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On dispersion relations for acoustic flows in finite alternating domains

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Abstract

A one-dimensional treatment of solutions to the flow-acoustic equations for a composed medium consisting of a finite number of alternating materials layers A and B (for example water and air) is presented. Special emphasis is given to mathematical properties of the solutions (more precisely, the stability of the "trivial" zero-solution) and the possible wavenumber values. The discussion is intimately related to the discussion of bandgap formation in the dispersion relations as the number of $A-B$ layers n increases towards infinity. Estimates are given for the distribution of wavenumber values (β) as a function of n, in cases when the trivial solution is either stable or unstable. We emphasize that the present analytical analysis allows determination of distributed (in space domain) as well as surface solutions in one calculation.

1 Introduction

In this paper we study the acoustic problem for periodically arranged materials (materials A and B) in the 1-dimensional case, i.e., periodicity being extended only in one direction. We assume that a fluid flows in one or both of the constituent media. We study an acoustic band gap structure for the case, when the domain consists of a finite (yet big) number of layers n. As the periodicity is 1-dimensional, we can use simple analytical techniques to completely describe the dispersion relations.

We first show that the acoustic equations in question are linear Hamiltonian equations with periodic coefficients. Thus, the question of whether the solution for acoustic equations is bounded (in the infinite domain) reduces to the well-known problem of stability of the zero solution of linear Hamiltonian equations: if the zero solution is stable, then there cannot be unbounded solutions. The zero solution is stable, if the absolute value of the trace of the monodromy matrix A is less than 2, and we calculate this matrix for our system.

Suppose that the number of layers n is big enough. Then the asymptotics of the dispersion structure is the following. In the stable areas, where $|Tr(\mathrm{A})| < 2$, the distance between successive solutions is in the order of order $1/n$. In these areas, regions may also exist where this distance can grow to $O(1/\sqrt{n})$. In the unstable areas, where $|Tr(\mathrm{A})| > 2$, the distance between successive solutions is in the order of 1. Both the stable and the unstable regions do not depend on n. An immediate consequence is that for the stable regions the distance between successive solutions is not uniformly distributed.

2 Equations of flow acoustics

Under the monofrequency condition, the linearized equations for the acoustic flow are

$$\begin{aligned} \frac{i\omega}{c^2}p + \frac{v_0}{c^2}\frac{\partial p}{\partial x} + \rho_0\Big(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\Big) = 0, \\ i\omega v_x + v_0\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_0}{\partial y} + v_z\frac{\partial v_0}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x}, \\ i\omega v_y + v_0\frac{\partial v_y}{\partial x} = -\frac{1}{\rho_0}\frac{\partial p}{\partial y}, \\ i\omega v_z + v_0\frac{\partial v_z}{\partial x} = -\frac{1}{\rho_0}\frac{\partial p}{\partial z}, \end{aligned} \tag{1}$$

see Ref. [1] for details. Here $\mathbf{v}_0 = (v_0(y,z),0,0)$ is the background flow, $\rho_0(y,z)$ is the pressure of the background flow, p is the acoustic pressure, and $\mathbf{v} = (v_x, v_y, v_z)$ is the acoustic flow velocity.

We are looking for the solution of the following form:

$$\frac{\partial p}{\partial x} = i\beta p, \quad \frac{\partial \mathbf{v}}{\partial x} = i\beta \mathbf{v}. \tag{2}$$

Substituting relations (2) into equations (1), we get the following expression for the pressure p, cf. Ref. [1]:

$$\frac{1}{\rho_0(\omega+\beta v_0)^2}\frac{\partial^2 p}{\partial y^2}-\frac{1}{\rho_0^2(\omega+\beta v_0)^2}\frac{\partial\rho_0}{\partial y}\frac{\partial p}{\partial y}- \\ -\frac{2\beta}{\rho_0(\omega+\beta v_0)^3}\frac{\partial v_0}{\partial y}\frac{\partial p}{\partial y}+\frac{1}{\rho_0(\omega+\beta v_0)^2}\frac{\partial^2 p}{\partial z^2}- \\ -\frac{1}{\rho_0^2(\omega+\beta v_0)^2}\frac{\partial\rho_0}{\partial z}\frac{\partial p}{\partial z}-\frac{2\beta}{\rho_0(\omega+\beta v_0)^3}\frac{\partial v_0}{\partial z}\frac{\partial p}{\partial z}+ \\ +\Big(\frac{1}{c^2\rho_0}-\frac{\beta^2}{\rho_0(\omega+\beta v_0)^2}\Big)p=0. \qquad (3)$$

