\[ A, B > 0 \in \mathbb{R}^{n \times n} \text{ assures } \left( \frac{\text{tr}(A)}{n} \right)^{\frac{p}{n-1}} + \left( \frac{\text{tr}(B)}{n} \right)^{\frac{p}{n-1}} \leq \left( \frac{\text{tr}(A + pB)}{n} \right)^{\frac{p}{n-1}} \]

\leq \left( \frac{\text{tr}(A^p + B^p)}{n} \right)^{\frac{1}{n-1}} \text{ for } 1 \leq p < \infty

1 Introduction

The notion of mixed determinants for positive definite symmetric matrices gives an analog notion to the notion of mixed volumes for convex bodies in convex geometry. Many convex geometry notions and inequalities have been developed for more than a century (for references see Lutwak [4] and Schneider [6]). Also many notions in convex geometry have analogs in linear algebra and matrix theory (see [7, 8, 9] for some of such examples). These linear algebra (or matrix) analog notions can be employed with other notions of the theory of matrix inequalities in order to obtain new inequalities for positive definite symmetric matrices.

In this paper we employ the notion of mixed determinant for positive definite symmetric matrices. In particular, we use a unital positive linear map \( \Phi : A \mapsto \frac{A}{n} \) in conjunction with an operator concave function \( f : t \mapsto t^{1/p} \), \( p \geq 1 \) to obtain

\[ \left( \frac{\text{tr}(A + pB)}{n} \right)^{\frac{p}{n-1}} \leq \left( \frac{\text{tr}(A^p + B^p)}{n} \right)^{\frac{1}{n-1}} \]

and we employ the matrix analog of Firey’s extension of Brunn-Minkowski inequality:

\[ W_i^{\frac{p}{n-1}} (A + pB) \geq W_i^{\frac{p}{n-1}} (A) + W_i^{\frac{p}{n-1}} (B) \]

\[ \forall A, B > 0 \in \mathbb{R}^{n \times n}, 0 \leq i \leq n - 1, p \geq 1 \]

for the case \( i = 1 \) to obtain

\[ \left( \frac{\text{tr}(A)}{n} \right)^{\frac{p}{n-1}} + \left( \frac{\text{tr}(B)}{n} \right)^{\frac{p}{n-1}} \leq \left( \frac{\text{tr}(A + pB)}{n} \right)^{\frac{p}{n-1}} . \]

The main tools used here are the Aleksandrov inequality [1, 4, 6, 7], and operator monotonicity, operator convexity and operator concavity as found in the matrix inequalities book by Zhan [10].

Key Words: Mixed determinant, Aleksandrov inequality, Matrix analog of Firey’s extension of Brunn-Minkowski inequality, Operator monotone function, Operator convex function, Operator concave function, Unital positive linear map

October 9, 2006
2 Materials and Methods

We begin by stating the definition and axiomatic properties of mixed determinant and then we quote the very useful Aleksandrov inequality in Section 2.1. In Section 2.2 we introduce the concepts of operator monotone, operator convex and operator concave functions. Also we introduce the concepts of unital, positive and linear maps. Then we state the applications and relation between them. We use the materials of Sections 2.1 and 2.2 to obtain our main theorem, Theorem 20 in Section 2.3.

2.1 Cofactor Matrix Mixed Determinants and Aleksandrov Inequality

Definition 1 (Positive Semidefinite Symmetric Matrix [9]). An $n \times n$ matrix $A$ is said to be positive semidefinite symmetric if $x^T A x \geq 0$ for all $x \in \mathbb{R}^n$. An equivalent condition is that $A$ is symmetric and have all eigenvalues nonnegative. For symmetric matrices $A, B$ we write $B \preceq A$ or $A \succeq B$ to mean that $A - B$ is positive semidefinite symmetric.

