2005 WSEAS Int. Conf. on DYNAMICAL SYSTEMS and CONTROL, Venice, Italy, November 2-4, 2005 (pp214-219)

Control and Estimation on Manifolds

KARLHEINZ SPINDLER
Fachhochschule Wiesbaden, Fachbereich 02/AB
Kurt-Schumacher-Ring 18, D-65197 Wiesbaden
GERMANY
SpindlerGG@web.de

Abstract: In this paper we show how differential geometric methods can be used to solve both control and estimation problems on manifolds which result from real-life applications, in particular from problems in spacecraft attitude determination and control. The constructive character of the algorithms presented is emphasized.

Key words: Geometric control theory, optimal control, spacecraft attitude control, statistics on manifolds, estimation of constrained parameters, spacecraft attitude determination.

1. Introduction

Geometric control theory, using differential geometric methods to deal with control problems on arbitrary manifolds (in particular Lie groups and their symmetric spaces), has provided new methods which are particularly useful for motion planning problems in areas such as robotics, spacecraft attitude control and the control of underwater vehicles, but also in quantum computing and related fields, as the kinematics involved in such problems can always be represented as a dynamical system on a Lie group of admissible motions. In this paper we show how geometric methods in optimal control theory can be used to find constructive solutions to various motion planning problems. Our main tool will be the following version of (a weak form of) Pontryagin's Principle (cf. [10], Theorem 2) which is tailored to control problems on Lie groups in which the governing differential equation is invariant under the group (whereas the cost functional can arbitrarily depend on both the state and the controls).

Theorem 1. *Let G be a Lie group with Lie algebra $\mathfrak{g}$ and let $(E_1, \ldots, E_n)$ be a vector space basis of $\mathfrak{g}$. Consider a right-invariant dynamical system $\dot{g}(t) = U(t)g(t)$ evolving on G where $U(t)$ is the sum of a controlled term $\sum_{i=1}^{m} u_i(t)E_i$ and a drift term $\sum_{i=m+1}^{n} u_iE_i$ where $u_{m+1}, \ldots, u_n$ are given constants. Given elements $g_0 \in G$ and $g_1 \in G$, controls $t \mapsto u_i(t)$ (where $1 \leq i \leq m$) are sought which steer the system from $g(t_0) = g_0$ to $g(t_1) = g_1$ while minimizing a cost functional of the form $\int_{t_0}^{t_1} \Phi(g(t), u(t), t)\mathrm{d}t$. If $t \mapsto u^*(t)$ is an optimal control and if $t \mapsto g^*(t)$ is the resulting state trajectory in G so that $\dot{g}^*(t) = U^*(t)g^*(t)$ then there exist a curve $t \mapsto p^*(t)$ in $\mathfrak{g}^* \setminus \{0\}$ and a number $\varepsilon \leq 0$ such that*

$$\dot{p}^*(t) = -\varepsilon \frac{\partial \Phi}{\partial g}\big(g^*(t), u^*(t), t\big) - p^*(t) \circ \mathrm{ad}\big(U^*(t)\big),$$

$$p^*(t)E_i \;=\; -\varepsilon \frac{\partial \Phi}{\partial u_i}\big(g^*(t), u^*(t), t\big) \quad (1 \leq i \leq m).$$

The control laws derived from Theorem 1 are in open-loop form. As soon as one converts them to closed-loop form, one is faced with the problem of feeding data into the loop which are corrupted by measurement noise which needs to be filtered out. Since the state variables are elements of a nonlinear manifold, this immediately leads to questions concerning statistics on manifolds. Of particular interest to us is the problem of estimating system parameters which are constrained to lie on a manifold whose dimension is lower than that of the space in which the system equations are formulated. Typical examples arising in space flight dynamics applications are as follows:

- the determination of attitude quaternions q_1, q_2, q_3, q_4 which have to satisfy the constraint $q_1^2+q_2^2+q_3^2+q_4^2 = 1$;
- the simultaneous determination of the rotational motion, the locations of observable landmarks and the gravitational potential of a small solar system body such as a comet, where the relevant parameters are not independent, but satisfy functional relations;
- the simultaneous determination of gravitational constants of solar system bodies and of the scaling factors used to convert position and velocity data from ephemerides to the units used in the orbit determination program (cf. the solar and the lunar constraint discussed in [8], pp. 25/26, 111/112, 120/121).

In these situations the question arises how the constraints can be properly incorporated into a classical estimation scheme in which, either through least-squares batch processing or through Kalman filtering, linear updates to pre-existing estimates are calculated. One possible approach is outlined in this paper.