We consider now the 1-dimensional case, when the functions ρ_0 and v_0 do not depend on y, and we assume that the pressure p does not depend on y either. We have also assumed periodicity in z, i.e., both v_0 and ρ_0 are 2π-periodic functions of z. Then Equation (3) can be written as

$$\frac{\partial}{\partial z}\Big(\frac{1}{\rho_0(\omega+\beta v_0)^2}\frac{\partial p}{\partial z}\Big)+\Big(\frac{1}{c^2\rho_0}- \\ -\frac{\beta^2}{\rho_0(\omega+\beta v_0)^2}\Big)p=0. \qquad (4)$$

This is the Lagrange equation, which describes the linearized pendulum motion with the vibrating suspension point. We can put Equation (4) into the Hamiltonian form by introducing the canonical conjugate variable

$$q=\frac{1}{\rho_0(\omega+\beta v_0)^2}\frac{\partial p}{\partial z}.$$

Equation (4) now becomes

$$q'=-\frac{\partial H}{\partial p},\quad p'=\frac{\partial H}{\partial q}, \qquad (5)$$

where $(\cdot)'=\partial/\partial z$, and the Hamiltonian equals

$$H(p,q)=\frac{1}{2}\Big(\rho_0(\omega+\beta v_0)^2q^2+\Big(\frac{1}{c^2\rho_0}- \\ -\frac{\beta^2}{\rho_0(\omega+\beta v_0)^2}\Big)p^2\Big).$$

The Hamiltonian system (5) is integrable: this is a consequence of the Lyapunov-Floquet theorem that states that there exists a linear transformation $p,q\to P,Q$, which is 2π-periodic in z, that sends the linear system (5) with 2π-periodic in z coefficients to the linear system with the constant coefficients. However, for arbitrary functions ρ_0 and v_0, one cannot find this variable transformation *explicitly*.

3 The monodromy matrix and the wavevector

To determine the value of the wavevector x-component β, one has to solve Equation (4) with the following boundary conditions: for $z=z_L$ and $z=z_R$, the z-component of the pressure gradient should be zero:

$$(\nabla p,\mathbf{e}_z)=\partial p/\partial z=0.$$

Let $p=p(z,p(z_L),p'(z_L))$ be the solution to Equation (4). Then the equation for β becomes

$$p'(z_R,1,0)=0. \qquad (6)$$

Indeed, this equation is linear in $p(z_L)$, thus this value can be set to 1.

Let n denote the number of the layers of materials A and B. For big values of n, we may expect, that the number of solutions β on a bounded interval may depend on this interval. In the other words, the "density" of solutions on an interval $(\beta_{min}, \beta_{max})$ may vary.

We denote by ρ_0^A, v_0^A and c_A the values of the background density, fluid velocity and the velocity of sound in the material A, and by ρ_0^B, v_0^B, c_a the corresponding values in the material B:

$$\rho_0 = \rho_0^A, \quad v_0 = v_0^A, \quad c = c_A \quad \text{for} \quad z \in [0, z_1),$$
$$\rho_0 = \rho_0^B, \quad v_0 = v_0^B, \quad c = c_a \quad \text{for} \quad z \in [z_1, 2\pi)$$