In particular, $A \succeq 0 \in \mathbb{R}^{n \times n}$ indicates that $A$ is positive semidefinite symmetric. If $A$ is positive definite symmetric, that is positive semidefinite symmetric and invertible, we write $A > 0 \in \mathbb{R}^{n \times n}$.

Definition 2 (Cofactor Matrix). The cofactor matrix of $A$, $CA$, is the transpose of the classical adjoint of $A$, thus it is defined by

$$
(CA)_{ij} := (-1)^{i+j} D(A_{ij})
$$

where $A_{ij}$ denotes the $(n - 1) \times (n - 1)$ matrix obtained by deleting row $i$ and column $j$ of matrix $A$.

Definition 3 (Mixed Determinant\cite{7, 8}). Let $A_1, \ldots, A_r$ be $n \times n$ symmetric matrices, $\lambda_1, \ldots, \lambda_r$ be nonnegative real numbers. Then the determinant of $\lambda_1 A_1 + \cdots + \lambda_r A_r$ can be written as

$$
D(\lambda_1 A_1 + \cdots + \lambda_r A_r) = \sum \lambda_1^{i_1} \cdots \lambda_r^{i_r} D(A_{i_1, \ldots, i_n}),
$$

where the sum is taken over all $n$-tuples of positive integers $(i_1, \ldots, i_n)$ whose entries do not exceed $r$. The coefficient $D(A_{i_1, \ldots, i_n})$, with $A_{ik}, 1 \leq k \leq n$ from the set $\{A_1, \ldots, A_r\}$, is called the mixed determinant of the matrices $A_{i_1, \ldots, i_n}$.

Properties of Mixed Determinants \cite{7, 8}: Let $A_1, \ldots, A_n, A, B$ and $B'$ be $n \times n$ symmetric matrices, $\lambda_1, \ldots, \lambda_n$ be nonnegative real numbers.

1. $D(A, \ldots, A, B) = D(A, \ldots, A, B, A)
   \begin{align*}
   &= \cdots \\
   &= D(A, B, A, \ldots, A) \\
   &= D(B, A, \ldots, A).
   \end{align*}

   In fact, the mixed determinant is symmetric in its arguments, so in a larger generality one has:

   \begin{equation}
   D(A, A, \ldots, A, B, \ldots, B) = D(B, B, \ldots, B, A, \ldots, A),
   \end{equation}

   We use the notation $D(A, n - k; B, k)$ to represent any of $D(A, A, \ldots, A, B, \ldots, B)$, $D(B, B, \ldots, B, A, \ldots, A)$ in (1).

2. $D(\lambda_1 A_1, \ldots, \lambda_n A_n) = \lambda_1 \cdots \lambda_n D(A_1, \ldots, A_n)$.

3. $D(A_1, \ldots, A_{n-1}, B + B') = D(A_1, \ldots, A_{n-1}, B)
   \begin{align*}
   &= \cdots \\
   &= D(A_1, \ldots, A_{n-1}, B')
   \end{align*}

   In particular,

   \begin{equation}
   D(A, A_1, \ldots, A_{n-1}, B, B') = D(A, A, \ldots, A_{n-1})
   \begin{align*}
   &= \cdots \\
   &= D(A, A, \ldots, A_{n-1}, B').
   \end{align*}

   The properties in (2) and (3) follow from the $n$-linearity of the mixed determinant.

   One can show that for $n \times n$ symmetric matrices $A$ and $B$:

   \begin{equation}
   D(A, \ldots, A, B) = \frac{1}{n} \begin{pmatrix}
   a_1 \\
   \vdots \\
   a_{n-1} \\
   b_n
   \end{pmatrix} \begin{pmatrix}
   b_1 \\
   a_2 \\
   \vdots \\
   a_n
   \end{pmatrix}
   \end{equation}

   of which the generalization gives an alternative definition of the mixed determinant as in the following remark:

\footnote{The authors choose to quote this definition of mixed determinant in a way analogous to the definition of mixed volume in convex geometry \cite{4, 6}.}
Remark 4 ([7, 8]). A mixed determinant $D(A_1, A_2, \ldots, A_n)$ of $n \times n$ symmetric matrices $A_1, A_2, \ldots, A_n$ can be regarded as the arithmetic mean of the determinants of all possible matrices which have exactly one row from the corresponding rows of $A_1, A_2, \ldots, A_n$. 