2. Attitude Control

The *attitude* or *orientation* of a spacecraft (modelled as a rigid body) is the matrix $g \in \mathrm{SO}(3)$ whose rows are the directions of the body's principal axes with respect to some reference coordinate system. Let us denote by I_1, I_2, I_3 the moments of inertia, by $\omega_1, \omega_2, \omega_3$ the angular velocities and by T_1, T_2, T_3 the exerted torques about the principal axes. Then the attitude kinematics of the spacecraft are described by the equation

$$\dot{g}(t) \;=\; (\omega_1(t)E_1 + \omega_2(t)E_2 + \omega_3(t)E_3)\, g(t)$$

where

$$E_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, E_2 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, E_3 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

whereas the dynamics are governed by Euler's equations

$$\begin{aligned} I_1\dot{\omega}_1(t) &= (I_2 - I_3)\,\omega_2(t)\,\omega_3(t) \;+\; T_1(t), \\ I_2\dot{\omega}_2(t) &= (I_3 - I_1)\,\omega_3(t)\,\omega_1(t) \;+\; T_2(t), \\ I_3\dot{\omega}_3(t) &= (I_1 - I_2)\,\omega_1(t)\,\omega_2(t) \;+\; T_3(t). \end{aligned}$$

It is clear that the kinematical equation has the form of the state equation considered in Theorem 1, namely that of a right-invariant system on a Lie group. In the following subsections we present different scenarios in which Theorem 1 can be used to devise control laws to produce desirable attitude motions of the spacecraft.

2.1 Optimality of Eigenaxis Slews

An *eigenaxis slew* is an attitude manœuvre during which the direction of the angular velocity vector remains fixed in space. Eigenaxis slews are routinely employed because they are easily understood geometrically and can also be easily implemented in attitude control systems. The following result shows that they are also characterized by a certain optimality condition.

Theorem 2. *Suppose that the angular velocities of a rigid body about all three axes can be controlled. If the body is steered from an initial attitude $g(t_0)$ to a target attitude $g(t_1) = g_1$ via controls $t \mapsto \omega_i^\star(t)$ (where $i = 1,2,3$) which minimize a cost functional*

$$\int_{t_0}^{t_1} q(t)\big(\omega_1(t)^2 + \omega_2(t)^2 + \omega_3(t)^2\big)\mathrm{d}t$$

then there are constants c_i such that $\omega_i^\star(t) = c_i/q(t)$ for $1 \le i \le 3$.

Proof. See [10], Corollary 3. ∎

For the functions $\omega_i^\star$ in Theorem 2, the attitude kinematical equation can be explicitly integrated, yielding

$$g^\star(t) \;=\; \exp\big(\big(Q(t) - Q(t_0)\big)C\big)\, g_0$$

where Q is any antiderivative of $1/q$ and where $C = c_1E_1 + c_2E_2 + c_3E_3$. Thus the target condition $g(t_1) = g_1$ becomes an equation for the three unknowns c_i which can be solved analytically. The result is as follows.

Theorem 3. *In the situation of Theorem 2, let $\gamma := g_1 g_0^{-1}$ and $a := \arccos\big((\mathrm{tr}[\gamma] - 1)/2\big)$ and let Q be any antiderivative of Q. Then the optimal controls are given by the condition*

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \;=\; \frac{a}{2\sin(a)\big(Q(t_1) - Q(t_0)\big)} \begin{bmatrix} \gamma_{23} - \gamma_{32} \\ \gamma_{31} - \gamma_{13} \\ \gamma_{12} - \gamma_{21} \end{bmatrix}.$$

Proof. See [10], Theorem 4. ∎

2.2 Underactuated Axisymmetric Spacecraft

Here we consider the problem of manœuvring an axisymmetric spacecraft in the case that no torques about the symmetry axis can be exerted, for example due to a thruster failure. An application of Theorem 1 in this case yields explicit control laws which steer such a spacecraft between states of pure spin between prescribed attitudes.

Theorem 4. *Suppose that the angular velocities of a rigid body satisfying $I_1 = I_2$ about the first two axes can be controlled whereas the body spins with constant angular velocity ω_3 about the third axis. Let q be a positive weight function and let Q be an antiderivative of $1/q$. If the controls $t \mapsto \omega_i^*(t)$ (where $i = 1,2$) steer the body from an initial attitude $g(t_0) = g_0$ to a target attitude $g(t_1) = g_1$ while minimizing the cost functional*

$$\int_{t_0}^{t_1} \Big(q(t)\big(\omega_1(t)^2 + \omega_2(t)^2\big)\Big)\mathrm{d}t$$

then there are constants $r \ge 0$, α and β such that

$$\begin{aligned} \omega_1^*(t) &= r\,\cos\big(\alpha Q(t) - \omega_3 t + \beta\big)/q(t), \\ \omega_2^*(t) &= r\,\sin\big(\alpha Q(t) - \omega_3 t + \beta\big)/q(t). \end{aligned}$$

Proof. See [12], Theorem 3. ∎

To be useful in practice, we have to identify the constants r, α and β in terms of the initial attitude g_0 and the target attitude g_1. This can be done in a constructive and easily implementable way.