Both ρ_0^A, v_0^A, c_A, ρ_0^B, v_0^B and c_B are the functions of z, and we assume that they are smooth. However, the functions ρ_0, v_0 and c need not even to be continuous. Equation (4) reads

$$p' = F_1^A q, \quad q' = -F_2^A p, \quad z \in [0, z_1),$$
$$p' = F_1^B q, \quad q' = -F_2^B p, \quad z \in [z_1, 2\pi), \qquad (7)$$

where the functions $F_{1,2}^{A,B}$ equal

$$F_1^A = \rho_0^A (\omega + \beta v_0^A)^2,$$
$$F_2^A = \frac{1}{c_A^2 \rho_0^A} - \frac{\beta^2}{\rho_0^A (\omega + \beta v_0^A)^2},$$
$$F_1^B = \rho_0^B (\omega + \beta v_0^B)^2,$$
$$F_2^B = \frac{1}{c_B^2 \rho_0^B} - \frac{\beta^2}{\rho_0^B (\omega + \beta v_0^B)^2} \qquad (8)$$

When we glue the solutions in the materials A and B together, we demand that *the canonical coordinates* q, p *should be continuous*, rather than p, p' (cf., e.g., [2]). This is important for the derivative

$$\frac{\partial}{\partial z}\Big(\frac{1}{\rho_0(\omega + \beta v_0)^2} \frac{\partial p}{\partial z}\Big)$$

to be defined (in the generalized sense). To this extend, our model closely resembles the Ben Daniel-Duke model.

Remark. One can see that the Hamiltonian formalism is very useful for this kind of systems. First, the canonical variables are continuous, while the derivative of the pressure pressure p' is not. Second, the Hamilton equations are regular when we tend both β and ω to zero, which is important if we want to determine the speed of sound in the composite media.

Our boundary conditions are now $q = 0$ for $z = z_L$, $z = z_R$, and $p = 1$ for $z = z_L$. Let $z_L, z_R \in (0, z_1)$ (mod 2π).

A mapping $p, q|_{z=0} \to p, q|_{z=2\pi}$ is linear and thus can be written as

$$\begin{pmatrix} p(2\pi) \\ \\ q(2\pi) \end{pmatrix} = \mathrm{A} \begin{pmatrix} p(0) \\ \\ q(0) \end{pmatrix},$$

where A is a constant matrix (i.e., $\partial \mathrm{A}/\partial z = 0$). This matrix is called the *monodromy matrix*, and we write it as a product of two matrices A_L and A_R, where the matrix A_L defines the mapping $p, q|_{z=z_L} \to p, q|_{z=2\pi}$, and the matrix A_R defines the mapping $p, q|_{z=0} \to p, q|_{z=z_R}$. The equation for β now reads

$$\begin{pmatrix} 0\,,\,1 \end{pmatrix} \mathrm{A}_R \mathrm{A}^n \mathrm{A}_L \begin{pmatrix} 1 \\ \\ 0 \end{pmatrix} = 0 \qquad (9)$$

4 Wavenumber distribution for the stable case: $|Tr\mathbf{A}| < 2$

We now assume that the trace $|Tr\mathrm{A}| < 2$ for any values of the parameters. Then in the appropriate coordinates the matrix A can be put into the following form:

$$\mathrm{B}^T \mathrm{A} \mathrm{B} = \begin{pmatrix} \cos\psi & \lambda \sin\psi \\ \\ -\frac{1}{\lambda}\sin\psi & \cos\psi \end{pmatrix}$$

where B is the orthogonal matrix. The n-th power A^n becomes now:

$$\mathrm{A}^n = \mathrm{B}(\mathrm{B}^T \mathrm{A} \mathrm{B})^n \mathrm{B}^T = \mathrm{B} \begin{pmatrix} \cos n\psi & \lambda \sin n\psi \\ \\ -\frac{1}{\lambda}\sin n\psi & \cos n\psi \end{pmatrix} \mathrm{B}^T$$

Let us rewrite Equation (9) as

$$\lambda_1(\beta) \cos n\psi(\beta) + \lambda_2(\beta) \sin n\psi(\beta) = 0 \qquad (10)$$

Suppose that $\lambda_1^2 + \lambda_2^2 \neq 0$ for all β (this is a general case). Then Equation (10) can be written as

$$\sin\Big(n\psi + \arccos\Big(\frac{\lambda_1}{\sqrt{\lambda_1^2 + \lambda_2^2}}\Big)\Big) = 0 \qquad (11)$$

