2005 WSEAS Int. Conf. on DYNAMICAL SYSTEMS and CONTROL, Venice, Italy, November 2-4, 2005 (pp604-608)

MATRIX ANALYSIS OF ANCHORED STRUCTURES

NIKOS MASTORAKIS
Head of the Department of Computer Science
Military Inst. of University Education / Hellenic Naval Academy
Terma Hatzikyriakou 18539,
Piraeus, GREECE
http://www.wseas.org/mastorakis

OLGA MARTIN
Department of Mathematics
University "Politehnica" of Bucharest
Splaiul Independentei 313
Bucharest, ROMANIA

Abstract: - The purpose of this paper is to analyze a structure consisting of beam, bars and anchored cable, which hold the transmission lines, when is subjected to a static loading. For this study we use the matrix displacement method. Because the elastic cables are elements capable to resist only in axial tension forces, the numerical algorithm will have two steps. In the first step we use the displacement method for the beams, bars and all cables. In the second step, we repeat the calculus, after we remove the compressed cables.

Key-Words: matrix displacement method, static loading , anchored cables.

1 Introduction

The primary function of any structure is to support and to transfer externally applied loads to the reaction points while at the same time being subjected to some specified constraints.

Using the concept of stiffness matrix, which has been dealt with by J.S. Przemieniecky, [5], O.C. Zienkiewicz,[6], Brebbia C.A.[1], the proposed numerical procedure can be used to determine the equilibrium configuration of any 3-dimensional assembly with cable components. An analysis of the suspension cables and the truss-systems was presented in [2], [3].

The general assumptions used in this paper are:

- displacements of a structural element are not very large and the geometry of the system is well defined before the analysis is attempt;
- displacements and strains of the loaded structure are small and hence, linear elasticity theory applies.

The algorithm is presented for the assembly shown in Fig. 3. The basic structure (nodes **1-9**) is clamped in the points **1** and **8** and is hinged in the nodes **9-16**. All members that lie between nodes **1-8**, are considered beams. The two stiffeners: **3-7** and **2-7** and the cables**: 2-14, 2-13**, **4-15**, **4-16**, **7-9**, **7-10**, **4-11**, **4-12** are pin-jointed bars.

2 Stiffness properties of elements

In the local coordinate system (Fig. 1), the 12×12 stiffness matrix for a beam element bounded by the nodes **N1**, **N2** is $ks(A, I_z, l)$, where: A – the cross-sectional area, I_z – the axial moment of inertia of the section about z axes and l – the length of the beam. The form of this matrix is presented in [5].

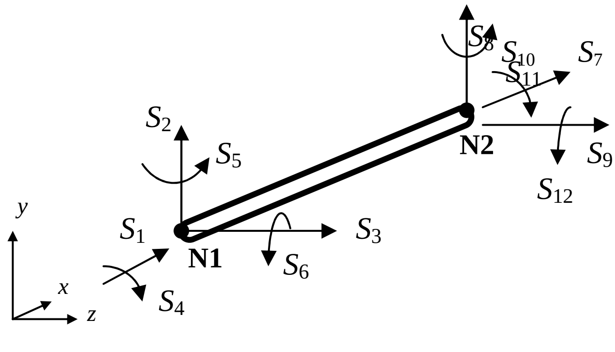

Fig.1

For a pin-jointed element, which is shown in the Fig.2, the stiffness matrix is of the form

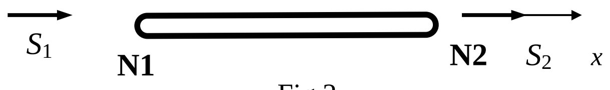

Fig.2

$$kf(A,l) = \frac{E \cdot A}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad (1)$$

where: A – the cross-sectional area, moment of and l – the length of the bar.

3 Transformation of coordinate axes

In order to determine the stiffness property of the complete structure, a common datum must be established for all unassembled structural elements so that all the displacements and their corresponding forces will be referred to a common coordinate system.

Since the stiffness matrices ks and kf are initially calculated in local coordinates, (x, y, z), it is necessary to introduce transformation matrices changing the frame of reference from a local to a datum coordinate system, (X, Y, Z). This relationship is expressed by the matrix equation

$$u = R \cdot \bar{u} \quad (2)$$

where R is rotator matrix between the element displacements u in the local system and the element displacements $\bar{u}$ in the datum system. The elements of R are the cosines of angles between the local and datum coordinate system.