Theorem 5. *In the situation of Theorem 4, choose the antiderivative Q of $1/q$ such that $Q(t_1) + Q(t_0) = 0$. Moreover, let $\gamma := g_1 g_0^{-1} \in \mathrm{SO}(3)$. Then the constants r, α and β in Theorem 4 can be determined as follows.*

- *If $\gamma_{33} = 1$ (so that $\gamma_{13} = 0$ and hence $\gamma_{11}^2 + \gamma_{12}^2 = 1$) let φ be such that $\gamma_{11} = \cos\varphi$ and $\gamma_{12} = \sin\varphi$. Then β is arbitrary, and the parameters α and r are given as follows. Let $f := 2\pi/\big(Q(t_1) - Q(t_0)\big)$ and let α_0 be the unique number in the interval $[0, f)$ such that $\alpha_0 \equiv \big(\omega_3(t_1 - t_0) - \varphi\big)/\big(Q(t_1) - Q(t_0)\big)$ modulo f. If $\alpha_0 = 0$, then $r = 0$, and α is arbitrary. If $\alpha_0 > 0$ we have*

$$\alpha = \begin{cases} \alpha_0 - f & \text{if } \alpha_0 \le f/2, \\ \alpha_0 & \text{if } \alpha_0 \ge f/2; \end{cases} \qquad r = \sqrt{(f+\alpha)(f-\alpha)}.$$

• *If* $\gamma_{33} = -1$ *(so that* $\gamma_{13} = 0$ *and hence* $\gamma_{11}^2 + \gamma_{12}^2 = 1$*) let* φ *be such that* $\gamma_{11} = \cos\varphi$ *und* $\gamma_{12} = \sin\varphi$*. Then* $r = \pi/\big(Q(t_1) - Q(t_0)\big)$*,* $\alpha = 0$ *and* $\beta \equiv (\varphi + \omega_3 t_0 + \omega_3 t_1)/2$ *modulo* 2π*.*

• *If* $|\gamma_{33}| < 1$ *then* r*,* α *and* β *can be found by applying the following algorithm:*

Step 1. *Determine the angle* u *(uniquely determined modulo* 2π*) such that*

$$\begin{bmatrix} \cos u \\ \sin u \end{bmatrix} = \frac{1}{1-\gamma_{33}^2} \begin{bmatrix} \gamma_{31} & -\gamma_{32} \\ \gamma_{32} & \gamma_{31} \end{bmatrix} \begin{bmatrix} \gamma_{13} \\ \gamma_{23} \end{bmatrix}$$

and let $\beta := (u+\omega_3 t_0+\omega_3 t_1)/2$ *and* $v := (u+\omega_3 t_1-\omega_3 t_0)/2$*.*

Step 2. *Determine the smallest number* $\theta > 0$ *which satisfies one of the two equations*

$$\begin{aligned} &\frac{1}{\sqrt{1-\gamma_{33}}} \begin{bmatrix} \cos v & \sin v \\ \sin v & -\cos v \end{bmatrix} \begin{bmatrix} \gamma_{31} \\ \gamma_{32} \end{bmatrix} \\ &= \begin{bmatrix} \cos h(\theta) & \mp \sin h(\theta) \\ \pm \sin h(\theta) & \cos h(\theta) \end{bmatrix} \begin{bmatrix} \pm\sqrt{\gamma_{33} - \cos\theta} \\ \sin\theta/\sqrt{1-\cos\theta} \end{bmatrix} \end{aligned}$$

where $h(\theta) := (\theta/2)\sqrt{\gamma_{33} - \cos\theta}/\sqrt{1-\cos\theta}$*; this number lies in the interval* $[\arccos\gamma_{33}, 2\pi - \arccos\gamma_{33}]$.

Step 3. *Let*

$$\begin{aligned} \alpha &:= \frac{\pm\theta}{Q(t_1) - Q(t_0)} \sqrt{\frac{\gamma_{33} - \cos\theta}{1-\cos\theta}} \quad \text{and} \\ r &:= \frac{\theta}{Q(t_1) - Q(t_0)} \sqrt{\frac{1-\gamma_{33}}{1-\cos\theta}} \end{aligned}$$

where in the definition of α *the upper/lower sign has to be chosen if* θ *satisfies the equation with the upper/lower sign in Step 2, respectively.*

Proof. See [12], Theorem 5. ∎

If $\omega_3 \neq 0$ it is not really useful to target for a specific attitude g_1, but rather for any attitude which has a prescribed third row $g_1^T e_3$ (namely a prescribed target direction for the spin axis at the end of the manœuvre). In the next theorem we will show how this can be done in an optimal way. (We emphasize that both Theorem 4 and Theorem 5 are valid no matter whether $\omega_3 = 0$ or $\omega_3 \neq 0$.)

