2(\Omega)) = \left\{v \in L^2(\mathcal{Q}) \mid \frac{\partial^j v}{\partial t^j} \in L^2(\mathcal{Q}), j = 1,\ldots,m\right\}$. Then, for any pair of real number $r, s \geq 0$, we may introduce the Sobolev space $H^{r,s}(\mathcal{Q}) = L^2(0,T,H^r(\Omega)) \cap H^s(0,T,L^2(\Omega))$, which is a Hilbert space normed by

$$\left(\int_0^T \| v \|^2_{H^r(\Omega)} dt + \| v \|^2_{H^s(0,T,L^2(\Omega))}\right)^{1/2},$$

where $H^s(0,T,L^2(\Omega)) = [H^m(0,T,L^2(\Omega)), L^2(\mathcal{Q})]_\theta$, where $\theta \in (0,1)$, $s = (1-\theta)m$.

Remark 1 *For $v \in H^{r,s}(\mathcal{Q})$ the trace functions of v: $\frac{\partial^j v}{\partial n^j}$ on $\Sigma = \partial\Omega \times (0,T)$ for an integer j such that $j \in [0, r-\frac{1}{2}]$ exist and satisfy $\frac{\partial^j v}{\partial n^j} \in H^{r_j,s_j}(\Sigma)$ where $r_j = r - j - \frac{1}{2}$ and $s_j = \frac{s(r-j-1/2)}{r}$. Moreover the functions $v \mapsto \frac{\partial^j v}{\partial n^j}$ are continuous linear mappings from $H^{r,s}(\mathcal{Q})$ into $H^{r_j,s_j}(\Sigma)$ (see e.g. [12]).* □

Lemma 1 *Let $\Omega \subset I\!R^m$, $m \geq 1$, be an open and bounded set with a smooth boundary and q be a nonnegative integer. We have the following results*
(i) $H^q(\Omega) \subset L^p(\Omega)$, $\forall p \in [1, \frac{2m}{m-2q}]$, with continuous embedding (with the exception that if $2q = m$, then $p \in [1, +\infty[$ and if $2q > m$, then $p \in [1, +\infty]$).
(ii) (Gagliardo-Nirenberg inequalities) There exists $C > 0$ such that

$$\| v \|_{L^p} \leq C \| v \|^{\theta}_{H^q} \| v \|^{1-\theta}_{L^2}, \forall v \in H^q(\Omega),$$

where $0 \leq \theta < 1$ and $p = \frac{2m}{m-2\theta q}$ (with the exception that if $q - m/2$ is a nonnegative integer, then θ is restricted to 0).

For the proof of this lemma see, for instance, Adams [1]. □

We can now introduce the following spaces:
$\mathcal{H}_i = L^\infty(0, T, H^{i-1}(\Omega))$, $\mathcal{V}_i = L^2(0, T, H^i(\Omega))$, $\mathcal{W}_i = \mathcal{H}_i \cap \mathcal{V}_i$, (for $i = 1, 2$), $\mathcal{H}^1_2 = H^1(0, T, L^2(\Omega))$, $\mathcal{H}^1_1 = H^1(0, T, V')$ and $\mathcal{W}^1_i = \mathcal{H}^1_i \cap \mathcal{V}_i$, (for $i = 1, 2$).

Remark 2 *(i) $\mathcal{W}^1_i$ is compactly embedded into $\mathcal{V}_{i-1}$, for $i = 1, 3$ (see e.g. [14]).*
(ii) $\mathcal{W}^1_i \subset C^0([0, T], H^{i-1}(\Omega))$, for $i = 1, 3$ (see e.g. [12]). □

Definition 1 *A real valued function Φ defined on $\mathcal{Q} \times I\!R^q$, $q \in I\!N^*$, is a Carathéodory function iff $\Phi(., \mathbf{v})$ is measurable for all $\mathbf{v} \in I\!R^q$ and $\Phi(x, t, .)$ is continuous for almost all $(x, t) \in \mathcal{Q}$.*

Now we state the main hypotheses on F_i, $i = 1, 2$:
(**H1**) *F_i is a Carathéodory function on $\mathcal{Q} \times I\!R$. For almost all $(x, t) \in \mathcal{Q}$, $F_i(x, t, .)$ is Lipschitz and bounded satisfying :*

(i) $|F_i(x, t, r)| \leq M_1$, $\forall r \in I\!R$ and a.e. in $\mathcal{Q}$,

(ii) F_i is differentiable. The partial derivative $F'_{ix}(., ., r)$ and $G_i = F'_{ir}(., ., r)$ are Lipschitz continuous in $\mathcal{Q}$, for all $r \in I\!R$, and are globally bounded in $\mathcal{Q} \times I\!R$.