For a beam element we have $R = R1$, with

$$R1 = \begin{bmatrix} C & 0 & 0 & 0 \\ 0 & C & 0 & 0 \\ 0 & 0 & C & 0 \\ 0 & 0 & 0 & C \end{bmatrix}, \quad C = \begin{bmatrix} l_{ox} & m_{ox} & n_{ox} \\ l_{oy} & m_{oy} & n_{oy} \\ l_{oz} & m_{oz} & n_{oz} \end{bmatrix}$$

$$\begin{aligned} &l_{ox} = \cos(Ox, OX); \quad m_{Ox} = \cos(Ox, OY); \\ &n_{ox} = \cos(Ox, Oz); \quad l_{oy} = \cos(Oy, OX); \\ &m_{oy} = \cos(Oy, OY); \quad n_{oy} = \cos(Oy, OZ); \\ &l_{oz} = \cos(Oz, OX); \quad m_{oz} = \cos(Oz, OY); \end{aligned} \quad (3)$$

and $n_{oz} = \cos(Oz, OZ)$.

The matrix $R = R2$ for a pin-jointed bar is

$$R2 = \begin{bmatrix} l_{ox} & m_{ox} & n_{ox} & 0 & 0 & 0 \\ 0 & 0 & 0 & l_{ox} & m_{ox} & n_{ox} \end{bmatrix} \quad (4)$$

with the above notations. Then, as explained in [5], we obtain the relationship between the local stiffness matrix, ks or kf and these stiffness matrix written in a datum coordinate system, $\overline{ks}$ and $\overline{kf}$, respectively:

$$\overline{ks} = R1^T \cdot ks \cdot R1 \quad (5)$$

$$\overline{kf} = R2^T \cdot kf \cdot R2 \quad (6)$$

In the next step, we built the stiffness matrix for the complete structure, K, summing all the overlapping terms of the matrix $\overline{ks}$ or $\overline{kf}$, which correspond to the adjacent elements. Thus, we must number the nodes, which make the connection between the cable and the foundation to be after the nodes of beam elements.

Finally, in order to obtain the nodal displacements of the considered structure, we must find out the solution of the following matrix relation

$$K \cdot U = F \quad (7)$$

where the external loading matrix F, correspond to the displacements U. As the above relation has been established for a free structure, the matrix F will also contain the reactions and K will be a singular matrix. In order to calculate the unknown displacements U_A, in the active forces direction, F_A, as well as the reactions, F_R and the forces due to the imposed displacements, F_I, the equation (7) will be written as follows

$$\begin{bmatrix} F_A \\ F_I \\ F_R \\ 0 \end{bmatrix} = \begin{bmatrix} K_{AA} & K_{IA}^T & K_{RA}^T & K_{CA}^T \\ K_{NA} & K_{II} & K_{RI}^T & K_{CI}^T \\ K_{RA} & K_{RI} & K_{RR} & K_{CR}^T \\ K_{CA} & K_{CI} & K_{CR} & K_{CC} \end{bmatrix} \cdot \begin{bmatrix} U_A \\ U_I \\ 0 \\ U_C \end{bmatrix} \quad (8)$$

Here U_C is the unknown displacements of nodes in the directions without the imposed displacements and the active forces. From the above system (8), U_C, U_A, F_R, F_I can be immediately obtained with the next relations:

$$\begin{aligned}
\overline{K}_{AA} &= K_{AA} - K_{CA}^{T} \cdot K_{CC}^{-1} \cdot K_{CA} \\
\overline{K}_{RA} &= K_{AR} - K_{CR}^{T} \cdot K_{CC}^{-1} \cdot K_{CA} \\
\overline{K}_{RI} &= K_{RI} - K_{CR}^{T} \cdot K_{CC}^{-1} \cdot K_{CI} \\
\overline{K}_{IA} &= K_{IA} - K_{CI}^{T} \cdot K_{CC}^{-1} \cdot K_{CA} \\
U_{A} &= \overline{K}_{AA}^{-1} \cdot (F_{A} - \overline{K}_{IA}^{T} \cdot U_{I})
\end{aligned} \tag{9}$$

$$\begin{aligned}
U_{C} &= -K_{CC}^{-1} \cdot K_{CA} \cdot \overline{K}_{AA}^{-1} \cdot F_{A} + \\
&\quad + K_{CC}^{-1} \cdot (K_{CA} \cdot \overline{K}_{AA}^{-1} \cdot K_{IA}^{T} - K_{CI}) \cdot U_{i} \\
F_{R} &= \overline{K}_{RA} \cdot \overline{K}_{AA}^{-1} \cdot F_{A} - \\
&\quad - (\overline{K}_{RA} \cdot \overline{K}_{AA}^{-1} \cdot \overline{K}_{IA}^{T} - \overline{K}_{RI}) \cdot U_{I}
\end{aligned}$$

