Approximate Solution of Singular Integro-Differential Equations by Reduction Methods in Generalized Hölder spaces

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Abstract: In present paper we elaborated the numerical schemes of reduction methods for approximative solution of Singular Integro- Differential Equations with kernels of Cauchy type. The equations are defined on the arbitrary smooth closed contours of complex plane. The researched methods are based on the Faber-Laurent polynomials. Theoretical background of reduction methods has been obtained in Generalized Hölder spaces.

Singular Integro- differential equations, Faber-Laurent polynomials, Generalized Hölder spaces

1 Introduction

Singular Integro- Differential Equations with Cauchy kernels (SIDE) model many problems in elasticity theory, aerodynamics, mechanics, thermoelasticity, queueing system analysis, etc. [1]-[4]

The general theory of SIDE has been widely investigated in last decades [5]-[7]. It is well known that the exact solution for SIDE can be found in rare spacial cases. Even in these cases, evaluating the solution numerically can be very complicated and laborious.

In this connection it is interesting to develop the approximative methods for the solution of SIDE with the corresponding theoretical background. Note this problem was studied in [8]-[9]. The SIDE was defined on the unit circle

The case, however, when the contour of integration can be an arbitrary closed smooth curve (not unit circle), has not been studied enough. Transition to another contour, different from the standard one, implies many difficulties. It should be noted that the conformal mapping from the arbitrary contour to the unit circle using the Riemann function does not solve the problem, but only makes it more difficult.

We note that theoretical background of reduction methods for approximate solution of SIDE in classical Hölder spaces has been obtained in [15],[16] and theoretical background of collocation methods for approximate solution of SIDE in Generalized Hölder spaces has been obtained in [17]. The equations have been defined on arbitrary smooth closed contours.

2 Theorems on Approximation of Functions by Partial Sums of Faber-Laurent Series

Let Γ be an arbitrary closed smooth contour bounding a connected domain D^+ in the complex plane, let $z = 0 \in D^+$, and let $D^- = C \setminus \{D^+ \bigcup \Gamma\}$, where C is the complex plane. The class of such contours will be denoted by $\Lambda^{(1)}$.

Let $z = \psi(w)$ and $z = \phi(w)$ be functions conformably mapping the exterior of the unit circle Γ_0 onto D^- and D^+, respectively, such that $\psi(\infty) = \infty$, $\psi^{'}(\infty) > 0$ and $\phi(\infty) = 0$, $\phi^{'}(\infty) > 0$. The inverse functions of $z = \psi(w)$ and $z = \phi(w)$ will be denoted by $w = \Phi(z)$ and $w = F(z)$, respectively. We assume that [10]

$$\lim_{z\to\infty} z^{-1}\Phi(z) = 1 \quad \text{and} \quad \lim_{z\to\infty} zF(z) = 1. \quad (1)$$

It follows from (1) that the functions $w = \Phi(z)$ and $w = F(z)$ admit the following expansions in neighborhoods of the points $z = \infty$ and $z = 0$, respectively:

$$\Phi(z) = z + \sum_{k=0}^{\infty} r_k z^{-k}, \quad F(z) = \frac{1}{z} + \sum_{k=0}^{\infty} v_k z^k.$$

By $\Phi_n(z)$, $n = 0, 1, 2, \ldots$, we denote the set of terms with nonnegative powers of z in the expansion of $\Phi^n(z)$, and by $F_n(1/z)$, $n = 1, 2, \ldots$, we denote the set of terms with negative powers of z in the expansion of $F^n(z)$. The polynomials $\Phi_n(z), n = 0, 1, 2, \ldots$ and $F_n(1/z), n = 1, 2, \ldots,$ are the Faber-Laurent polynomials for the contour Γ [10]. By S_n we denote the reduction operator over the system of Faber-Laurent polynomials:

$$(S_n g)(t) = \sum_{k=0}^{n} a_k^* \Phi_k(t) + \sum_{k=0}^{n} b_k^* F_k(\frac{1}{t}) \quad t \in \Gamma,$$

where a_k^* and b_k^* are the Faber-Laurent coefficients of the function $g(t)$:

$$a_k^* = \frac{1}{2\pi i} \int_{\Gamma_0} \frac{g(\psi(w))}{w^{k+1}}, \quad k = 0, 1, 2, \ldots,$$

$$b_k^* = \frac{1}{2\pi i} \int_{\Gamma_0} \frac{g(\phi(w))}{w^{k+1}}, \quad k = 1, 2, \ldots.$$

3 Numerical Schemes of the Reduction Methods

We introduce the main function spaces and classes in which the equations will be considered.