Suppose that functions $\psi, \lambda_1, \lambda_2$ are smooth. The structure of solutions to Equation (6) is described by the following results. In the neighbourhood of a "regular" point, i.e., such β that

$$\frac{\partial\psi}{\partial\beta} \neq 0,$$

the distance between the solutions is at most of order $1/n$. More precisely, the following Theorem is true:

Theorem 1. *Consider an interval L, such that there exists a constant $c > 0$, such that for any $\beta \in L$*

$$|\frac{\partial\psi}{\partial\beta}| \geq c$$

Then there is a constant $C > 0$, that does not depend on n, such that on any interval $(\beta, \beta + C/n) \in L$ there is at least one solution to Equation (6).

The proof rests on the following Proposition:

Proposition 1. *Let β_0 be such that*

$$\frac{\partial\psi}{\partial\beta}(\beta_0) \neq 0$$

Then there is a constant $\tilde{C} > 0$, that does not depend on n, such that for any $n \gg 1$ there exists a solution $\tilde{\beta}$ to Equation (6) that satisfies

$$|\tilde{\beta} - \beta_0| \leq \frac{\tilde{C}}{n}.$$

Proof. If $(\lambda_1^2 + \lambda_2^2)(\beta_0) = 0$, then $\tilde{\beta} = \beta_0$ and the Proposition is proved. Let now $(\lambda_1^2 + \lambda_2^2)(\beta_0) \neq 0$.

Let $k \in \mathbb{Z}$ be such that

$$|n\psi(\beta_0) + \arccos\Big(\frac{\lambda_1}{\sqrt{\lambda_1^2 + \lambda_2^2}}\Big)(\beta_0) - \pi k| = \alpha < \pi$$

Let $\tilde{\beta} = \beta_0 + \frac{1}{n}\beta_1$. Substituting this relation to Equation (11), we get

$$n\psi(\beta_0) + \frac{\partial\psi}{\partial\beta}(\tilde{\beta}_0)(\beta_1) + \\ + \arccos\Big(\frac{\lambda_1}{\sqrt{\lambda_1^2 + \lambda_2^2}}\Big)(\tilde{\beta}) - \pi k = 0$$

where $\tilde{\beta}_0 \in [\beta_0, \beta_0 + \frac{1}{n^{1/m}}\beta_1]$. Notice that this is an exact equation. One can see that the expression

$$n\psi(\beta_0) + \frac{\partial\psi}{\partial\beta}(\tilde{\beta}_0)\beta_1 + \arccos\Big(\frac{\lambda_1}{\sqrt{\lambda_1^2 + \lambda_2^2}}\Big)(\tilde{\beta}) - \pi k$$

changes sign on the interval $\beta_1 \in [-2\pi/c, 2\pi/c]$, if n is sufficiently big, where $c = |\partial\psi/\partial\beta(\beta_0)|$. Thus, on this interval there is a solution to our equation, and the interval itself does not depend on n. □

If we drop the condition $|\partial\psi/\partial\beta| \geq c$, then the distance between the solutions may be of order $1/\sqrt{n}$ instead of $1/n$: this is the case for a typical "degenerate" point, i.e., such β that

$$\frac{\partial\psi}{\partial\beta} = 0, \quad \frac{\partial^2\psi}{\partial\beta^2} \neq 0.$$

Theorem 2. *Let β_0 be such that*

$$\frac{\partial\psi}{\partial\beta}(\beta_0) = 0, \quad \frac{\partial^2\psi}{\partial\beta^2}(\beta_0) \neq 0$$

Then there is an interval L_1, $mes(L_1) > c/\sqrt{n}$ for some $c > 0$ and the distance between L_1 and β_0 being at most of order $1/n$, such that there are no solutions to Equation (6) in L_1.

The point β_0 needs not to belong the interval L_1: there may be some "exceptional" values of n, such that there are solutions in an $O(1/n)$ of the degenerate point β_0.