(iii) $F'_{ix}.n = 0$, on Σ. □

Remark 3 *If U and ϕ are sufficiently regular and satisfy $\frac{\partial U}{\partial n} = \frac{\partial \phi}{\partial n} = 0$ then, thanks to (H1)(ii), we have $\nabla(F_1(x, t, \phi)).n = \nabla(UF_2(x, t, \phi)).n = 0$.* □

The paper is organized as follows. In the next section, we give some preliminary results. In section 3, we prove that system (1) has a unique solution and we obtain a stability result. In section 4, we introduce the perturbed problem and, in section 5, we study the Fréchet differentiability of the solution with respect to perturbations, this property being necessary to analyse the optimal control problem. In section 6, we study the optimal control problem by introducing a minimization problem for a given functional which depends on control variables. The existence of an optimal solution is proven and necessary optimality conditions are given. The optimality system requires calculation of gradients which are also necessary to elaborate a numerical algorithm which solves the optimal control problem.

From now on we will always denote by C some positive constant which may be different at each occurrence. Now we give the weak formulation associated to system (1).
Multiplying the first part of (1) by $v \in V$ and the second part by $q \in V$ and integrating over Ω lead to the classical weak formulation :

$$\left\{ \begin{array}{l} \displaystyle\int_\Omega \frac{\partial \phi}{\partial t} v dx + \nu \int_\Omega \nabla\phi.\nabla v dx = \int_\Omega F_1(., t, \phi) v dx \\ \displaystyle\qquad\qquad + \int_\Omega (U - U_a) F_2(., t, \phi) v dx, \\ \displaystyle\int_\Omega \frac{\partial U}{\partial t} q dx + \alpha \int_\Omega \frac{\partial \phi}{\partial t} q dx + \mu \int_\Omega \nabla U.\nabla q dx \\ \displaystyle\qquad\qquad = \int_\Omega f q dx, \\ (\phi(0), U(0)) = (\phi_0, U_0) \end{array} \right. \quad (2)$$

Now we are going to show that system (2) has a unique solution. □

2 State system

2.1 Existence and stability results

Theorem 1 *Let assumption (H1) be fulfilled.*
For any $(\phi_0, U_0) \in (L^2(\Omega))^2$ and $(f, U_a) \in (L^2(\mathcal{Q}))^2$, there exists a weak solution (ϕ, U) of (2) such that $(\phi, U) \in (\mathcal{W}^1_1)^2$.

Proof. Existence of a solution results from the classical Faedo-Galerkin method. Thanks to the properties of the different terms appearing in the weak formulation and under the previous assumptions, we obtain a priori estimates sufficient to prove the convergence of the approximate solution (U_m, ϕ_m) for the weak topology of $L^2(0, T, H^1)$ and the weak-star topology of $L^\infty(0, T, L^2)$. Then we obtain the existence of a solution. For more details we refer to [3]. □

Theorem 2 *Let assumption (H1) be fulfilled.*
For any $(\phi_0, U_0) \in (H^1(\Omega))^2$ and $(f, U_a) \in L^2(\mathcal{Q}) \times L^\infty_a(\mathcal{Q})$, there exists a unique pair of functions $(\phi, U) \in \mathcal{W}^1_2 \times \mathcal{W}^1_2$ solution of system (2). Moreover, we have the following stability result:
Let $(\phi_{01}, U_{01}, f_1, U_{a1})$ and $(\phi_{02}, U_{02}, f_2, U_{a2})$ be two functions of $(H^1)^2 \times L^2(\mathcal{Q}) \times L^\infty_a(\mathcal{Q})$. Let (ϕ_1, U_1) and (ϕ_2, U_2) be two functions of $\mathcal{W}^1_2(\mathcal{Q})$ solutions of (1). Then if (ϕ_1, U_1) (resp. (ϕ_2, U_2)) is solution of (1) with

the data $(\phi_{01}, U_{01}, f_1, U_{a1})$ *(resp.* $(\phi_{02}, U_{02}, f_2, U_{a2})$*), we have the following estimates:*

$$\|\phi\|^2_{\mathcal{W}^1_2} + \|U\|^2_{\mathcal{W}^1_2} \leq$$
$$C(\| \phi_0 \|^2_{H^1} + \| U_0 \|^2_{H^1} + \| f \|^2_{L^2(\mathcal{Q})} + \| U_a \|^2_{L^2(\mathcal{Q})}),$$

where $\phi = \phi_1 - \phi_2$, $U = U_1 - U_2$, $\phi_0 = \phi_{01} - \phi_{02}$, $U_0 = U_{01} - U_{02}$, $f = f_1 - f_2$ *and* $U_a = U_{a1} - U_{a2}$.

For the proof, we again refer to [3].