Theorem 6. *Suppose that the angular velocities of a rigid body satisfying* $I_1 = I_2$ *about the first two axes can be controlled whereas the body spins with constant angular velocity* ω_3 *about the third axis. Let* q *be a positive weight function and let* Q *be an antiderivative of* $1/q$ *satisfying* $Q(t_1) + Q(t_0) = 0$*. Moreover, let* d *be a given unit vector. Assume that the controls* $t \mapsto \omega_i^*(t)$ *(where* $i = 1, 2$*) steer the body from an initial attitude* $g(t_0) = g_0$ *to an attitude* $g(t_1) = g_1$ *satisfying* $g_1^T e_3 = d$ *while minimizing the cost functional given in Theorem 4. Let*

$$\begin{bmatrix} \gamma_{31} \\ \gamma_{32} \\ \gamma_{33} \end{bmatrix} := g_0 d = g(t_0) d\,.$$

Then the constants r*,* α *and* β *in the control law in Theorem 4 which give rise to the optimal control can be determined as follows.*

• *If* $\gamma_{33} = 1$ *then* $r = 0$*, and* α, β *are arbitrary.*

• *If* $\gamma_{33} = -1$ *then* $r = \pi/\big(Q(t_1) - Q(t_0)\big)$*,* $\alpha = 0$ *and* β *is arbitrary.*

• *If* $|\gamma_{33}| < 1$ *then, denoting by* v *the angle (uniquely determined modulo* 2π*) such that*

$$\begin{bmatrix} \cos v \\ \sin v \end{bmatrix} = \frac{1}{\sqrt{1-\gamma_{33}^2}} \begin{bmatrix} -\gamma_{32} \\ \gamma_{31} \end{bmatrix},$$

we have $r = \dfrac{\arccos(\gamma_{33})}{Q(t_1) - Q(t_0)}$*,* $\alpha = 0$ *and* $\beta = v + \omega_3 t_0$*.*

Proof. See [12], Theorem 6. ∎

2.3 Slews Avoiding a Forbidden Direction

In many space missions it is important that certain body axes stay away from forbidden space directions, a typical example being the re-pointing of a space telescope from one observation target to the next during which the telescope must not be pointed towards bright objects even for a short time. Typically, control specifications to effect such manœuvres are not determined by the on-board software, but in the control centre and then uplinked to the spacecraft, and avoidance of forbidden directions is ensured in a rather roundabout way by performing, if necessary, not a single slew, but a sequence of slew manœuvres connecting the initial attitude to the target attitude via a sequence of intermediate attitudes chosen in such a way that forbidden directions are avoided. As one step towards a higher level of automation and on-board autonomy, we address the following problem: Steer a rigid spacecraft within a given time interval from rest to rest between prescribed attitudes in such a way that a given body-fixed direction b is guaranteed to avoid a given space-fixed direction d during the motion. We solve this problem using optimal control theory with a cost functional which penalizes both high angular velocities and proximity to forbidden attitudes.

Theorem 7. *Let* $g_0, g_1 \in \mathrm{SO}(3)$ *be given attitudes, let* $[t_0, t_1]$ *be a given time interval, let* $b \in \mathbb{R}^3$ *the body coordinates of a body-fixed direction ("telescope direction") and let* $d \in \mathbb{R}^3$ *be the space coordinates of a given space-fixed direction ("forbidden direction"). Let* $q : (t_0, t_1) \to (0, \infty)$ *be a function which is absolutely continuous on each closed subinterval of* (t_0, t_1)*, satisfies* $q(t) \to \infty$ *as* $t \to t_0$ *and* $t \to t_1$ *and also satisfies the condition* $\int_{t_0}^{t_1} q(t)^{-1} \mathrm{d}t = 1$*. Moreover, let* $\chi : I \to (0, \infty)$ *be an absolutely continuous function defined on some interval* $I \subseteq [-1, 1]$ *containing* $\langle b, g_0 d \rangle$ *and* $\langle b, g_1 d \rangle$*. Let the angular velocities* $t \mapsto \omega_i(t)$ *of a rotating rigid body be such that*

$$\int_{t_0}^{t_1} \chi\big(\langle b, g(t) d \rangle\big) q(t) \big(\omega_1(t)^2 + \omega_2(t)^2 + \omega_3(t)^2\big)\, \mathrm{d}t$$

becomes minimal. Writing $\Omega := (q\omega_1, q\omega_2, q\omega_3)^T$, $c := \langle b, gd\rangle$, $F := \chi'[c]/(2q\chi[c])$ *and* $\varphi := b \times gd$ *(so that in particular* $c(t) = \cos\big(\theta(t)\big)$ *where* $\theta(t)$ *is the angle between the telescope direction and the forbidden direction at any time* t*), we have*

$$\dot\Omega + 2F\langle\Omega,\varphi\rangle\Omega \;=\; F\|\Omega\|^2\varphi\,.$$