Proof. Take $\tilde{\beta} = \beta_0 + \frac{1}{\sqrt{n}}\beta_1$ and substitute into Equation (11). Then we get

$$n\Big(\psi(\beta_0) + \frac{1}{2}\frac{\partial^2\psi}{\partial\beta^2}(\beta_0)\frac{\beta_1^2}{n} + \dots\Big) + \\ + \arccos\Big(\frac{\lambda_1}{\sqrt{\lambda_1^2+\lambda_2^2}}\Big)(\beta_0) + \dots - \pi k = 0$$

As $\partial^2\psi/\partial\beta^2(\beta_0) \neq 0$, one can choose β_1 such that

$$|n(\psi(\beta_0) + \frac{1}{2}\frac{\partial^2\psi}{\partial\beta^2}(\beta_0)\frac{\beta_1^2}{n}) + \\ + \arccos\Big(\frac{\lambda_1}{\sqrt{\lambda_1^2+\lambda_2^2}}\Big)(\beta_0) - \pi k| > 1$$

In the set of such values of β, one can pick up an interval L_1, such that $\mathrm{mes}(L_1) > c/\sqrt{n}$, which follows by construction. □

Proposition 2. *Let β_0 be such that*

$$\frac{\partial\psi}{\partial\beta}(\beta_0) = 0, \quad \frac{\partial^2\psi}{\partial\beta^2}(\beta_0) \neq 0$$

Then there is a set of the parameters of positive Lebesgue measure, such that for any $N > 0$ there is $n > N$ such that there exists a pair of solutions $\tilde\beta_{1,2}$ to Equation (6) that both satisfy

$$|\tilde\beta_{1,2} - \beta_0| \leq \frac{C}{n}$$

where the constant C does not depend on n.

Proof. Let $\partial^2\psi/\partial\beta^2(\beta_0) > 0$. For an arbitrary small constant $c > 0$, there is a set of parameters of positive measure, such that the Diofantine condition

$$|n\psi(\beta_0) + \arccos\Big(\frac{\lambda_1}{\sqrt{\lambda_1^2+\lambda_2^2}}\Big)(\beta_0) - \pi k| > \frac{c}{n} \quad (12)$$

is not satisfied for some sequence of values n. Take such $n \gg 1$, and consider a solution of the form $\tilde\beta = \beta_0 + \frac{1}{n}\beta_1$. As above, we have to solve the equation

$$n\psi(\beta_0) + \frac{1}{2n}\frac{\partial^2\psi}{\partial\beta^2}(\tilde\beta_0)(\beta_1)^2 + \\ + \arccos\Big(\frac{\lambda_1}{\sqrt{\lambda_1^2+\lambda_2^2}}\Big)(\tilde\beta) - \pi k = 0 \quad (13)$$

Neglecting the terms of order $1/n^2$ and higher, we get a quadratic equation for β_1, that always have two real solutions due to condition (12). The rest of the proof is similar to Proposition 1. □

Summarizing the results for the stable case, i.e., when $|Tr\mathrm{A}| < 2$, we see that most of wavenumbers β are $1/n$-close to each other. There also can be gaps in wavenumbers, that are of order $1/\sqrt{n}$ and are located in an $1/\sqrt{n}$-neighbourhood of zeroes of the derivative $\partial\psi/\partial\beta$.

5 Distribution for wavenumbers for the unstable case: $|Tr\mathbf{A}| > 2$

The situation is different in the unstable case: although the gaps have the similar position, they are now of order 1. Moreover, the distance between any two solutions is also of order 1.

The the matrix A can still be put into the eigenvector-form:

$$B^T AB = \begin{pmatrix} \cosh\psi & \lambda\sinh\psi \\ \frac{1}{\lambda}\sinh\psi & \cosh\psi \end{pmatrix}$$

where B is the orthogonal matrix. The n-th power A^n becomes now:

$$A^n = B \begin{pmatrix} \cosh n\psi & \lambda\sinh n\psi \\ -\frac{1}{\lambda}\sinh n\psi & \cosh n\psi \end{pmatrix} B^T$$

As above, we rewrite Equation (9) as

$$\lambda_1(\beta)\cosh n\psi(\beta) + \lambda_2(\beta)\sinh n\psi(\beta) = 0 \qquad (14)$$

Suppose that $\lambda_1^2 - \lambda_2^2 \neq 0$ for all β (this is a general case), and also that $\lambda_1 > 0$. Then Equation (14) can be written as

$$\sinh(n\psi + \phi) = 0, \quad \cosh\phi = \frac{\lambda_1}{\sqrt{\lambda_1^2 - \lambda_2^2}} \qquad (15)$$

For $n \to \infty$, this equation has only solutions that are close to zeroes of the function $\psi(\beta)$.