Remark 4 *If we suppose that, for almost all* $(x,t) \in \mathcal{Q}$, $F_1(x,t,.) = F_2(x,t,.) = 0$ *in* $]-\infty, 0] \cup [1, +\infty[$, *then, for the initial data* ϕ_0 *is such that* $0 \leq \phi_0 \leq 1$, *a.e. in* Ω, *the weak solution* $\phi \in \mathcal{W}^1_1$ *of (2) satisfies, for all* $t \in (0,T)$: $0 \leq \phi(.,t) \leq 1$, *a.e. in* Ω *(see [3]).* □

3 Perturbed problem

Below, the solution $\mathbf{U} = (\phi, U)$ of system (1), with the data (ϕ_0, U_0, f, U_a), will be treated as the target function. Thus we are interested in the optimal regulation of the deviation of the state from the desired target $\mathbf{U}$. Now we are going to analyse the full nonlinear equations which govern large perturbations $\mathbf{u} = (\varphi, u)$ to the target : this means that $\mathbf{U} + \mathbf{u}$ is solution of (1), with the data $(\phi_0 + \varphi_0, U_0 + u_0, f + \tilde{f}, U_a + u_a)$. Hence we consider the following system:

$$\left\{ \begin{array}{l} \dfrac{\partial \varphi}{\partial t} - \nu \Delta \varphi = (F_1(x,t,\varphi + \phi) - F_1(x,t,\phi)) \\ \quad +(u - u_a) F_2(x,t,\varphi + \phi) \\ \quad +(U - U_a)(F_2(x,t,\varphi + \phi) - F_2(x,t,\phi)) \ \ in \ \mathcal{Q}, \\ \dfrac{\partial u}{\partial t} + \alpha \dfrac{\partial \phi}{\partial t} - \mu \Delta u = \tilde{f} \ \ in \ \mathcal{Q}, \\ \dfrac{\partial \varphi}{\partial n} = \dfrac{\partial u}{\partial n} = 0 \ \ in \ \Sigma, \\ (\varphi(0), u(0)) = (\varphi_0, u_0) \ \ in \ \Omega. \end{array} \right. \tag{3}$$

We notice that regularity of $\mathbf{U}$ is given by theorem 2. If we set $\tilde{F}_1(x,t,\varphi) = F_1(x,t,\varphi + \phi) - F_1(x,t,\phi)$, $\tilde{F}_2(x,t,\varphi) = F_2(x,t,\varphi + \phi)$, $V_a = U - U_a$, then (3) is reduced to

$$\left\{ \begin{array}{l} \dfrac{\partial \varphi}{\partial t} - \nu \Delta \varphi = \tilde{F}_1(x,t,\varphi) + (u - u_a)\tilde{F}_2(x,t,\varphi) \\ \qquad +V_a(\tilde{F}_2(x,t,\varphi) - \tilde{F}_2(x,t,0)) \ \ in \ \mathcal{Q}, \\ \dfrac{\partial u}{\partial t} + \alpha \dfrac{\partial \varphi}{\partial t} - \mu \Delta u = \tilde{f} \ \ in \ \mathcal{Q}, \\ \dfrac{\partial \varphi}{\partial n} = \dfrac{\partial u}{\partial n} = 0 \ \ in \ \Sigma, \\ (\varphi(0), u(0)) = (\varphi_0, u_0) \ \ in \ \Omega. \end{array} \right. \tag{4}$$

Remark 5 *(i) We easily verify that* $(\tilde{F}_i, i = 1,2)$ *satisfy an hypothesis similar to (H1).*
(ii) If $U \in \mathcal{W}^1_2$ *et* $U_a \in L^\infty_a(\mathcal{Q})$ *then* $U \in L^\infty(0,T,H^1) \subset L^\infty(0,T,L^6)$ *and* $V_a = U - U_a \in L^\infty(0,T,L^6)$.
(iii) For sake of simplicity, we omit the "~" on $(\tilde{f}, \tilde{F})$ *for (4).* □

Now we give the weak formulation associated to system (4).
Multiplying the first part of (4) by $v \in V$ and the second part by $q \in V$ and integrating over Ω and using the third part of (4) give the classical weak formulation:

$$\left\{ \begin{array}{l} \displaystyle\int_\Omega \frac{\partial \varphi}{\partial t} v dx + \nu \int_\Omega \nabla \varphi . \nabla v dx = \int_\Omega F_1(.,t,\varphi) v dx \\ \qquad \displaystyle + \int_\Omega (u - u_a) F_2(.,t,\varphi) v dx \\ \qquad \displaystyle + \int_\Omega V_a(F_2(.,t,\varphi) - F_2(.,t,0)) v dx, \\ \displaystyle\int_\Omega \frac{\partial u}{\partial t} q dx + \int_\Omega \mu \nabla u . \nabla q dx + \alpha \int_\Omega \frac{\partial \varphi}{\partial t} q dx \\ \qquad \displaystyle = \int_\Omega f q dx, \\ (\varphi(0), u(0)) = (\varphi_0, u_0). \end{array} \right. \tag{5}$$

