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A Novel Potential Field based Domain Mapping Method

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Abstract:- A new approach is proposed for mapping a geometrically complicated domain to a simple well-shaped domain using the potential field theory. This bijective *domain mapping* gives a *parameterization* which is useful in handling many application problems like path planning, shape matching, morphing, etc. Harmonic function is chosen for establishing the potential field as it will never have local minima within the domain. This paper presents the domain mapping method and an application of the same to the robot motion planning problem. Potential field along with the streamlines provides two parameters required for representation of any point in the domain. Once the domain is mapped, any post-processing like path planning is easy as the domain of operation is convex. for example, path planning boils down to finding a straight line joining two points in a convex domain and mapping it back to the original domain. Results show that domain mapping is an effective method for shape transformation. For on-line applications, this method is extremely useful since after mapping computational effort required is very less in the query phase.

Key-Words:- Domain mapping, Harmonic function, Path planning.

1 Introduction

This work presents a novel approach called *Domain Mapping*. Domain mapping is the process of establishing a bijective mapping between an irregular domain and a geometrically well-shaped domain. We encounter irregular shapes/domains in many application problems. Shape matching, tiling, motion planning etc. are some of the examples. It is difficult and only few methods exist for handling such problems. So, one way to handle these complex domains is to map the original irregular/non-convex domain to an appropriate chosen well-shaped/convex one, perform all the required mathematical manipulations in the new domain, and transfer the results back to the original one. A sensible domain mapping method should be able to guarantee a bijective mapping between the two domains. It is the "quality" of the mapping which matters most in the entire process.

In this paper, we show our approach to domain mapping and application of the same to robot motion planning problem. Robot motion planning involving the task of planning an optimal path between two points in the workspace of robot without touching obstacles and boundaries, is in general a complicated problem. By using the domain mapping, the problem can be simplified and solved elegantly.

For domain mapping, a Finite Element Method (FEM) approach which treats the given domain as composed of an assemblage of elastic triangular rubber sheets sewn together along their edges, has been tried by the authors' group, Suryawamshi et al [1]. Domain mapping using Artificial Potential Field (APF) approach for two dimensional cases is presented here. For path planning, many artificial potential functions have been proposed by Khatib [2], Barraquand and Latombe [3] etc. Potential field methods have been criticized for their *local minima* problem which causes the robot to reach wrong locations. A remedy for this is suggested by Wang and Chirikjian [4] who used an analogy of the heat transfer problem with variable thermal conductivity. Sundar and Shiller [5] used Hamilton-Jacobi-Bellman theory for establishing a potential field. Another solutions to the local minima problem of APF approach is suggested by Zhnag et al [6] who combined simulated annealing algorithm with APF for path planning problem. Harmonic functions which are solutions of Laplace's equation, completely eliminate local minima, as they satisfy the maximum principle. Elegant properties of harmonic functions attracted many researchers like Connolly et al [7], Kim [8], Alvarez et al [9]. However no one exploited the strength of the harmonic functions completely. We show that potential filed approach withe some modifications can be used for domain mapping. In case of robot motion planning problem, obstacles can be handled by applying different boundary conditions. Application of Dirichlet and Neumann boundary conditions for path planning problem using harmonic functions has been studied by Karnik et al [10]. The above methods use potential field (potential values) to compute gradient and plan the path. But we utilize the potential gradients for tracking streamlines and further for domain mapping with potential value (ϕ) and angle made by the streamline at the centre (θ) as parameters. After mapping, we can perform any geometric operation with ease. Path planning becomes a trivial problem of finding a straight line between two points in a convex domain. Once mapped, any number of paths can be generated without any extra cost.

2 Theory

A potential function with no local minima within the domain guarantees a bijective mapping. Harmonic functions possess this property. A twice differentiable real valued function $f : U \to R$, where $U \subseteq R^n$ is some domain, is called harmonic if its *Laplacian* vanishes on U i.e. if $\nabla^2 f = 0$. In other words, a function satisfying the Laplace's equation,

$$\nabla^2 \phi = \sum_{i=1}^{n} \frac{\partial^2 \phi}{\partial x_i^2} = 0, \qquad (1)$$

where x_i is the i-th Cartesian coordinate and n is the dimension of the domain, is called harmonic function. Laplace's equation arises in many important physical applications such as electrostatics, fluid dynamics etc. Typically, the problem is posed with a set of boundary conditions and the solution is sought for a unique scalar field that satisfies both the differential equation and boundary conditions. Such solution is a harmonic function. Harmonic functions possess interesting properties.