By $C(\Gamma)$ we denote the space of continuous functions on Γ with norm

$$||g(t)||_C = \max_{t \in \Gamma} |g(t)|.$$

By $C^{(r)}(\Gamma)$, $C^{(0)}(\Gamma) = C(\Gamma)$, we denote the space of r times continuous-differentiable functions on Γ, where r is a nonnegative integer, with norm

$$||g(t)||_{C^{(r)}} = \sum_{k=0}^{r} ||g^{(k)}(t)||_C.$$

By $\omega(\delta)$ ($\delta \in (0, h]$, $h = \mathrm{diam}(\Gamma)$) we denote the arbitrary module of continuity.

By $H_\omega(\Gamma)(= H(\omega))$ we denote the generalized Hölder spaces [11], [14] with norm

$$||g||_\omega = ||g||_C + H(g; \omega), \tag{2}$$

where

$$H(g; \omega) = \sup_{\delta \in (0, h]} \left(\frac{\omega(g; \delta)}{\omega(\delta)} \right),$$

here $\omega(g; \delta)$ is the module of continuity for function $g(t)$ defined on Γ. We consider only the spaces $H(\omega)$ with the module of continuity satisfying the Bari-Stechkin conditions: [11], [14]

$$\int_0^h \frac{\omega(\xi)}{\xi} d\xi < \infty \tag{3}$$

or

$$\int_0^\delta \frac{\omega(\xi)}{\xi} d\xi + \delta \int_\delta^h \frac{\omega(\xi)}{\xi^2} d\xi = O(\omega(\delta)), \delta \to 0. \tag{4}$$

In this case the Cauchy operator

$$(S\phi^*)(t) = \frac{1}{\pi i} \int_\Gamma \frac{\phi^*(\tau)}{\tau - t} d\tau$$

is bounded in $H(\omega)$ [11].

By $H^{(r)}(\omega)$, $r \geq 0$ ($H^{(0)}(\omega) = H(\omega)$ we denote the space of r times continuous-differentiable functions. The $r - th$ order derivatives of these functions are elements of space $H(\omega)$ and the norm in $H^{(r)}(\omega)$ is given by formula

$$||g||_{\omega, r} = ||g(t)||_{C^{(r)}} + H(g^{(r)}; \omega). \tag{5}$$

We note that if $\omega(\delta) = \delta^\alpha$, $\alpha \in (0, 1]$, then $H(\omega) = H_\alpha$ is the classical Hölder space with exponent α.

The space $H(\omega)$ is a Banach nonseparable space. So the approximation of the whole class of functions $H(\omega)$ by the norm (2) with the help of finite-dimensional approximation is impossible. But the problem can be solved in some subset of $H(\omega)$.

Let ω_1 and ω_2 be the modules of continuity satisfying (3) or (4). We suppose that $\tilde{\Phi}(\delta) = \omega_2(\delta)/\omega_1(\delta)$ is nondecreasing and $\lim_{\delta \to 0} \tilde{\Phi}(\delta) = 0$.

The following theorem gives the deviation of reduction operator and function in generalized Hölder spaces [14]:

Theorem 1 *Let ω_1 and ω_2 be two modules of continuity satisfying the conditions (3) or (4). If*

$$\lim_{n \to \infty} \tilde{\Phi}\left(\frac{1}{n}\right) \ln n = 0,$$

then for any function $g(t) \in H(\omega_2)$, the following inequality holds:

$$||S_n g - g||_{\omega_1} \leq c_2 \tilde{\Phi}\left(\frac{1}{n}\right) \ln(n) H(g; \omega_2).$$

where c_2 is a constant.