With a proof similar to that of theorem 2, we have the following results:

Theorem 3 *Let assumption (H1) be fulfilled. Then For any* $(\varphi_0, u_0) \in (H^1(\Omega))^2$ *and* $(f, u_a, V_a) \in L^2(\mathcal{Q}) \times L^\infty_a(\mathcal{Q}) \times L^\infty(0,T,L^6)$, *there exists a unique pair of functions* $(\varphi, u) \in \mathcal{W}^1_2 \times \mathcal{W}^1_2$ *solution of system (4). Moreover if* (φ_1, u_1) *(resp.* (φ_2, u_2)*) is solution of (4) with the data* $(\varphi_{01}, u_{01}, u_{a1}, f_1)$ *(resp.* $(\varphi_{02}, u_{02}, u_{a2}, f_2)$*) in* $(H^1)^2 \times L^\infty_a(\mathcal{Q}) \times L^2(\mathcal{Q})$, *we have the following estimate:*

$$\|\varphi\|^2_{\mathcal{W}^2_2} + \|u\|^2_{\mathcal{W}^2_2} \leq$$
$$C\left(\| \varphi_0 \|^2_{H^1} + \| u_0 \|^2_{H^1} + \| f \|^2_{L^2(\mathcal{Q})} + \| u_a \|^2_{L^2(\mathcal{Q})}\right),$$

where $\varphi = \varphi_1 - \varphi_2$, $u = u_1 - u_2$, $\varphi_0 = \varphi_{01} - \varphi_{02}$, $u_0 = u_{01} - u_{02}$, $f = f_1 - f_2$ *and* $u_a = u_{a1} - u_{a2}$. □

4 Fréchet differentiability

Before investigating the Fréchet differentiability of the function $\mathcal{F} : (\varphi_0, u_0, u_a, f) \longrightarrow \mathbf{u} = (\varphi, u)$, which maps the source term $(\varphi_0, u_0, u_a, f) \in (H^1)^2 \times L^\infty_a(\mathcal{Q}) \times L^2(\mathcal{Q})$ of system (4) to the corresponding solution $(\varphi, u) \in \mathcal{W}^1_2 \times \mathcal{W}^1_2$, we study the following problem: to find (ψ, w) such that

$$(\mathcal{P}_I)\quad \begin{cases} \dfrac{\partial \psi}{\partial t} - \nu\Delta\psi = G_1(x,t,\varphi)\psi \\ +(w-w_a)F_2(x,t,\varphi) + W_a G_2(x,t,\varphi)\psi \;\; in\ \mathcal{Q}, \\ \dfrac{\partial w}{\partial t} + \alpha\dfrac{\partial \psi}{\partial t} - \mu\Delta\psi = g \;\; in\ \mathcal{Q}, \\ \dfrac{\partial \psi}{\partial n} = \dfrac{\partial w}{\partial n} = 0 \;\; in\ \Sigma, \\ \psi(0) = \psi_0, w(0) = w_0 \;\; in\ \Omega, \end{cases}$$

where $W_a = u - u_a + V_a$.

Theorem 4 *Let assumption (H1) be fulfilled. If $(\varphi, u) \in (\mathcal{W}_2^1)^2$ and $W_a \in L^\infty(0,T,L^6)$ then*
(i) For any $(\psi_0, w_0, w_a, g) \in (H^1)^2 \times L_a^\infty(\mathcal{Q}) \times L^2(\mathcal{Q})$, there exists a unique pair of functions $(\psi, w) \in (\mathcal{W}_2^1)^2$ solution of system $(\mathcal{P}_I)$, such that
$\| \psi \|_{\mathcal{W}_2^1}^2 + \| w \|_{\mathcal{W}_2^1}^2 \leq C_e(\| \psi_0 \|_{H^1}^2 + \| w_0 \|_{H^1}^2 + \| w_a \|_{L^2(\mathcal{Q})}^2 + \| g \|_{L^2(\mathcal{Q})}^2).$
(ii) Let $(\psi_{0i}, w_{0i}, w_{ai}, g_i)$, $i = 1,2$ be two couples of $(H^1)^2 \times L_a^\infty(\mathcal{Q}) \times L^2(\mathcal{Q})$. If (ψ_i, w_i) is the solution of $(\mathcal{P}_I)$, with the data $(\psi_{0i}, w_{0i}, w_{ai}, g_i)$, $i = 1,2$, then
$\| \psi \|_{\mathcal{W}_2^1}^2 + \| w \|_{\mathcal{W}_2^1}^2 \leq C_e(\| \psi_0 \|_{H^1}^2 + \| w_0 \|_{H^1}^2 + \| w_a \|_{L^2(\mathcal{Q})}^2 + \| g \|_{L^2(\mathcal{Q})}^2),$
where $\psi = \psi_1 - \psi_2$, $w = w_1 - w_2$, $\psi_0 = \psi_{01} - \psi_{02}$, $w_0 = w_{01} - w_{02}$, $w_a = w_{a1} - w_{a2}$ and $g = g_1 - g_2$.