4 Numerical results

Figure 3 shows a structure subjected to the forces and the moments in the nodes **4** and **5**. Because this loading, we neglect the displacements: u_4, u_5, u_6, u_7 for the beam elements, which have very small values. Hence, four degrees of freedom per node are necessary. For a node i we note: $u_1 = u_i$, $u_2 = v_i$, $u_3 = w_i$, and $u_6 = \theta_i$ (Fig.1). The elements: **2-6**, **3-7** and the cables have two degrees of freedom per node.

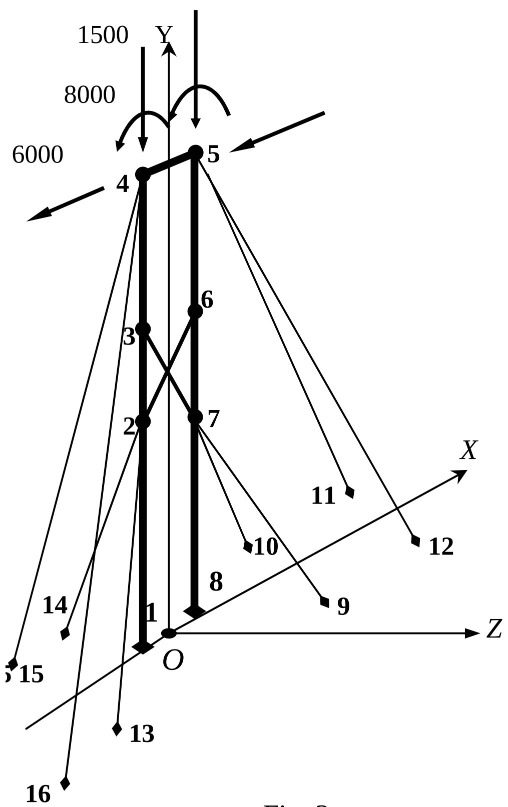

Fig. 3

The nodes coordinates in a datum coordinate system, (X, Y, Z), the cross-sectional area, A and I_z inertial moment of the elements are presented in the tables:

Table 1

Node	X	Y	Z
	m	*m*	*m*
1	-1.5	0	0
2	-1.5	10	0
3	-1.5	14	0
4	-1.5	20	0
5	1.5	20	0
6	1.5	14	0
7	1.5	10	0
8	1.5	0	0
9	9	0	2
10	9	0	-2
11	11	0	-2
12	11	0	2
13	-9	0	2
14	-9	0	-2
15	-11	0	-2
16	-11	0	2

Table 2

Elements	$A[m^2]$	$I_z[m^4]$
beams	0.071	0.0004
cables	0.000113	-
stiffeners	0.015	-

The modulus of elasticity is $E = 2.1 \cdot 10^{10}$ daN/m^2. For each beam element we have

$$l^3 ks(A, I_z, l)/E =$$

$$= \begin{bmatrix}
Al^2 & 0 & 0 & 0 & -Al^2 & 0 & 0 & 0 \\
0 & 12I & 0 & 6I \cdot l & 0 & -12I & 0 & 6I \cdot l \\
0 & 0 & 12I & 0 & 0 & 0 & 12I_z & 0 \\
0 & 6I \cdot l & 0 & 4I \cdot l^2 & 0 & -6I \cdot l & 0 & 2I \cdot l^2 \\
-A \cdot l^2 & 0 & 0 & 0 & A \cdot l^2 & 0 & 0 & 0 \\
0 & 12I & 0 & -6I \cdot l & 0 & 12I & 0 & -6I \cdot l \\
0 & 0 & -12I & 0 & 0 & 0 & 12I & 0 \\
0 & 6I \cdot l & 0 & 2I \cdot l^2 & 0 & -6I \cdot l & 0 & 4I \cdot l^2
\end{bmatrix}$$

and $R1$ is of the next form, excepting the element **4-5** for which $R1$ equals with the unity matrix(6x6),

$$R1=\begin{bmatrix}0&1&0&0&0&0&0&0\\-1&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&0&1&0&0\\0&0&0&0&-1&0&0&0\\0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&1\end{bmatrix}$$

and then we use (5) to obtain ks in a datum coordinate system, hence $\overline{ks}$.