In complex space $H(\omega)$ we consider the singular integro- differential equation (SIDE)

$$M\varphi \equiv \sum_{r=0}^{q} \Big[c_r(t)\varphi^{(r)}(t) + d_r(t) \frac{1}{\pi i} \int_\Gamma \frac{\varphi^{(r)}(\tau)}{\tau - t} d\tau +$$

$$+ \frac{1}{2\pi i} \int_\Gamma h_r(t, \tau) \varphi^{(r)}(\tau) d\tau \Big] = f(t), \quad t \in \Gamma, \tag{6}$$

where $c_r(t)$, $d_r(t)$, $h_r(t,\tau)$, $r=\overline{0,q}$, $f(t)$ are known functions on Γ; $\varphi^{(0)}(t)=\varphi(t)$ is the unknown function, $\varphi^{(r)}(t)=\dfrac{\varphi^{(r)}(t)}{dt^r}$, $r=\overline{1,q}$, q is an integer.

We search for a solution of equation (6) in the class of functions, satisfying the conditions

$$\frac{1}{2\pi i}\int\limits_{\Gamma}\varphi(\tau)\tau^{-k-1}d\tau=0,\quad k=\overline{0,q-1},\tag{7}$$

on Γ. Using the Riesz operators $P=\frac{1}{2}(I+S)$, $Q=I-P$, (where I is an identical operator and S is a singular operator with Cauchy nucleus) we rewrite the equation (6) in the following form:

$$\sum_{r=0}^{q}\Bigg[a_r(t)P\varphi^{(r)}(t)+b_r(t)Q\varphi^{(r)}(t)+$$

$$+\frac{1}{2\pi i}\int\limits_{\Gamma}h_r(t,\tau)\varphi^{(r)}(\tau)d\tau\Bigg]=f(t),\quad t\in\Gamma,\tag{8}$$

where $a_r(t)=c_r(t)+d_r(t)$, $b_r(t)=c_r(t)-d_r(t)$, $r=\overline{0,q}$. Equation (8) with conditions (7) will be denoted as "problem (8)-(7)".

Let $\overset{o}{H}{}^{(q)}(\omega_2)$ be subspace of space $H^{(q)}(\omega_2)$, the elements from $\overset{o}{H}{}^{(q)}(\omega_2)$ satisfy conditions (7) and the norm is defined as in $H^{(q)}(\omega_2)$. By X_n we denote the subspace of the functions of the form $\sum\limits_{k=-n}^{n}\gamma_k t^k$, $t\in\Gamma$ (where γ_k are complex coefficients) and the norm is defined as in $H(\omega_2)$. By

$$\overset{o}{X}_n=\left\{t^q\sum_{k=0}^{n}\xi_k t^k+\sum_{k=-n}^{-1}\xi_k t^k\right\}$$

(where ξ_k are complex coefficients) and the norm is defined as in $H^{(q)}(\omega_2)$.

We search for approximate solution of problem (8)-(7) in the form of a polynomial

$$\varphi_n(t)=t^q\sum_{k=0}^{n}\alpha_k^{(n)}\Phi_k(t)+$$

$$+\sum_{k=1}^{n}\alpha_{-k}^{(n)}F_k\left(\frac{1}{t}\right),\quad t\in\Gamma,\tag{9}$$

with unknown coefficients $\alpha_k=\alpha_k^{(n)}$, $k=\overline{-n,n}$; obviously, $\varphi_n(t)$ constructed by formula (9) satisfies conditions (7) and so $\varphi_n(t)\in\overset{o}{H}{}^{(q)}(\omega_2)$.

The unknown coefficients α_k, $k=\overline{-n,n}$, are found from the condition $S_n[M\varphi_n-f]=0$, which is treated as the operator equation

$$S_nMS_n\varphi_n=S_nf\tag{10}$$

for the unknown function $\varphi_n(t)$ in the subspace $\overset{o}{X}_n$. Note that Eq. (10) is a system of $2n+1$ linear equations with $2n+1$ unknowns α_k, $k=\overline{-n,n}$, whose explicit form is omitted for being cumbersome. Note that the matrix of this system is determined by the Faber-Laurent coefficients of the functions $a_r(t)$, $b_r(t)$ and

$$\frac{1}{2\pi i}\int\limits_{\Gamma}h_r(t,\tau)\Phi_k(\tau)\,d\tau,\quad k=\overline{0,n},$$

and

$$\frac{1}{2\pi i}\int\limits_{\Gamma}h_r(t,\tau)F_k\left(\frac{1}{\tau}\right)d\tau,\quad k=\overline{1,n},\quad r=\overline{0,q}.$$

In what follows, we give a theoretical background of the reduction methods, i.e., derive conditions providing the solvability of Eq. (10) and the convergence of the approximate solutions (9) to the exact solution $\varphi^*(t)$ of problem (8)-(7).