Proof. The proofs of existence, uniqueness and the results of the linear system $(\mathcal{P}_I)$ are similar to those of theorem 3, therefore we omit the details. □

We are now going to study the Fréchet differentiability of $\mathcal{F}$. For simplicity, we denote the space $(H^1(\Omega))^2 \times L_a^\infty(\mathcal{Q}) \times L^2(\mathcal{Q})$ by $\mathcal{U}_F$.

Theorem 5 *Let $X_d = (\varphi_0, u_0, u_a, f) \in \mathcal{U}_F$, and $H = (h_\varphi, h_u, h_a, h_f) \in \mathcal{U}_F$ with $\mathcal{F}(X_d)$ and $\mathcal{F}(X_d{+}H)$ being the corresponding solutions of (4). Then*

$$\|\mathcal{F}(X_d + H) - \mathcal{F}(X_d) - \mathcal{F}'(X_d)H\|_{\mathcal{W}_2^1 \times \mathcal{W}_2^1} \leq C \| H \|_{\mathcal{U}_F}^{3/2}, \quad (6)$$

where $\mathcal{F}'(X_d) : \mathcal{U}_F \to \mathcal{W}_2^1 \times \mathcal{W}_2^1$ is a linear operator defined in the following way: for any H the pair $(\psi, w) = \mathcal{F}'(X_d)H$ is the solution of

$$(\mathcal{P}_F)\quad \begin{cases} \dfrac{\partial \psi}{\partial t} - \nu\Delta\psi = G_1(x,t,\varphi)\psi + (w - h_a)F_2(x,t,\varphi) \\ \qquad +W_a G_2(x,t,\varphi)\psi \;\; in\ \mathcal{Q}, \\ \dfrac{\partial w}{\partial t} + \alpha\dfrac{\partial \psi}{\partial t} - \mu\Delta\psi = h_f \;\; in\ \mathcal{Q}, \\ \dfrac{\partial \psi}{\partial n} = \dfrac{\partial w}{\partial n} = 0 \;\; in\ \Sigma, \\ \psi(0) = h_\varphi, w(0) = h_u \;\; in\ \Omega, \end{cases}$$

where $W_a = u - u_a - V_a$.

The proof will be detailed in [3]. □

5 An optimal control problem

We suppose now that $\mathcal{X} = (\varphi_0, u_0, u_a, f) = (X_1, X_2)$ (where $X_1 = (\varphi_0, u_0)$ and $X_2 = (u_a, f)$) is decomposed into the control $Z = (Z_\varphi, Z_u) \in \mathcal{V}_{ad}$ (data assimilation) and the control $Y = (Y_a, Y_f) \in \mathcal{U}_{ad}$ where $\mathcal{V}_{ad} = (L^2(\Omega))^2$ and $\mathcal{U}_{ad} = L_a^\infty(\mathcal{Q}) \times L^2(\mathcal{Q})$ i.e. $X_2^t = \mathcal{C}Y^t$ and $X_1^t = \mathcal{B}Z^t$, where $\mathcal{C} = diag(C_a, C_f)$ (resp. $\mathcal{B} = diag(B_\varphi, B_u)$) are bounded operators on $\mathcal{U}_{ad}$ to $\mathcal{U}_{ad}$ (resp. to $H^1 \times H^1$). So the function (φ, u) is related to the disturbance Z and control Y by the system (see (4)):

$$\begin{cases} \dfrac{\partial \varphi}{\partial t} - \nu\Delta\varphi = F_1(x,t,\varphi) + (u - C_a Y_a)F_2(x,t,\varphi) \\ \qquad +V_a(F_2(x,t,\varphi) - F_2(x,t,0)) \;\; in\ \mathcal{Q}, \\ \dfrac{\partial u}{\partial t} + \alpha\dfrac{\partial \varphi}{\partial t} - \mu\Delta u = C_f Y_f \;\; in\ \mathcal{Q}, \\ \dfrac{\partial \varphi}{\partial n} = \dfrac{\partial u}{\partial n} = 0 \;\; in\ \Sigma, \\ \varphi(0) = B_\varphi Z_\varphi \;\; in\ \Omega, \\ u(0) = B_u Z_u \;\; in\ \Omega. \end{cases} \quad (7)$$