Proof. This follows from a direction application of Theorem 1 in conjunction with the bracket relations $[E_1, E_2] = -E_3$, $[E_2, E_3] = -E_1$ and $[E_3, E_1] = -E_2$. ∎

The system derived in Theorem 7 for the function $t \mapsto \Omega(t)$ turns out to be completely integrable.

Theorem 8. *There are constants* C_1, C_2 *and* C_3 *such that along the trajectories of the system* $\dot\Omega + 2F\langle\Omega,\varphi\rangle\Omega = F\|\Omega\|^2\varphi$ *we have*

$$\|\Omega\|^2 = \frac{C_1}{\chi(c)},\ \langle\Omega, b\rangle = \frac{C_2}{\chi(c)},\ \langle\Omega, gd\rangle = \frac{C_3}{\chi(c)}\,.$$

Proof. We first verify that each of the functions $\Phi_1(t) := \|\Omega(t)\|^2$, $\Phi_2(t) := \langle\Omega(t), b\rangle$ and $\Phi_3(t) := \langle\Omega(t), g(t)d\rangle$ is a solution of the differential equation

$$\dot\Phi + 2F\langle\Omega,\varphi\rangle\Phi \;=\; 0\,.$$

But this equation has $\chi[c]$ as an integrating factor, since $2F\langle\Omega,\varphi\rangle$ equals $(\mathrm{d}/\mathrm{d}t)\big(\ln\chi[c]\big)$, and hence can be rewritten in the form $(\mathrm{d}/\mathrm{d}t)\big(\chi[c]\Phi\big) = 0$ which yields $\chi[c]\Phi = C$. ∎

Remark. Plugging the equation $\|\Omega\|^2 = C_1/\chi(c)$ into the cost functional in Theorem 7 yields $\int_{t_0}^{t_1} C_1 q(t)^{-1}\mathrm{d}t = C_1$; hence the constant C_1 indicates the control effort. As a possible comparison manœuvre which avoids the forbidden direction we can always choose an eigenaxis slew with angular velocities $\omega_i(t) = c_i/q(t)$ where the c_i are constants such that $\sqrt{c_1^2 + c_2^2 + c_3^2} < 2\pi$ (see Theorem 3); hence any available upper bound χ^* for $\chi\big(\langle b, g(t)d\rangle\big)$ yields the estimate $C_1 \le 4\pi^2\chi^*$ for C_1.

Theorem 8 can be used to write the optimal angular velocities at any time as a function of the current attitude. Ideally, we would like to plug the expression thus obtained for $\omega(t) = \widehat\omega\big(g(t), t; C_1, C_2, C_3\big)$ into the kinematical equation for the attitude, integrate the resulting differential equation for g (starting from the initial condition $g(t_0) = g_0$) to obtain $t \mapsto g(t; C_1, C_2, C_3)$ and then determine the constants C_i by matching $g(t_1; C_1, C_2, C_3)$ with g_1. Unfortunately, the integration cannot be carried out by quadratures. Therefore, we derive three equivalent scalar equations which, at least in principle, can be used to determine the constants of integration. It turns out that the function $t \mapsto c(t)$ satisfies the differential equation

$$\dot c^2 \;=\; \frac{C_1(1-c^2)\chi[c] + 2C_2C_3c - C_2^2 - C_3^2}{q^2\chi[c]^2}$$