Theorem 2 *Let the following conditions be satisfied :*

1) *the functions $a_r(t),b_r(t)$, and $h_r(t,\tau)$, $r=0,\dots,q$, belong to the space $H(\omega_2)$;*
2) $a_q(t)b_q(t)\neq 0,\ t\in\Gamma$;
3) $\operatorname{ind}a_q(t)=0,\ \operatorname{ind}b_q(t)=q$;
4) *the operator $M:\dot{H}^{(q)}_{\omega_1}\to H(\omega_1)$ has a bounded inverse.*

If $\lim\limits_{\delta\to+0}\tilde{\Phi}(\delta)\ln^2\delta=0$, then for numbers n large enough, the equation of the reduction methods (10) is uniquely solvable. The approximate solutions $\varphi_n(t)$, given by formula (9) converge in the norm of the space $H^{(q)}(\omega_1)$ as $n\to\infty$ to the exact solution $\varphi^*(t)$ of problem (8)-(7) for an arbitrary right-hand side $f(t)\in H(\omega_2)$. The following relation holds:

$$||\varphi^*-\varphi_n||^{(q)}_{\omega_1}=O\left(\tilde{\Phi}\left(\frac{1}{n}\right)\ln^2 n\right)\tag{11}$$

4 Auxiliary Results

We formulate one result from [12], establishing the equivalence (in sense of solvability) of problem (8)-(7) and the singular integral equation.

Using the integral representations [12]

$$\frac{d^q(P\varphi)(t)}{dt} = \frac{1}{2\pi i}\int_\Gamma \frac{v(\tau)}{\tau - t}d\tau, \tag{12}$$

$$\frac{d^q(Q\varphi)(t)}{dt} = \frac{t^{-q}}{2\pi i}\int_\Gamma \frac{v(\tau)}{\tau - t}d\tau, \quad t \in \Gamma,$$

we reduce problem (8)-(7) to the equivalent singular integral equation (SIE)

$$\Theta v \equiv A_q(t)(Pv)(t) + B_q(t)(Qv)(t) +$$

$$+\frac{1}{2\pi i}\int_\Gamma K(t,\tau)v(\tau)d\tau = f(t), \quad t \in \Gamma, \tag{13}$$

for the unknown function $v(t)$ where

$$A_q(t) = \frac{1}{2}[a_q(t) + t^{-q}b_q(t)],$$

$$B_q(t) = \frac{1}{2}[a_q(t) - t^{-q}b_q(t)],$$

the function $K(t,\tau)$ belongs to the class $H(\omega_2)$. The obvious form can be found in [12]. Note that the right-hand sides in (8) and (13) coincide by virtue of conditions (7).

Lemma *The singular integral equation (13) and problem (8)-(7) are equivalent in the sense of of solvability. That is, for each solution $v(t)$ of Eq. (13), there is the solution $\varphi(t)$ of problem (8)-(7), determined by the following formulas*

$$(P\varphi)(t) = \frac{(-1)^q}{2\pi i(q-1)!}\int_\Gamma v(\tau)\Big[(\tau - t)^{q-1}\ln\Big(1 - \frac{t}{\tau}\Big) +$$

$$+\sum_{k=1}^{q-1}\tilde{r}_k\tau^{q-k-1}t^k\Big]\,d\tau, \tag{14}$$

$$(Q\varphi)(t) = \frac{(-1)^q}{2\pi i(q-1)!}\int_\Gamma v(\tau)\tau^{-q}\Big[(\tau - t)^{q-1}\ln\Big(1 - \frac{\tau}{t}\Big)$$

$$+\sum_{k=0}^{q-2}\tilde{s}_k\tau^{q-k-1}t^k\Big]\,d\tau$$

(where $\tilde{r}_k$, $k = \overline{1, q-1}$ and $\tilde{s}_k$, $k = \overline{0, q-2}$ are real numbers) and vice versa for each solution $\varphi(t)$ of problem (8)-(7), there is the solution $v(t)$

$$v(t) = \frac{d^q(P\varphi)(t)}{dt^q} + t^q\frac{d^q(P\varphi)(t)}{dt^q}$$

for the (13). Furthermore, for a given set of linear-independent solutions of the singular integral equation (13), there are corresponding set of linear-independent solutions of the problem (8)-(7) and vise versa. In formulas (14) $\ln(1 - t/\tau)$ and $\ln(1 - \tau/t)$ are the branches that vanish at the points $t = 0$ and $t = \infty$, respectively)