In order to obtain the regularity given in theorem 3, we make the following assumptions: $(Y, Z) \in \mathcal{U}_{ad} \times \mathcal{V}_{ad}$ and $V_a \in L^\infty(0,T,L^6)$.
Let $\mathcal{P} : (Y, Z) \mapsto (\varphi, u)$ be the map: $\mathcal{U}_{ad} \times \mathcal{V}_{ad} \to \mathcal{W}_2^1 \times \mathcal{W}_2^1$ defined by (7) and introduce the cost function defined by

$$J(Y,Z) = \frac{a}{2} \| \varphi - \varphi_{obs} \|_{L^2(\mathcal{Q})}^2 + \frac{b}{2} \| u - u_{obs} \|_{L^2(\mathcal{Q})}^2 + \frac{\gamma}{2} \| Y \|_{\mathcal{U}_{ad}}^2 + \frac{\delta}{2} \| Z \|_{\mathcal{V}_{ad}}^2, \quad (8)$$

where a, b, γ, δ are fixed such that $\gamma, \delta \geq 0$ with $\gamma + \delta > 0$ and $a, b \geq 0$ with $a + b > 0$. The functions $u_{obs} \in L^2(\mathcal{Q})$ and $\varphi_{obs} \in L^2(\mathcal{Q})$ are given and represent the observations.
Let $\mathcal{K} = \mathcal{K}_1 \times \mathcal{K}_2$ such that $\mathcal{K}_1$ and $\mathcal{K}_2$ are (given) nonempty, closed, convex subsets of $\mathcal{U}_{ad} \times \mathcal{V}_{ad}$. The proposed problem consists in finding an optimal pair $(Y^*, Z^*) \in \mathcal{K}$ such that

$$J(Y^*, Z^*) = \inf_{(Y,Z)\in\mathcal{K}} J(Y,Z). \quad (9)$$

Theorem 6 *Let assumption (H1) be fulfilled.*
Then there exists an optimal control $(Y^*, Z^*) \in \mathcal{K}$, such that (Y^*, Z^*) is defined by (9) and $(\varphi^*, u^*) = \mathcal{P}(Y^*, Z^*)$ is solution of (7).

Proof: Let (Y_k, Z_k) and $(\varphi_k, u_k) = \mathcal{P}(Y_k, Z_k)$, be a minimizing sequence i.e. $\liminf_k J(Y_k, Z_k) = \min_{(Y,Z)\in\mathcal{K}} J(Y, Z)$. Then (Y_k, Z_k) is uniformly bounded in $\mathcal{K}$ and $(\mathcal{C}Y_k, \mathcal{B}Z_k)$ is uniformly bounded in $L^\infty(\mathcal{Q})\times L^2(\mathcal{Q}) \times H^1 \times H^1$ (see definition of operators $\mathcal{B}$ and $\mathcal{C}$). In view of theorem 3, we can deduce that the sequence (φ_k, u_k) is uniformly bounded in $\mathcal{W}_2^1 \times \mathcal{W}_2^1$. Therefore we can extract from $(Y_k, Z_k, \varphi_k, u_k)$ a subsequence also denoted by $(Y_k, Z_k, \varphi_k, u_k)$ and such that

$$
\begin{aligned}
&(Y_k, Z_k) \rightharpoonup (Y^*, Z^*) \text{ weakly in } \mathcal{U}_{ad} \times \mathcal{V}_{ad},\\
&(\varphi_k, u_k) \rightharpoonup (\tilde{\varphi}, \tilde{u}) \text{ weakly in } (\mathcal{W}_2)^2,\\
&(\varphi_k, u_k) \rightarrow (\tilde{\varphi}, \tilde{u}) \text{ strongly in } (L^2(\mathcal{Q}))^2.
\end{aligned}
$$

It is easy to prove that $(\tilde{\varphi}, \tilde{u}) = \mathcal{P}(Y^*, Z^*)$ and thanks to uniqueness of the solution of (7), we have $(\tilde{\varphi}, \tilde{u}) = (\varphi^*, u^*)$. Then, using lower semicontinuity of the cost functional J, we can deduce $J(Y, Z) \leq \liminf_k J(Y_k, Z_k)$. Thus (Y^*, Z^*) is a minimizer of J.