Proposition 3. *Let β_0 be such that*

$$\psi(\beta_0) = 0, \quad \frac{\partial\psi}{\partial\beta}(\beta_0) \neq 0.$$

Then there is an interval of order 1, containing β_0, such that on this interval there is precisely one solution to Equation (6). This solution is $1/n$-close to β_0.

Proof. We rewrite Equation (15) as

$$\psi(\beta) + \frac{1}{n}\phi(\beta) = 0,$$

and use the implicit function theorem: for $n \to \infty$ this equation has (locally) only one solution $\beta = \beta_0$, as $\partial\psi/\partial\beta(\beta_0) \neq 0$. For big, but finite values of n, the solution will be $1/n$-close to β_0, and it can be found by the same method as in the previous section. □

6 Bandgap structure

Numerical experiments show that the stable and the unstable areas may be mixed with each other. As an example, we consider the water-air mix, and we assume that in each of the media the background density ρ_0, the media background velocity v_0 and the velocity of sound c are constants. The background velocity in air is assumed to be zero. The band gap areas are the sets, where $|Tr(A)| > 2$, see fig.1. Along the horizontal line is the *number* of a solution, i.e., 1-st, 2-nd, ..., 100-th, starting from the first one that is greater than zero. On the vertical line are the corresponding values of β.

The distribution of solutions when $n \to \infty$ is the fol-

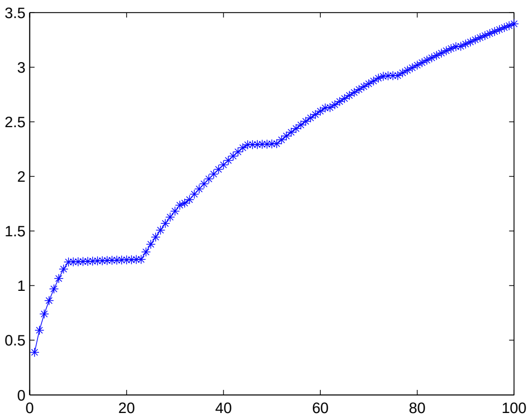

Figure 1: Bond gaps for air-water mix. The seemingly horizontal parts of the graph are the soltions that lie in the stability regions. Notice that these regions are very small compared with the unstable regions.

lowing. In the stable areas, where $|Tr(\mathrm{A})| < 2$, the solutions lie $1/n$-close to each other, and in these areas there also may be regions, where the distance between two successive solutions may grow to $O(1/\sqrt{n})$. In the unstable areas, where $|Tr(\mathrm{A})| > 2$, the distance between two successive solutions is of order 1. It is important to note that the stable and unstable regions themselves do not depend on n.

7 Conclusions

An analytical study of solutions to the flow-acoustic problem for a material composed of alternating layers of materials A and B has been presented. Depending on the trace of the monodromy matrix, that solutions are stable (trace less than 2) or unstable (trace larger than 2). Asymptotics of distances in wavenumber space have been given as the number of $A-B$ layers in the medium goes to infinity, including a discussion of bandgap formation well-known from solid-state physics applications. Finally, the present analysis gives information about possible pressure solutions including distributed and localized pressure eigenstates in space domain.

The present analysis for phononic bandgap structures shows many similarities with the corresponding analysis of photonic bandgap structures. In actual fact under some assumptions, they both obey the Helmholtz wave equation for propagation in homogeneous steady media. Hence, many of the mathematical implications obtained in the present work can to some degree be carried over to photonics applications.

References

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