5 Proof of theorem

We will show that if n is large enough, then the operator S_nMS_n defined in (10) is invertible. The operator acts from the subspace $\overset{o}{X}_n$ to the space X_n .

In a similar way, using the formulas (12), we represent the functions $d^q(P(\varphi_n)(t))/dt^q$, $d^q(Q(\varphi_n)(t))/dt^q$ by Cauchy type integrals with the same density $v_n(t)$:

$$\frac{d^q(P(\varphi_n)(t))}{dt^q} = \frac{1}{2\pi i}\int_\Gamma \frac{v_n(\tau)}{\tau - t}d\tau, \quad t \in F^+,$$

$$\frac{d^q(Q(\varphi_n)(t))}{dt^q} = \frac{t^{-q}}{2\pi i}\int_\Gamma \frac{v_n(\tau)}{\tau - t}d\tau, \quad t \in F^-. \tag{15}$$

Using the formulas $(Px)^{(r)}(t) = P(x^{(r)})(t)$ $(Qx)^{(r)}(t) = Q(x^{(r)})(t)$, $r = 1, 2, \ldots$, and the relations [13]

$$(t^{k+q})^{(r)} = \frac{(k+q)!}{(k+q-r)!}t^{k+q-r}, \quad k = \overline{0, n},$$

$$(t^{-k})^{(r)} = (-1)^r\frac{(k+r-1)!}{(k-1)!}t^{-k-r}, \quad k = \overline{1, n},$$

from (15), we obtain

$$v_n(t) = \sum_{k=0}^{n}\frac{(k+q)!}{k!}t^k\xi_k +$$

$$+(-1)^q\sum_{k=1}^{n}\frac{(k+q-1)!}{(k-1)!}t^{-k}\xi_{-k}$$

Consequently, $v_n(t) \in X_n$; here we have used the fact that the polynomials $\varphi_n(t)$ given by (9) can be represented uniquely in the form

$$t^q\sum_{k=0}^{n}\xi_kt^k + \sum_{k=-n}^{-1}\xi_kt^k.$$

Using of the relations (15), Eq. (10), as well as problem (8)-(7), can be reduced to the equivalent equation (in the sense of solvability)

$$S_n\Theta S_nv_n = S_nf, \tag{16}$$

treated as an equation in the subspace X_n of polynomials with the same norm as in $H(\omega_2)$. Obviously, Eq. (16) is the equation of the reduction methods over Faber-Laurent polynomials for the singular integral equation (13). The reduction methods over Faber-Laurent polynomials for singular integral equations were considered in [14], where sufficient conditions for the solvability and convergence of this method were obtained. Assumptions of Theorem 2 provide the validity of all assumptions of Theorem 1 in [14]; consequently, Eq. (16) with n large enough is uniquely solvable; moreover, the approximate solutions $v_n(t)$ of this equation converge to the exact solution $v^*(t)$ of the singular integral equation (13) in the norm of the space $H(\omega_1)$ as $n \to \infty$:

$$\|v_n - v^*\|_{\omega_1} = O\left(\Phi\left(\frac{1}{n}\right)\ln^2 n\right) \tag{17}$$

The function $\varphi_n(t)$ can be expressed via the function $v_n(t)$ by formulas (14). Then using the definition of the norm in the space $H^{(q)}(\omega_1)$, together with (17), we obtain estimate (11). The proof of Theorem 2 is complete.

6 Conclusions

The results from this article generalized the results from [15], [16].

The theoretical background of reduction methods for approximative solution of SIDE in Generalized Holder spaces has been proved. The equations are defined on the arbitrary smooth closed contours.

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