Now consider the bars and for this we have (1) and

$$R2(x_1,y_1,z_1,x_2,y_2,z_2)=$$

$$\begin{bmatrix}\frac{x_1-x_2}{l}&\frac{y_1-y_2}{l}&\frac{z_1-z_2}{l}&0&0&0\\0&0&0&\frac{x_1-x_2}{l}&\frac{y_1-y_2}{l}&\frac{z_1-z_2}{l}\end{bmatrix}$$

and with (6) we calculate $\overline{kf}$.

For our example, we find the stiffness matrix for the complete structure with next relations.

1. If $1\le N1\le 8$, $N2\le 8$, $i=1,2,..,6$, $j=1,2,..,6$ for the beam elements and $i=1,2,3$, $j=1,2,3$ for the stiffeners:

$$K[6(N1-1)+i,6(N1-1)+j]=$$
$$=K[6(N1-1)+i,6(N1-1)+j]+\bar{k}_1(i,j)$$
$$K[6(N1-1)+i,6(N2-1)+j]=$$
$$=K[6(N1-1)+i,6(N2-1)+j]+\bar{k}_2(i,j)$$
$$K[6(N2-1)+i,6(N1-1)+j]=$$
$$=K[6(N2-1)+i,6(N1-1)+j]+\bar{k}_3(i,j)$$
$$K[6(N2-1)+i,6(N2-1)+j]=$$
$$=K[6(N2-1)+i,6(N2-1)+j]+\bar{k}_4(i,j)$$

2. If $1\le N1\le 8$, $N2\ge 9$, $i=1,2,3$, $j=1,2,3$ for the cable:

$$K[6(N1-1)+i,3(N2-1)+48+j]=$$
$$=K[6(N1-1)+i,3(N2-1)+j+48]+\bar{k}_2(i,j)$$
$$K[6(N1-1)+i,6(N1-1)+j]=$$
$$=K[6(N1-1)+i,6(N1-1)+j]+\bar{k}_1(i,j)$$
$$K[3(N2-1)+48+i,6(N1-1)+j]=$$
$$=K[3(N2-1)+48+i,6(N1-1)+j]+\bar{k}_3(i,j)$$
$$K[3(N2-1)+48+i,3(N2-1)+48+j]=$$
$$=K[3(N2-1)+48+i,3(N2-1)+48+j]+\bar{k}_4(i,j)$$

where we consider for each element, that the $\overline{ks}$ or $\overline{kf}$ matrix is of the form

$$\begin{bmatrix}\bar{k}_1&\bar{k}_2\\\bar{k}_3&\bar{k}_4\end{bmatrix}\qquad(10)$$

Writing a program in MathCAD, where we used the relations (9) and the division matrix method, we find the displacements in step 1 (all cables):

$$\begin{bmatrix}U_2\\V_2\\W_2\\\theta_2\\U_3\\V_3\\W_3\\\theta_3\\U_4\\V_4\\W_4\\\theta_4\end{bmatrix}=10^{-4}\cdot\begin{bmatrix}-210\\-1.7\\0\\4.1\\-220\\-2.1\\0\\20\\-460\\-2.13\\0\\30\end{bmatrix},\quad\begin{bmatrix}U_5\\V_5\\W_5\\\theta_5\\U_6\\V_6\\W_6\\\theta_6\\U_7\\V_7\\W_7\\\theta_7\end{bmatrix}=10^{-4}\cdot\begin{bmatrix}-460\\1.7\\0\\-20\\-20\\1.8\\0\\-20\\-21\\1.5\\0\\5.3\end{bmatrix}$$

and the reactions:

- in the nodes **1, 8**:

$$\mathbf{R_1}=\begin{bmatrix}1900\\25450\\0\\-9900\end{bmatrix},\quad\mathbf{R_2}=\begin{bmatrix}2400\\-22400\\0\\11500\end{bmatrix},$$