Proposition 1 *The function $\mathcal{P}$ is continuously Fréchet differentiable. The derivative operator $\mathcal{P}'(Y, Z)$ is defined in the following way, if we set $\mathcal{P}'(Y, Z) : \mathcal{H} = (H, K) \mapsto \mathbf{w} = (\psi, w)$, where $H = (h_a, h_f)$ and $K = (k_\varphi, k_u)$,then the pair $\mathbf{w} = (\psi, w)$ is solution of the linear system:*

$$
(\mathcal{P}_{LP})\quad
\begin{cases}
\dfrac{\partial\psi}{\partial t} - \nu\Delta\psi = G_1(x,t,\varphi)\psi + W_aG_2(x,t,\varphi)\psi \\
\qquad +(w - C_ah_a)F_2(x,t,\varphi) \ \ in\ \mathcal{Q},\\
\dfrac{\partial w}{\partial t} + \alpha\dfrac{\partial\psi}{\partial t} - \mu\Delta\psi = C_fh_f \ \ in\ \mathcal{Q},\\
\dfrac{\partial\psi}{\partial n} = \dfrac{\partial w}{\partial n} = 0 \ \ in\ \Sigma,\\
\psi(0) = B_\varphi k_\varphi \ \ in\ \Omega,\\
w(0) = B_uk_u \ \ in\ \Omega,
\end{cases}
$$

where $W_a = u - u_a - V_a$. Futhermore, for all $(Y, Z) \in \mathcal{U}_{ad} \times \mathcal{V}_{ad}$, there exists a constant C_e such that

$$\| \mathcal{P}'(Y, Z) \|_{\mathcal{L}(\mathcal{U}_{ad}\times\mathcal{V}_{ad},\mathcal{W}_2^1\times\mathcal{W}_2^1)} \leq C_e.$$

Proof. The proof of this proposition is a consequence of the nature of the operators $\mathcal{B}$, $\mathcal{C}$ and of the results of theorem 5. □

In order to obtain the necessary optimality conditions which have to be satisfied by the solution of the optimal control problem, we introduce the following adjoint system:

$$
\begin{cases}
-\dfrac{\partial p}{\partial t} - \alpha\dfrac{\partial q}{\partial t} - \nu\Delta p - G_1(x,t,\varphi)p \\
\qquad -W_aG_2(x,t,\varphi)p = a(\varphi - \varphi_{obs}) \ \ in\ \mathcal{Q},\\
-\dfrac{\partial q}{\partial t} - \mu\Delta q - F_2(x,t,\varphi)p = b(u - u_{obs}) \ \ in\ \mathcal{Q},\\
\dfrac{\partial p}{\partial n} = \dfrac{\partial q}{\partial n} = 0 \ \ in\ \Sigma,\\
p(T) = q(T) = 0 \ \ in\ \Omega,
\end{cases}
\tag{10}
$$

where $(\varphi, u) = \mathcal{P}(Y, Z)$ and $W_a = u - u_a - V_a$.

Remark 6 *The adjoint system (10) is a linear system, then it is quite easy to prove that (p, q) exists and is unique in the same manner as in theorem 4.* □

We can now give the first-order optimality conditions for the optimal control problem (9).

Theorem 7 *Let assumption (H1) be satisfied, $(Y^*, Z^*) \in \mathcal{K}$ be an optimal control of (9) and $(u^*, \varphi^*) \in \mathcal{W}_2^1 \times \mathcal{W}_2^1$ such that (Y^*, Z^*) is defined by (9) and $(u^*, \varphi^*) = \mathcal{P}(Y^*, Z^*)$ solution of (7). Then, for all $(Y, Z) \in \mathcal{K}$ we have*

$$\int_0^T\int_\Omega (\mathcal{L}^*(p^*, \varphi^*, q^*) + \gamma Y^*)(Y^* - Y)dxdt \geq 0$$

and

$$\int_\Omega (\mathcal{M}^*(p^*, \varphi^*, q^*) - \delta Z^*)(Z^* - Z)dx \geq 0,$$

where (p^*, q^*) is the solution of the adjoint system (10) corresponding to (φ^*, u^*), $\mathcal{L}^*(p^*, \varphi^*, q^*) = (-C_a^*p^*F_2(.,.,\varphi^*), C_f^*q^*)$ and $\mathcal{M}^*(p^*, \varphi^*, q^*) = (B_\varphi^*(p^*(0) + \alpha q^*(0)), B_u^*q^*(0))$.

Proof. The cost function J is a composition of differentiable maps then J is differentiable and we have, for all $\mathcal{H} = (H, K) \in \mathcal{K}$,

$$
\begin{aligned}
J'(Y, Z).\mathcal{H} = &\iint_{\mathcal{Q}} (a(\varphi-\varphi_{obs})\psi + b(u-u_{obs})w)dxdt\\
&+ \gamma < Y, H >_{\mathcal{U}_{ad}} + \delta < Z, K >_{\mathcal{V}_{ad}},
\end{aligned}
$$

where $(\psi, w) = \mathcal{P}'(Y, Z).\mathcal{H}$ solution of system $(\mathcal{P}_{LP})$.