which can be explicitly integrated if χ is suitably chosen; one possible choice is $\chi(x) := 1/(1-x)$. It turns out that for this choice there are only two possibilities for the function c: either it is strictly monotonic (which means that the angle between the telescope axis and the forbidden direction either increases or decreases throughout the manœuvre), or else $\dot c$ is negative for the first part of the manœuvre and positive for the second (which means that the telescope axis first moves away from the forbidden direction for some time and then approaches it again until the end of the manœuvre). The exact determination of the constants C_1, C_2 and C_3 from the initial attitude and the target attitude is too technical to be described (see [13] for details); suffice it to say that the calculations boil down to solving a quartic polynomial equation and making several case-by-case distinctions, with the consequence that the control algorithm does not require iterative methods and hence can be executed in a fixed number of steps which is known *a priori*; this is important if the algorithm is supposed to be autonomously executed by the on-board software. Moreover, since the angle between the telescope direction and the forbidden direction during the manœuvre is never smaller than at the beginning or at the end of the manœuvre, there is no need to prescribe a safety margin for this angle during the manœuvre. This observation also provides the upper bound $\chi^* := \max\{\chi\big(\langle b, g_0d\rangle\big), \chi\big(\langle b, g_1d\rangle\big)\}$ for $\chi\big(\langle b, g(t)d\rangle\big)$. Applying Theorem 7 in conjunction with the *a priori* estimate $C_1 \le 4\pi^2\chi^*$ given in the above remark, we can derive an *a priori* bound on the required torque $t \mapsto T(t)$ in terms of the spacecraft characteristics, the desired attitude change and the manœuvre duration D. Conversely, if there is an actuator constraint $\|T(t)\| \le T_{\max}$ then solving for D yields a duration in which an admissible manœuvre resulting in the desired attitude change can be carried out.

3. Parking a Car

As a second application of the approach proposed in the introduction, we consider the problem of parking a car. With respect to a fixed Cartesian coordinate system, we denote by (x, y) the position of the car's centre of mass and by φ the angle between the car's axis and the horizontal; moreover, we denote by u the velocity and by ω the angular velocity of the car. Then the kinematics are governed by the equations

$$\begin{bmatrix}\dot x(t)\\ \dot y(t)\end{bmatrix} = u(t)\begin{bmatrix}\cos\varphi(t)\\ \sin\varphi(t)\end{bmatrix},\quad \dot\varphi(t) = \omega(t).$$

To exhibit the symmetry of this control system, it is useful to recast it as a control system on the group $G = \mathrm{SE}(2,\mathbb{R})$ of planar motions, which we represent as the matrix Lie group

$$G \;=\; \{\begin{pmatrix} D & 0\\ v^T & 1\end{pmatrix} \mid D \in \mathrm{SO}(2,\mathbb{R}), v \in \mathbb{R}^2\}\,.$$

The Lie algebra $\mathfrak{g}$ of G is spanned by the generators

$$E_0 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ E_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},\ E_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

which satisfy the bracket relations $[E_0, E_1] = -E_2$, $[E_0, E_2] = E_1$ and $[E_1, E_2] = 0$. Associating with each trajectory $t \mapsto \big(x(t), y(t), \varphi(t)\big)$ of the original system the trajectory $t \mapsto g(t)$ in G defined by

$$g(t) := \begin{bmatrix} \cos\varphi(t) & \sin\varphi(t) & 0 \\ -\sin\varphi(t) & \cos\varphi(t) & 0 \\ x(t) & y(t) & 1 \end{bmatrix},$$

we find that

$$\dot g(t) = \big(\omega(t)E_0 + u(t)E_1\big)g(t),$$

which is a right-invariant control system on the Lie group G. We now ask for the controls $t \mapsto u(t)$ and $t \mapsto \omega(t)$ which steer this control system from a given initial state $g(t_0) = g_0$ to a given target state $g(t_1) = g_1$ while minimising a cost functional of the form

$$\int_{t_0}^{t_1} q(t)\big(\alpha u(t)^2 + \beta\omega(t)^2\big)\mathrm{d}t$$

with a given function $q : (t_0, t_1) \to (0, \infty)$ with $q(t) \to \infty$ for $t \to t_0$ and $t \to t_1$ and given positive constants $\alpha, \beta \in \mathbb{R}$. (A typical choice would be the mass of the vehicle for α and its moment of inertia about the z-axis for β.) An application of Theorem 1 leads to the following result.

Theorem 9. *If the control functions $t \mapsto u(t)$ and $t \mapsto \omega(t)$ for the dynamical system considered are such that the above cost functional is minimised and if Q denotes an arbitrary antiderivative of $1/q$, then there are constants E, C and k such that*

$$u(t) = -\sqrt{\beta}\frac{E}{q(t)}\mathrm{sn}_k\left(\frac{\sqrt{\alpha}E}{k}Q(t) + C\right),$$
$$\omega(t) = \sqrt{\alpha}\frac{E}{q(t)}\mathrm{cn}_k\left(\frac{\sqrt{\alpha}E}{k}Q(t) + C\right),$$

where sn_k and cn_k denote Jacobian elliptic functions.

Proof. See [6], Theorem 2. ∎

The determination of the constants E, C and k in terms of the initial and the target state is quite technical; we refer to [6].