- in the nodes **9, 10,.....,16**:

$$\begin{bmatrix} X_9 \\ Y_9 \\ Z_9 \\ X_{10} \\ Y_{10} \\ Z_{10} \\ X_{11} \\ Y_{11} \\ Z_{11} \\ X_{12} \\ Y_{12} \\ Z_{12} \end{bmatrix} = \begin{bmatrix} 1395 \\ -1860 \\ 372 \\ 1395 \\ -1860 \\ -372 \\ 961 \\ -1902 \\ 190 \\ 904 \\ -1902 \\ -190 \end{bmatrix}, \quad \begin{bmatrix} X_{13} \\ Y_{13} \\ Z_{13} \\ X_{14} \\ Y_{14} \\ Z_{14} \\ X_{15} \\ Y_{15} \\ Z_{15} \\ X_{16} \\ Y_{16} \\ Z_{16} \end{bmatrix} = \begin{bmatrix} 1307 \\ 1862 \\ -372 \\ 1397 \\ 1862 \\ 372 \\ 905 \\ 1906 \\ -191 \\ 905 \\ 1906 \\ 191 \end{bmatrix}.$$

Analyzing the sign of the cable reactions results that these are compressed if $Y > 0$ and in tension if $Y < 0$.

Now we repeat the previous calculus for the structure defined by the **1, 2,….., 12** nodes and the same matrix F. The new values of the displacements and of the reactions are

$$\begin{bmatrix} U_2 \\ V_2 \\ W_2 \\ \theta_2 \\ U_3 \\ V_3 \\ W_3 \\ \theta_3 \\ U_4 \\ V_4 \\ W_4 \\ \theta_4 \end{bmatrix} = 10^{-4} \cdot \begin{bmatrix} -360 \\ -2.7 \\ 0 \\ 8 \\ -360 \\ -3.4 \\ 0 \\ 20 \\ -630 \\ -4 \\ 0 \\ 30 \end{bmatrix}, \quad \begin{bmatrix} U_5 \\ V_5 \\ W_5 \\ \theta_5 \\ U_6 \\ V_6 \\ W_6 \\ \theta_6 \\ U_7 \\ V_7 \\ W_7 \\ \theta_7 \end{bmatrix} = 10^{-4} \cdot \begin{bmatrix} -630 \\ 2 \\ 0 \\ -20 \\ -380 \\ 2 \\ 0 \\ -20 \\ -340 \\ 1.7 \\ 0 \\ -20 \end{bmatrix}$$

$$R_1 = \begin{bmatrix} 3228 \\ 39900 \\ 0 \\ -16810 \end{bmatrix}, \quad R_2 = \begin{bmatrix} 2419 \\ -25450 \\ 0 \\ 13780 \end{bmatrix},$$

$$\begin{bmatrix} X_9 \\ Y_9 \\ Z_9 \\ X_{10} \\ Y_{10} \\ Z_{10} \\ X_{11} \\ Y_{11} \\ Z_{11} \\ X_{12} \\ Y_{12} \\ Z_{12} \end{bmatrix} = \begin{bmatrix} 2252 \\ -3003 \\ 601 \\ 2252 \\ -3003 \\ -601 \\ 1236 \\ -2602 \\ 260 \\ 1236 \\ -2602 \\ -260 \end{bmatrix}.$$

5 Conclusion

A completely procedure for the static analysis of structure which include cable elements has been presented. The algorithm has the great advantage that may be used helpless a finite element program and it may be extended to analyze these composite structures for any external loading. In this case a single modification appears: the matrix *ks* will have the corresponding form to 6 degrees of freedom per node, [5].

References

[1] C. A. Brebbia, C. A. Telles, I.C. Wrobel, *Boundary Element Techniques,* Springer-Verlag, New York, 1984.

[2] O'Brien T., General Solution of suspended Cable Problems, *Journal of the Structural Division*, ASCE, Proceedings, febr. 1967.

[3] Peyrot A. H., Goulois A. M., Analysis of cable structures, *Computers & Structures*, vol.10, pp. 805-813, Pergamon Press Ltd., 1979.

[4] Pissarenko G., Yakovlev A., *Aide memoire de résistance des matériaux*, Editions de Moscou, 1987.

[5] J.S. Przemieniecky, *Theory of matrix structural analysis*, New York, McGraw Hill Book Co., 1968.

[6] O. C. Zienkiewicz, Finite Element *Procedures in the Solution of Plate and Shell*, John Wiley & Sons, Inc., New York, 1965.