We can easily prove that

$$
\lim_{\varepsilon \to 0^+} \frac{D(A + \varepsilon B) - D(A)}{\varepsilon} = nD(A, n - 1; B, 1)
$$

where the last equality was by (4) and

$$
A \cdot B := \sum_{i,j} (A)_{ij} (B)_{ij}
$$

with $(A)_{ij}$ standing for the $ij$th entry of $A$ and $CA$ the cofactor matrix of $A$.

Definition 5 (Matrix Firey $p$ Summation). Let $A, B \in M_n^+$ positive definite symmetric matrices. Then the matrix Firey $p$ sum of $A$ and $B$, denoted $A +_p B$, is defined by

$$
(5) \quad A +_p B := (A^p + B^p)^{1/p}.
$$

The commutativity and the associativity of $+_p$ are obvious.

Definition 6 (Firey $p$ Scalar Multiplication [9]). Let $\varepsilon$ be positive scalar and $A$ be an $n \times n$ positive definite symmetric matrix. Firey $p$ scalar multiplication $\varepsilon \cdot A$ is defined as $\varepsilon \cdot A := \varepsilon^{1/p} A$.

Now we state a well-known theorem for inequalities of mixed determinants of positive definite symmetric matrices. This very useful theorem, which has many applications, is called the Aleksandrov inequality is the matrix version of the Aleksandrov-Fenchel inequality of convex geometry.

Theorem 7 (Aleksandrov Inequality [1, 4, 6, 7]). For any $n \times n$ positive definite symmetric matrices $A$ and $B$. The Aleksandrov inequality states that

$$
D^s(A, s + t; \Xi)D^t(B, s + t; \Xi) \leq D^{s+t}(A, s; B, t; \Xi)
$$

where $\Xi$ is any $(n - s - t)$ tuples of matrices. 

2.2 Maps on Matrix Spaces-Operator Monotonicity, Operator Convexity, Operator Concavity and Unital Positive Linear Map.

Definition 8 (Operator Monotonicity, Operator Convexity, Operator Concavity [2, 10]). A real-valued continuous function $f(t)$ defined on a real interval $\Omega$ is said to be operator monotone if

$$
A \leq B \implies f(A) \leq f(B)
$$

for all such Hermitian matrices $A, B$ of all orders whose eigenvalues are contained in $\Omega$. $f$ is called operator convex if for any $0 < \epsilon < 1$,

$$
f(\epsilon A + (1 - \epsilon) B) \leq \epsilon f(A) + (1 - \epsilon) f(B)
$$

holds for all Hermitian matrices $A, B$ of all orders with eigenvalues in $\Omega$. $f$ is called operator concave if $-f$ is operator convex.

In general we have the following useful integral representations for operator monotone and operator convex functions.

Theorem 9 (Integral Representations of Operator Monotone Function and Operator Convex Function [2, 3, 10]). If $f$ is an operator monotone function on $[0, \infty)$, then there exists a positive measure $\mu$ on $[0, \infty)$ such that

$$
(7) \quad f(t) = \alpha + \beta t + \int_0^\infty \frac{st}{s + t} d\mu(s)
$$

where $\alpha$ is a real number and $\beta \geq 0$. If $g$ is an operator convex function on $[0, \infty)$ then there exists a positive measure $\mu$ on $[0, \infty)$ such that

$$
(8) \quad g(t) = \alpha + \beta t + \gamma t^2 + \int_0^\infty \frac{st^2}{s + t} d\mu(s)
$$

where $\alpha, \beta, \gamma$ are real numbers and $\gamma \geq 0$. 