4. Quantum Computing

Data which are processed in a quantum computer undergo a series of unitary transformations, and designing a universal quantum computer requires the ability to synthesize arbitrary unitary transformations from simple (one-, two- or three-bit) quantum gates; obtaining such a synthesis is a problem in constructive controllability. The dynamical system to be controlled is defined by the time-dependent Schrödinger equation

$$\dot U(t) = -iH(t)U(t),\ U(0) = \mathbf{1}$$

where the Hamiltonian H can be decomposed as

$$H = H_0 + \sum_{i=1}^{m} u_i(t)H_i$$

with a drift term H_0 internal to the system and externally controllable Hamiltonians H_i. (See [5].) For finite-dimensional quantum systems the Schrödinger equation is a right-invariant system on the (special) unitary group. Since synthesizing controls in a time-optimal way seems to be a difficult problem (see [5]), it may be worth studying cost functionals as in Theorem 1 which may be easier to work with and can still be time-efficient. Problems of a similar nature also arise in spectroscopy; see [4].

5. Estimation of Constrained Parameters

Let us consider a system whose evolution depends on a number of parameters $p_1, \ldots, p_n$ which are not independent, but are constrained to lie on an m-dimensional embedded submanifold M of $\mathbb{R}^n$. We are given a number N of noisy measurements which we combine to a measurement vector $\widehat{\mu} \in \mathbb{R}^N$, and we can, for any parameter estimate $p \in M$, compare the actually obtained measurement vector $\widehat{\mu}$ with the theoretically expected measurement vector $\mu(p)$, thereby forming the residual vector $\rho(p) := \widehat{\mu} - \mu(p) \in \mathbb{R}^N$. We try to find an estimate p^* which is optimal in the sense that it minimises the size of the residual vector (in which the entries are weighted according to the assumed quality of the measurements by using as a weighting matrix W the inverse of the measurement covariance matrix). This is typically done in an iterative way in which, in each step, an "old" estimate $p_{\text{old}} \in M$ is replaced by a "new" (and presumably better) estimate p_{new}. We show how this can be done in the situation of constrained parameters, leaving aside all technicalities such as the use of *a priori* information or the use of consider parameters. (Cf. [11].)

Step 1. Calculate the partial derivative matrix $A = \partial\mu/\partial p \in \mathbb{R}^{N\times n}$ at the point p_{old} as if $p_1, \ldots, p_n$ were independent parameters.

Step 2. Find a basis $(v_1, \ldots, v_m)$ of the tangent space $T_{p_{\text{old}}}M$ of M at the "old" estimate $p_{\text{old}} \in M$ and form the matrix $S \in \mathbb{R}^{n\times m}$ whose columns are $v_1, \ldots, v_m$. (In a vicinity of any point $p \in M$, here $p = p_{\text{old}}$, the constraints are given by equations $f_k(p) = 0$ (where $1 \le k \le n - m$) whose gradients are linearly indendent at p and hence can be shown to form a basis of $(T_pM)^\perp$;

thus $v_1, \ldots, v_m$ can be found as a basis of the orthogonal complement of the vector space spanned by the gradients of the functions f_k at the point p.)

Step 3. Postmultiply A by S to obtain the matrix $\widehat{A} := AS \in \mathbb{R}^{N\times m}$ and solve the estimation problem associated with this matrix. In the normal equations formulation, this means computing the vector $\Delta := (\widehat{A}^T W \widehat{A})^{-1} \widehat{A}^T W \rho(p_{\text{old}}) \in \mathbb{R}^m$ which has the covariance matrix $\text{Cov}(\Delta) = (\widehat{A}^T W \widehat{A})^{-1}$.

Step 4. Calculate the increment $\delta p := \sum_{k=1}^m \Delta_k v_k = S\Delta$ and perform the nonlinear update step $p_{\text{new}} := \exp_{p_{\text{old}}}(\delta p)$ where exp is the exponential function of the manifold M (defined by $\exp_p(v) := \gamma(1)$ where γ is the unique geodesic satisfying $\gamma(0) = p$ and $\dot\gamma(0) = v$). The accuracy of the new estimate is given by the covariance matrix $\text{Cov}(\delta p) = S\text{Cov}(\Delta)S^T$.

To see an application in spacecraft attitude determination, let $Q = \{(q_1, q_2, q_3, q_4) \in \mathbb{R}^4 \mid q_1^2 + q_2^2 + q_3^2 + q_4^2 = 1\}$. An orthonormal basis of the tangent space of Q at an element $q = (q_1, q_2, q_3, q_4)$ is given by the columns of the matrix

$$S(q) := \begin{bmatrix} -q_2 & -q_3 & -q_4 \\ q_1 & q_4 & -q_3 \\ -q_4 & q_1 & q_2 \\ q_3 & -q_2 & q_1 \end{bmatrix}.$$