Multiplying the first part of $(\mathcal{P}_{LP})$ by p and the second part by q, using Green's formula and integrating by time we obtain (owing to the homogeneous Neumann boundary conditions)

$$
\begin{aligned}
&\iint_{\mathcal{Q}} \left(-\frac{\partial p}{\partial t} - \nu\Delta p - G_1(.,t,\varphi)p - W_aG_2(.,t,\varphi)p\right)\psi dxdt\\
&\quad = \iint_{\mathcal{Q}} F_2(.,t,\varphi)pwdxdt - \iint_{\mathcal{Q}} F_2(.,t,\varphi)p\ C_ah_adxdt\\
&\qquad -\int_\Omega p(T)\psi(T)dx + \int_\Omega p(0)\psi(0)dx,\\
&\iint_{\mathcal{Q}} (-\frac{\partial q}{\partial t} - \mu\Delta q)wdxdt = \iint_{\mathcal{Q}} \alpha\frac{\partial q}{\partial t}\psi dxdt\\
&\qquad -\int_\Omega q(T)(\alpha\psi(T) + w(T))dx\\
&\qquad +\int_\Omega q(0)(\alpha\psi(0) + w(0))dx + \iint_{\mathcal{Q}} qC_fh_fdxdt,\\
&\psi(0) = B_\varphi k_\varphi, w(0) = B_uk_u.
\end{aligned}
$$

Since (p,q) is solution of (10), we have

$$a \iint_Q (\varphi - \varphi_{obs})\psi dxdt = \iint_Q F_2(.,t,\varphi)pwdxdt - \iint_Q F_2(.,t,\varphi)p\, C_a h_a dxdt + \int_\Omega p(0)B_\varphi k_\varphi dx - \iint_Q \alpha \frac{\partial q}{\partial t}\psi dxdt,$$

$$b \iint_Q (u - u_{obs})wdxdt + \iint_Q F_2(.,t,\varphi)pwdxdt = \iint_Q \alpha \frac{\partial q}{\partial t}\psi dxdt + \iint_Q qC_f h_f dxdt + \alpha\int_\Omega q(0)B_\varphi k_\varphi dx + \int_\Omega q(0)B_u k_u dx,$$

hence

$$b \iint_Q (u - u_{obs})wdxdt + a \iint_Q (\varphi - \varphi_{obs})\psi dxdt = -\iint_Q F_2(.,t,\varphi)pC_a h_a dxdt + \iint_Q qC_f h_f dxdt + \int_\Omega p(0)B_\varphi k_\varphi dx + \alpha\int_\Omega q(0)B_\varphi k_\varphi dx + \int_\Omega q(0)B_u k_u dx.$$

From the expression of $J'(Y,Z).\mathcal{H}$ we can deduce that

$$J'(Y,Z).\mathcal{H} = \iint_Q (\mathcal{L}^*(p,\varphi,q) + \gamma Y)Hdx + \int_\Omega (\mathcal{M}^*(p,\varphi,q) - \delta Z)Kdxdt,$$

where,

$$\mathcal{L}^*(p,\varphi,q) = (-C_a^* pF_2(.,.,\varphi), C_f^* q)$$

and

$$\mathcal{M}^*(p,\varphi,q) = (B_\varphi^*(p(0) + \alpha q(0)), B_u^* q(0)).$$

Since (Y^*,Z^*) is an optimal solution we have, for all $(Y,Z) \in \mathcal{K}$,

$$\frac{\partial J}{\partial Y}(Y^*,Z^*).(Y^* - Y) \geq 0 \text{ and } \frac{\partial J}{\partial Z}(Y^*,Z^*).(Z^* - Z) \geq 0,$$

hence, for all $(Y,Z) \in \mathcal{K}$ we finally get

$$\begin{aligned} &\iint_Q (\mathcal{L}^*(p^*,\varphi^*,q^*) + \gamma Y^*)(Y^* - Y)dxdt \geq 0, \\ &\int_\Omega (\mathcal{M}^*(p^*,\varphi^*,q^*) - \delta Z^*)(Z^* - Z)dx \geq 0. \end{aligned} \quad (11)$$

This completes the proof. □

6 Conclusion

This paper is an attempt to set an optimal control approach for a class of problems related to non-isothermal solidification of metals. Other choices of control variables and disturbances can be envisaged. Numerical aspects will be presented in a forthcoming paper. It is clear that the amount of computation is crucial in solution of control problems (for an existing approach based on POD we refer to [15]). This remark also applies to a game problem associated with the worst case control of this model as it will be shown in a forthcoming paper [3]. □

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