The three concepts of operator monotone, operator convex and operator concave functions are intimately related. One of such examples is as follows, which we will state without proof:

Theorem 10 (Relation between Operator Monotone and Operator Concave Functions [10]). A nonnegative continuous function on $[0, \infty)$ is operator monotone if and only if it is operator concave.

Definition 11 (Positive Map [2, 10]). A map $\Phi : M_n \to M_m$ is called positive if it maps positive semidefinite matrices to positive semidefinite matrices: $A \geq 0 \implies \Phi(A) \geq 0$. 

Definition 12 (Unital Map [2, 10]). Denote by $I_n$ the identity matrix in $M_n$. A map $\Phi : M_n \rightarrow M_m$ is called unital if $\Phi(I_n) = I_m$.

Definition 13 (Linear Map). A map $L : M_n \rightarrow M_m$ is called linear if for all $A$ and $B$ in $M_n$ and every scalar $\alpha$

$$T(\alpha A + B) = \alpha T(A) + T(B).$$

For the following material in this subsection we follow the treatment as in [10]. We first derive some inequalities involving unital positive linear maps with operator monotone functions and with operator convex functions, then use these results to obtain our main inequalities of Theorem 20 in the next section.

Lemma 14 ([10]). Let $A > 0$. Then

$$(A \quad B^* \quad C) \geq 0$$

if and only if the Schur complement

$$C - B^* A^{-1} B \geq 0.$$ 

Lemma 15 ([10]). Let $\Phi$ be a unital positive linear map from $M_m$ to $M_n$. Then

$$\Phi(A^2) \geq \Phi(A)^2 \quad (A \geq 0),$$

$$\Phi(A^{-1}) \geq \Phi(A)^{-1} \quad (A > 0).$$

□

Theorem 16 ([10]). Let $\Phi$ be a unital positive linear map from $M_m$ to $M_n$ and $f$ an operator monotone function on $[0, \infty)$. Then for every $A \geq 0$,

$$f(\Phi(A)) \geq \Phi(f(A)).$$

□

Theorem 17 ([10]). Let $\Phi$ be a unital positive linear map from $M_m$ to $M_n$ and $g$ an operator convex function on $[0, \infty)$. Then for every $A \geq 0$,

$$g(\Phi(A)) \leq \Phi(g(A)).$$

□

2.3 Main Results

Since $D(I, n - 1; A, 1) = \frac{1}{n} \mathbb{E} I \cdot A = \frac{1}{n} I \cdot A = \frac{1}{n} \text{tr} A$ and $D(I, n - 1; I, 1) = 1$ then the map $\Phi : A \mapsto D(I, n - 1; A, 1)I$ is unital positive linear.

Let $\Phi(A) = \sum_{i=1}^{n-1} \lambda_i$ be the eigenvalues of $A$ in increasing order.

$$f : t \mapsto t^{1/p}, p \geq 1$$

is operator concave since we have (see [9] Theorem 16)

$$(\varepsilon X + p (1 - \varepsilon) Y) \geq \varepsilon X + (1 - \varepsilon) Y, \quad 0 < \varepsilon < 1,$$

for all $X, Y > 0 \in \mathbb{R}^{n \times n}$ where $\varepsilon \cdot X := \varepsilon^{1/p} X$ is the Firey $p$ scalar multiplication. This is equivalent to

$$(\varepsilon X^p + (1 - \varepsilon) Y^p)^{1/p} \geq \varepsilon X + (1 - \varepsilon) Y$$

or with $A = X^p$ and $B = Y^p$ we have

$$(\varepsilon A + (1 - \varepsilon) B)^{1/p} \geq \varepsilon A^{1/p} + (1 - \varepsilon) B^{1/p}$$

which means that $f : A \mapsto A^{1/p}, p \geq 1$ is operator concave. By Theorem 10, $f$ is also operator monotone.