i. Mean value property: If $B(x, r)$ is a ball with centre x and radius r which is completely contained in U, then the value $f(x)$ of the harmonic function at the centre of ball is given by the average of surface of the ball.

ii. Maximum principle: Harmonic function cannot have local extrema. Using the above theorem or otherwise, we can prove this. By definition, at maxima (minima), the function value has to be higher (lower) than the surrounding points, which makes it impossible for the average of surrounding points to be equal to the value of function at that point. Even, without using the above theorem, a glance at Laplace's equation makes it clear. At local extrema all second order partial derivatives i.e. $\frac{\partial^2 \phi}{\partial x_1^2}$, $\frac{\partial^2 \phi}{\partial x_2^2}$ etc will have the same sign, making their sum non-zero and thus not satisfying the Laplace's equation.

Unique properties of the potential field established using harmonic functions, along with streamlines, facilitates the bijective mapping of one domain to the other.

3 Algorithm

3.1 Overview

Domain mapping aims at developing a bijective transformation between the two different regions. We map the original irregular domain to a topologically equivalent domain e.g. circle in 2-D, sphere in 3-D and so on. We need to choose a shape centre before proceeding, hence the mapping is unique with respect to the shape centre, up to a rigid rotation.

The input to our algorithm is a description of the domain. In the case of robot motion planning problem, configuration space[1] (C-space) of robot is the domain. The domain is discretized into pixels for solving the Laplace's equation numerically. After classification of the pixels and applying boundary conditions, potential values are computed. The established potential field allows us to track the streamlines which cut the equi-potential contours orthogonally and make a unique angle (θ) at the centre, which will be one of the two parameters in domain mapping along with the potential value (ϕ). To get a set of θ and ϕ values for each grid point, an interpolation is to be performed. This establishes the complete mapping between the two domains i.e. original irregular 2-D shape to a circle. The values of θ and ϕ always lie between $0^\circ \leq \theta \leq 360^\circ$ and $0 \leq \phi \leq 1$, respectively.

In the new domain, performing any geometric manipulation is easy. Now path planning is equivalent to finding a straight line between two points in a circle. The same is computed and mapped back to the original domain to get the desired path. All these steps are explained in detail in the following sections.

3.2 Pre-processing

The input domain can be a point cloud or an image with the shape centre marked in it. In higher dimensions, a standard algorithm is needed to find the shape centre. Since it is 2-D, we have chosen it manually. When input is a point cloud, each point has to be assigned a pixel in the grid by choosing an appropriate resolution based on input. In case the input is an image, an image processing method is to be used for representing the domain as a grid. In our implementation, we use the MATLAB's *imread* function for reading the image. This function converts the image into a matrix (grid) containing the RGB value of each pixel. To make the processing easier, inner space, outer space, shape centre are coloured differently. Depending on these RGB values and number of neighbours[2] each pixel is flagged as inner/outer/centre pixel.

[1]In the C-space, a point represents a robot configuration.

3.3 Boundary Conditions

After discretization and classification of pixels, we apply the boundary conditions before potential computation. Dirichlet boundary conditions ($\phi = constant$) are used for boundary and shape centre. We assign a low potential value ($\phi = 0$) to shape centre and a high potential value ($\phi = 1$) to the boundary. The pixels which are outside the domain are assigned a potential value of $(1 + \epsilon)$ where ϵ is a small value.

3.4 Potential Computation

Laplace's equation is solved over the domain between the boundary and shape centre to obtain the potential value for inner pixels, iteratively by the following procedure.

Numerical solution for the Laplace's equation can be easily derived from the finite difference method. If $f(x, y)$ is a proposed harmonic function, its second derivative can be derived using Taylor series expansion. After neglecting higher order terms, it is given by

$$\frac{\partial^2 f}{\partial x^2} = \frac{\phi(x_{i+1}, y_i) - 2\phi(x_i, y_i) + \phi(x_{i-1}, y_i)}{h^2} \quad (2)$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\phi(x_i, y_{i+1}) - 2\phi(x_i, y_i) + \phi(x_i, y_{i-1})}{k^2} \quad (3)$$

where h and k are the step sizes used along x and y directions, respectively. If h and k are equal, then the Laplacian over a 2-D discrete domain can be written as,

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{1}{h^2}[\phi(x_{i+1}, y_i) + \phi(x_{i-1}, y_i) + \phi(x_i, y_{i+1}) + \phi(x_i, y_{i-1}) - 4\phi(x_i, y_i)]$$

[2]In a 2-D grid, each pixel is surrounded by eight neighbours, which is used as the criterion to find out pixels on boundary of the domain.

substituting this in Laplace's equation, we obtain,

$$\phi(x_i, y_i) = \frac{1}{4}[\phi(x_{i+1}, y_i) + \phi(x_{i-1}, y_i) + \phi(x_i, y_{i+1}) + \phi(x_i, y_{i-1})].$$

If the potential value at a point (x_i, y_i) in j-th iteration is $\phi_j(x, y)$, in the $(j+1)$-th iteration it is given by

$$\phi_{j+1}(x_i, y_i) = \frac{1}{4}[\phi_j(x_{i+1}, y_i) + \phi_j(x_{i-1}, y_i) + \phi_j(x_i, y_{i+1}) + \phi_j(x_i, y_{i-1})]. \quad (4)$$

The termination criteria is given by

$$max\|\phi_{j+1} - \phi_j\| \leq \zeta \quad (5)$$

where ζ is the tolerance limit. Initially, all inner pixels are assigned a random value[3]. Iteratively updating ϕ at all the pixels using (4), till the termination criteria is met, gives us the potential value at each pixel. The value of ζ depends on the complexity of domain. For the simple convex domains, $\zeta = 0.001$ itself is enough, whereas very low value like, $\zeta = 10^{-7}$ is needed for general domains. For accurate results, ζ should be as low as possible.

Values of ϕ at boundary and shape centre remain same, as they are not altered during the iterations. It is interesting to observe the potential value variation over the domain. The variation is very less near the boundary and it increases as we move towards the shape centre. Near the boundary, there is a change only after fifth or sixth place of decimal. The contours as shown in the next section are convex near the centre and start distorting as we move towards the boundary. Value of ϕ is constant over a contour and serves as a parameter in domain mapping as its value is bounded between [0,1] in the domain.

3.5 Streamline Tracking

Unlike other potential field approaches, we go further ahead and track streamlines, which are nothing but the gradient lines of potentials. These streamlines start from the boundary, approaches the shape centre at unique angle by cutting the equi-potential contours orthogonally. The way in a circle, a radial line from a point on the circumference to the centre enters it at a unique angle and this angle can be used as a representation of that point on boundary, here also the corresponding angle (θ) made by streamline can be used to represent a point on the boundary. The difference is that, a streamline, unlike a radial line, need not be a straight line in the case of a general shape.

So, we track streamlines by starting from a point on the boundary[4], proceed towards the shape centre. When the streamline touches the shape centre, the procedure is terminated and the angle is recorded. This is to be repeated as many times as the number of streamlines needed. This streamline tracking problem boils down to solving an ordinary differential equation (ODE). If $X(t) = [x, y]^T$ is a coordinate vector, then

$$\dot{X}(t) = -\eta \nabla \phi[X(t)] \quad (6)$$

where η is a normalization parameter, represents the streamline equation. This is solved using *ode45* routine of the MATLAB which uses the Runge-Kutta method with adaptive step sizing. In the solution of ODE, we need the gradient value of ϕ at intermediate points of grid. This is handled by fitting a bilinear function,

$$\phi(x, y) = p_1 xy + p_2 x + p_3 y + p_4 \quad (7)$$

in a local neighbourhood. Using the 8 neighbour pixels, a rectangular linear system can be formed. When potentials are computed with enough accuracy, it can be observed that no two streamlines touch or intersect each other, even in a narrow region.

3.6 Mapping

Since potential values are computed over a grid, each grid point has a potential value, which is not

[3] It is observed that, convergence is faster with random value assignment compared to assigning zero potential value to inner pixels.

[4] One can do this in the reverse way (i.e. tracking from center to boundary) also.

the case with the streamline angle (θ). Streamline is a solution of differential equation, hence the angle is available at the time steps chosen while solving ODE. This calls for an interpolation to get the θ value for each grid point. Similar interpolation is needed during the reverse mapping also. These values are stored in a table, which is enough for any post-processing. By this we have a complete mapping between the original general domain and target domain (circle) which means that, for a given (x, y), there exists a unique (ϕ, θ) and vice versa.

3.7 Path Planning (Query)

The streamlines are pre-planned paths, in a way. That way, a feasible path between any two points is readily available after streamline tracking itself, but such path always passes through the shape center making it sub-optimal. So we plan the required optimal paths systematically as explained below.

For path planning between a source (s) and a destination (d) points, first (ϕ, θ) values for the s and d are found by table look-up. Then a straight line $l(\phi, \theta)$ in the new domain (circle) is planned. This straight line is mapped back to the original domain to get the actual path required.

In an irregular grid, interpolation poses some problems near the boundaries. This can be handled by an adaptive interpolation scheme. Another way of path planning without facing this problem when source/destination point is near the boundary is to utilize available streamlines and follow along the nearest streamline till some point away from boundary[5] and from there to plan the path as usual by interpolation. As shown in the results, the planned paths are always away from the domain boundary.

4 Results and Discussions

In this section we present some of the results. In all cases, the boundary conditions are same. Dirichlet boundary conditions are applied on outer boundary and shape centre. Computations are carried out on a PC with Pentium 4 processor. These results are enough for conclusion of domain mapping to path planning. The paths can be further smoothened by choosing more points on the straight line planned in new domain, while reverse mapping. Finer grid representation also helps in better mapping but increases the computational cost.

4.1 Case 1

This domain shown in Fig. 1 shows a C-space of a 2-DOF robot. Paths are planned in the domain from different starting points $(S1, S2, S3, S4)$ to destination points $(D1, D2, D3, D4)$. Fig. 1 shows equi-potential contours and streamlines. It can be observed that contours are distorting more and more as we move from centre to boundary. Only few of the tracked streamlines are shown. As expected, paths are well away from the boundary as shown in Fig. 2.

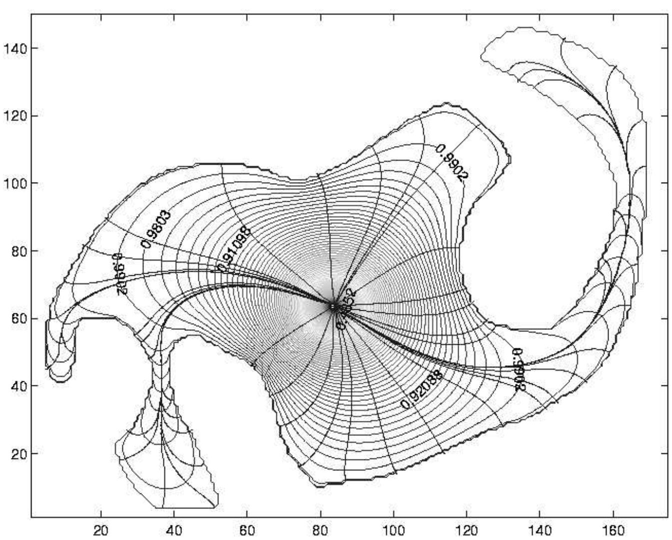

Figure 1: Case 1: Contours and streamlines

4.2 Case 2

This domain is a complicated one and took 11200 iterations for convergence during the potential computation with $\zeta = 10^{-7}$. The domain with the contours and streamlines are shown in

[5]Near the boundary, potential value is high. Roughly, following the streamline till $\phi = 0.95$ is suggested.

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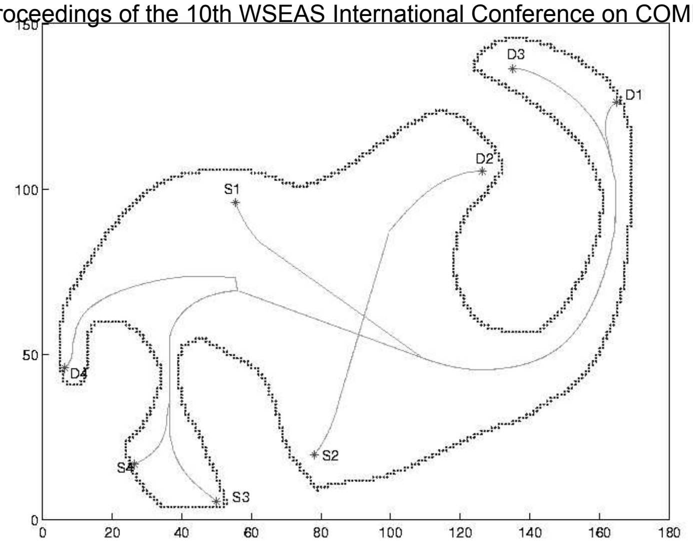

Figure 2: Case 1: Domain with paths

Fig. 3. Even in complicated cases, we can see that streamlines are not touching each other (except at shape center or singularity!). Different paths (GJ to K, J to AP, MZ to J, TN to UP) are planned by choosing different starting and destination points as shown in Fig. 4. Paths are well away from boundary, even when the starting points (MZ, AP) are in a narrow region.

5 Conclusions

Potential field approach is a good method for parameterizing the complex domains. This approach for mapping domains always guaranteed bijectivity. Domain mapping is computationally intensive but it is justified by the advantage it offers in the query phase. For example, path planning, there is no extra cost for planning an extra path. This feature makes this approach suitable for on-line robot motion planning. In a static environment, once the domain mapping is established off-line, path planning for any new task is very quick. A slowly changing dynamic environment can also be handled by updating the potential field accordingly.

Further, in principle, this work can be extended to higher dimensions. However, representing the domain with fine discretization will pose practical problems because of exponential

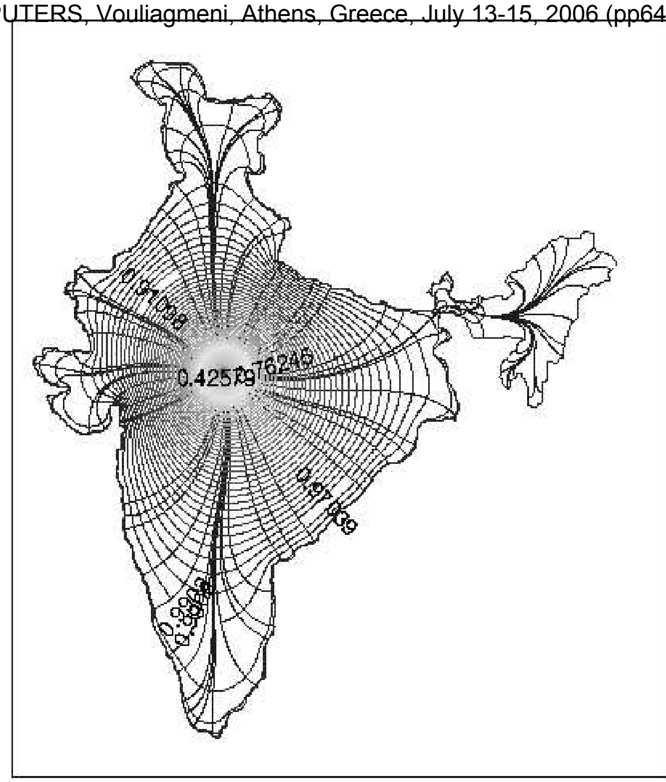

Figure 3: Case 2: Domain with contours and streamlines

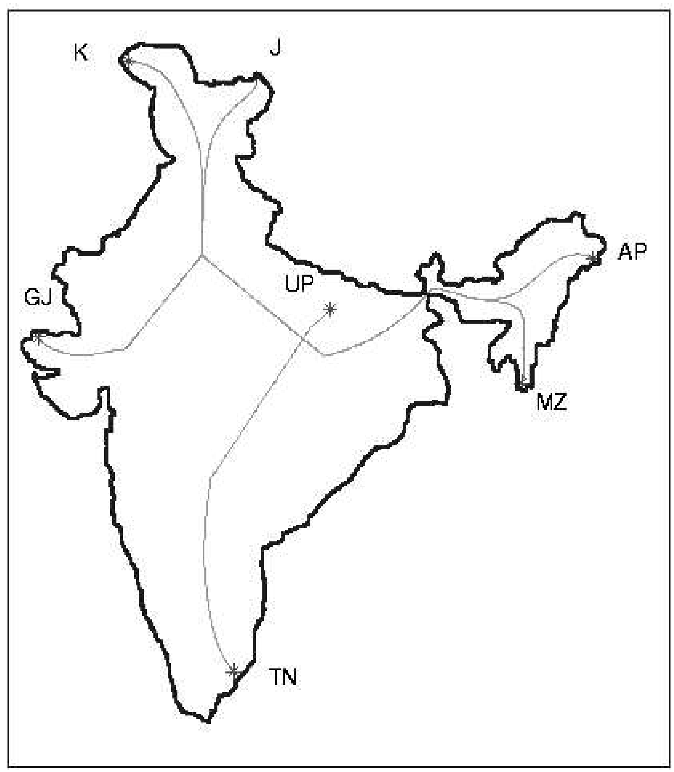

Figure 4: Case 2: Domain with different paths

complexity. Either Neumann or Dirichlet conditions *alone* are not sufficient for handling obstacles present in the C-space of a robot. However a two-stage approach involving successive application of Neumann and Dirichlet conditions will reduce obstacle to a radial line in the first stage and then to a point in the second stage.

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