Hence if $\mu(q_1, q_2, q_3, q_4) \in \mathbb{R}^N$ is a measurement vector and if we denote by $A(q) := \big(\partial\mu/\partial(q_1, q_2, q_3, q_4)\big)(q) \in \mathbb{R}^{N\times 4}$ the associated partial derivative matrix, then a typical update step is given as follows: determine $\Delta_1, \Delta_2, \Delta_3$ via

$$\Delta := \begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \end{bmatrix} := (\widehat{A}^T W \widehat{A})^{-1} \widehat{A}^T W \rho$$

where $\widehat{A} := A(q_{\text{old}})S(q_{\text{old}})$, let $\delta q := S(q_{\text{old}})\Delta$, i.e.,

$$\begin{bmatrix} \delta q_1 \\ \delta q_2 \\ \delta q_3 \\ \delta q_4 \end{bmatrix} := \Delta_1 \begin{bmatrix} -q_2 \\ q_1 \\ -q_4 \\ q_3 \end{bmatrix} + \Delta_2 \begin{bmatrix} -q_3 \\ q_4 \\ q_1 \\ -q_2 \end{bmatrix} + \Delta_3 \begin{bmatrix} -q_4 \\ -q_3 \\ q_2 \\ q_1 \end{bmatrix}\Bigg|_{q=q_{\text{old}}}$$

and calculate $\|\delta q\| = \sqrt{\Delta_1^2 + \Delta_2^2 + \Delta_3^2}$; then perform the nonlinear update step

$$q_{\text{new}} := \cos(\|\delta q\|)\, q_{\text{old}} + \sin(\|\delta q\|) \cdot \frac{\delta q}{\|\delta q\|}.$$

(Note that the geodesic γ of Q satisfying $\gamma(0) = q$ and $\dot\gamma(0) = v$ is given by $\gamma(t) = \cos\big(t\|v\|\big)q + \sin\big(t\|v\|\big)v/\|v\|$.) The treatment of estimation problems on more general manifolds requires the development of concepts and methods in the statistics of manifolds; cf. [1], [2], [9], [15].

References:

[1] Yasuko Chikuse: "Statistics on Special Manifolds"; Springer, New York 2003

[2] Nicholas I. Fisher, Toby Lewis, Brian J. J. Embleton: "Statistical analysis of spherical data"; Cambridge University Press 1987

[3] Luc Fraiture, Karlheinz Spindler: "Spherical Goniometry and Spacecraft Attitude Determination"; *Elemente der Mathematik* **57** (2002), pp. 137-157

[4] Dominique P. Frueh, Takuhiro Ito, Jr-Shin Li, Gerhard Wagner, Steffen J. Glaser, Navin Khaneja: "Sensitivity enhancement in NMR of macromolecules by application of optimal control theory"; *Journal of Biomolecular NMR* **32**, 2005, pp. 23-30

[5] Navin Khaneja, Steffen J. Glaser: "Cartan decomposition of SU(2^n) and control of spin systems"; *Chemical Physics* **267** (2001), pp. 11-23

[6] Evgenia Kirillova, Karlheinz Spindler: "Optimal Control of a Vehicle During Parking"; *Proc. Conf. Nonlinear Control Systems (NOLCOS) 2004*, Elsevier Science

[7] Naomi E. Leonhard, Jerrold E. Marsden: "Stability and drift of underwater vehicle dynamics: mechanical systems with rigid motion symmetry", *Physica D* **105** (1-3), 1997, pp. 130-162

[8] Theodore D. Moyer: "Mathematical Formulation of the Double-Precision Orbit Determination Program (DPODP)", Technical Report 32-1527, Jet Propulsion Laboratory, Pasadena 1971

[9] Xavier Pennec: "Probabilities and Statistics on Riemannian Manifolds: A Geometric Approach"; Research Report No. 5093, INRIA Sophia Antipolis, January 2004

[10] Karlheinz Spindler: "Optimal Control on Lie Groups With Applications to Attitude Control", *Mathematics of Control, Signals, and Systems* **11** (3), 1998, pp. 197-219

[11] Karlheinz Spindler: "Treatment of Constrained Parameters in Orbit and Attitude Determination"; *Proc. 15th Int. Symp. Space Flight Dynamics*, June 2000, Paper No. MS00/48

[12] Karlheinz Spindler: "Attitude Control of Underactuated Spacecraft", *European Journal of Control* **6** (3), 2000, pp. 229-242

[13] Karlheinz Spindler: "Attitude Manœuvres which Avoid a Forbidden Direction"; *Journal of Dynamical and Control Systems* **8** (1) 2002, pp. 1-22

[14] Gregory C. Walsh, Richard Montgomery, Shankar S. Sastry: "Optimal Path Planning on Matrix Lie Groups", *Proc. 33rd Conf. on Decision and Control*, Lake Buena Vista, Dec. 1994

[15] Geoffrey S. Watson: "Statistics on Spheres"; Wiley-Interscience, New York 1983