Using the fact that if $A, B$ are $n \times n$ positive definite symmetric matrices with $A \succeq B$ then $\lambda_k(A) \succeq \lambda_k(B)$ for all $k = 1, 2, \ldots, n$ if the respective eigenvalues of $A$ and $B$ are arranged in the same (increasing or decreasing) order and the fact that for any matrix $A \in \mathbb{R}^{n \times n}$,

$$W_0(A) = \lambda_1 \lambda_2 \cdots \lambda_n,$$

$$W_1(A) = \frac{1}{n} \sum_{i=1}^{n} \lambda_i,$$

$$\vdots$$

$$W_{n-3}(A) = \frac{1}{n} \sum_{i=1}^{n-3} \lambda_i \lambda_{i+2} \lambda_{i+3},$$

$$W_{n-2}(A) = \frac{1}{n} \sum_{i=1}^{n-2} \lambda_i \lambda_{i+2},$$

$$W_{n-1}(A) = \frac{1}{n} \sum_{i=1}^{n} \lambda_i,$$

where $W_n(A) = D(A, n - i; I, \varepsilon)$ the mixed determinant with $n - i$ copies of $A$ and $i$ copies of $I$.

We can easily see that for any $A, B > 0 \in \mathbb{R}^{n \times n}$

$$W_i(\varepsilon \cdot A + p (1 - \varepsilon) \cdot B) \geq W_i(\varepsilon A + (1 - \varepsilon) B) \geq (\varepsilon W_i(A)^{\frac{1}{n-i}} + (1 - \varepsilon) W_i(B)^{\frac{1}{n-i}})^{n-i},$$

$$0 \leq i \leq n - 1,$$

where the last inequality is by applying the Aleksandrov inequality of Theorem 7.
Theorem 18. If \( p \geq 1, A_0, B_0 > 0 \in \mathbb{R}^{n \times n} \) with \( W_i(A_0) = W_i(B_0) = 1 \) then
\[
W_i(\alpha \cdot A_0 + p (1 - \alpha) \cdot B_0) \geq 1 \quad \forall \alpha \in [0, 1], 
\]
\[0 \leq i \leq n - 1.\]

Proof. Apply (12) to \( A_0, B_0 \) with \( \varepsilon = \alpha. \)

The following theorem is an analog of Firey’s Extension of Brunn-Minkowski theorems (see Lutwak [5] for the original theorem in convex geometry).

Theorem 19 (Matrix Analog of Firey’s Extension of Brunn-Minkowski Inequality [9]). If \( A, B \in M_n, 0 \leq i \leq n - 1, p > 1, \) then
\[
W_i^{\frac{p}{n-i}}(A + p B) \geq W_i^{\frac{p}{n-i}}(A) + W_i^{\frac{p}{n-i}}(B)
\]
with equality if and only if \( A = c \cdot B, c > 0. \)

Proof. We apply Theorem 18 with
\[
A_0 = \frac{1}{W_i(A)^{\frac{p}{n-i}}} \cdot A,
\]
\[
B_0 = \frac{1}{W_i(B)^{\frac{p}{n-i}}} \cdot B,
\]
\[
\alpha = \frac{W_i(A)^{\frac{p}{n-i}}}{W_i(A)^{\frac{p}{n-i}} + W_i(B)^{\frac{p}{n-i}}}
\]
to obtain
\[
W_i \left( \frac{1}{W_i(A)^{\frac{p}{n-i}} + W_i(B)^{\frac{p}{n-i}}} \cdot A 
+ p \frac{1}{W_i(A)^{\frac{p}{n-i}} + W_i(B)^{\frac{p}{n-i}}} \cdot B \right)^{\frac{p}{n-i}} \geq 1
\]
\[
W_i \left[ \frac{A^p + B^p}{W_i(A)^{\frac{p}{n-i}} + W_i(B)^{\frac{p}{n-i}}} \right]^{\frac{1}{n-i}} \geq 1
\]
\[
\frac{1}{W_i(A)^{\frac{p}{n-i}} + W_i(B)^{\frac{p}{n-i}}} W_i(A^p + B^p)^{\frac{1}{n-i}} \geq 1
\]
\[
W_i(A^p + B^p)^{\frac{1}{n-i}} \geq W_i(A)^{\frac{p}{n-i}} + W_i(B)^{\frac{p}{n-i}}
\]
or
\[
W_i((A^p + B^p)^{\frac{1}{n-i}})^{\frac{n-i}{p}} \geq W_i(A)^{\frac{p}{n-i}} + W_i(B)^{\frac{p}{n-i}},
\]
that is,
\[
W_i(A + p B)^{\frac{p}{n-i}} \geq W_i(A)^{\frac{p}{n-i}} + W_i(B)^{\frac{p}{n-i}}.
\]
The equality part can be seen by directly substituting \( A = c \cdot B. \) This completes the proof.

Theorem 20. Let \( A, B \) be \( n \times n \) positive definite symmetric matrices. Then
\[
\left( \frac{\text{tr}(A)}{n} \right)^{\frac{p}{n-1}} + \left( \frac{\text{tr}(B)}{n} \right)^{\frac{p}{n-1}} \leq \left( \frac{\text{tr}(A + p B)}{n} \right)^{\frac{p}{n-1}}
\]
\[
\leq \left( \frac{\text{tr}(A^p + B^p)}{n} \right)^{\frac{1}{n-1}}
\]
for \( 1 \leq p < \infty. \)

Proof. The left inequality was by Theorem 19 with \( i = 1. \) Since \( f(t) = t^{1/p} (p \geq 1) \) is operator concave then by Theorem 10 it is also operator monotone. Hence, by applying Theorem 16 with the unitary positive linear map \( \Phi : A \mapsto D(I, n - 1; A, 1)I, \) we obtain
\[
\Phi(A^{1/p}) \leq \Phi(A)^{1/p}, p \geq 1,
\]
or
\[
D(I, n - 1; A^{1/p})I \leq D(I, n - 1; A)^{1/p}I.
\]
Applying this to \( A^p + B^p, A, B > 0 \in \mathbb{R}^{n \times n}, \) we have
\[
D(I, n - 1; (A^p + B^p)^{1/p})I \leq D(I, n - 1; A^p + B^p)^{1/p}I
\]
or
\[
D(I, n - 1; A + p B) \leq D(I, n - 1; A^p + B^p)^{1/p}.
\]
Since \( D(I, n - 1; A, 1) = \frac{1}{n} \text{tr} A, \) this is equivalent to
\[
\frac{\text{tr}(A + p B)}{n} \leq \left( \frac{\text{tr}(A^p + B^p)}{n} \right)^{1/p}
\]
and
\[
\left( \frac{\text{tr}(A + p B)}{n} \right)^{p/(n-1)} \leq \left( \frac{\text{tr}(A^p + B^p)}{n} \right)^{1/(n-1)}
\]
which is the right inequality. This completes the proof.

Acknowledgements: The first author dedicates this paper to his mother, Poolsuk Pranayanuntana, who was his first teacher of Mathematics and who always concerns of his well being, and to Associate Professor
Chandni Shah, Sep 13, 1959 - Apr 9, 2005, who was like a mother to him.

He wishes to thank Professor Erwin Lutwak, Professor Xingzhi Zhan and Assistant Professor Franziska Berger for some very useful conversations and suggestions.

The second author dedicates this paper to his parents and teachers.

The “one of the best” book by Professor Xingzhi Zhan [10] and such a wonderful publication by Professor Tsuyoshi Ando [2] have inspired both authors so much.

We wish to thank Associate Professor Kobchai Dejhan, Dean of the Faculty of Engineering, KMITL, Associate Professor Theerawat Mongkolaussavarat, Dean of the Faculty of Science, KMITL, for allocating funds for this research. We also thank their Faculty of Engineering and Faculty of Science for the resources provided.

References:


