Geodesic pancyclicity of crossed cubes

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Abstract:- For a pair of vertices $u, v \in V(G)$, a cycle is called a *geodesic cycle* with u and v if a shortest path of G joining u and v lies on the cycle. A graph G is *pancyclic* [12] if it contains a cycle of every length from 3 to $|V(G)|$ inclusive. Furthermore, a graph G is called *geodesic* k*-pancyclic* [3] if for each pair of vertices $u, v \in V(G)$, it contains a geodesic cycle of every integer length of l satisfying $2d_G(u, v) + k \le l \le |V(G)|$. Chang et al. [4] proved that CQ_n is pancyclic in the sense that a cycle of length l exists, $4 \le l \le |V(CQ_n)|$. In this paper, we study a new pancyclic property and show that Crossed cubes is geodesic 4-pancyclic.

Key–Words:- crossed cubes, panconnected, pancyclic, geodesic cycle, geodesic pancyclic.

1 Introduction

In this paper, a network is represented as a loopless undirected graph. For the graph definition and notation we follow [2]. $G = (V, E)$ is a graph if V is a finite set and E is a subset of $\{(u, v) \mid (u, v)$ is an unordered pair of $V\}$. We say that V is the *vertex set* and E is the *edge set*. Two vertices u and v are *adjacent* if $(u, v) \in E$. A path is a sequence of adjacent vertices, written as $\langle v(0), v(1), v(2), \ldots, v(m)\rangle$, in which all the vertices $v(0), v(1), \ldots, v(m)$ are distinct except possibly $v(0) = v(m)$. We also write the path $\langle v(0), P, v(m)\rangle$, where $P = \langle v(0), v(1) \ldots, v(m)\rangle$. The *length* of a path P, denoted by $len(P)$, is the number of edges in P. For convenience, we also use $(v(0), v(m))_i$ to denote the path joining $v(0)$ and $v(m)$ of length i. Let u and v be two vertices of G. The *distance* between u and v, denoted by $d_G(u, v)$ is the length of the shortest path of G joining u and v. The diameter of G is the maximum distance between any pair of vertices on G.

Path embedding problems have attracted much research attention [9, 13]. A graph G is *panconnected* if each pair of distinct vertices u, v are joined by a path of length l, $d_G(u, v) \le l \le |V(G)| - 1$. A *cycle* is a path with at least three vertices such that the first vertex is the same as the last one. A l-cycle is a cycle of length l. A ring structure is often used as an interconnection architecture for local area network and as a control and data flow structure in distributed networks due to its beneficial properties. The ring embedding problem, which deals with all the possible lengths of the cycles, is investigated in a lot of interconnection networks [3, 5, 12, 13]. In general, a graph is *pancyclic* if it contains a cycle of every length from 3 to $|V(G)|$ inclusive.

An n-dimensional crossed cube, CQ_n [6, 7], is a variation of hypercube Q_n and preserves many of its desirable properties. CQ_n has 2^n vertices and $n2^{n-1}$ links, same as hypercube Q_n. However, it has a small diameter $\lceil \frac{n+1}{2} \rceil$, about half that of Q_n.

In this paper, we consider the geodesic cycle embedding problem in CQ_n. The geodesic pancyclic property was proposed recently by Chan et. al. [3]. Herein, we will prove that CQ_n is geodesic 4-pancyclic for $n \ge 3$. The rest of this paper is organized as follows. In the next section we study necessary definitions and discuss some useful properties of the Crossed cubes. Section 3 then shows that CQ_n is geodesic 4-pancyclic. Finally, we present our conclusions and implications.

2 Preliminaries

In this section, we will give the relevant definitions in graph theory and for the Crossed cubes. To define the Crossed cubes, as proposed by Efe [6], the notion of so called "pair related" relation is introduced.

Definition 1 *[6] Let* $R = \{(00, 00), (10, 10), (01, 11), (11, 01)\}$. *Two two-digit binary strings* $u = u_1u_0$ *and* $v = v_1v_0$ *are pair related, denoted as* $u \sim v$, *if and only if* $(u, v) \in R$.

The following is the recursive definition of the n-dimensional Crossed cube CQ_n.

Definition 2 *[6] The Crossed cube* CQ_1 *is a complete graph with two nodes labelled by 0 and 1, respectively. For* $n \geq 2$, *an* n*-dimensional Crossed cube* CQ_n *consists of two* $(n-1)$*-dimensional sub-Crossed cubes,* CQ^0_{n-1} *and* CQ^1_{n-1}, *and a perfect matching between the nodes of* CQ^0_{n-1} *and* CQ^1_{n-1} *according to the following rule:*

Let $V(CQ^0_{n-1}) = \{0u_{n-2}u_{n-3}...u_0 : u_i = 0$ *or* $1\}$ *and* $V(CQ^1_{n-1}) = \{1v_{n-2}v_{n-3}...v_0 : v_i = 0$ *or* $1\}$. *The node* $u = 0u_{n-2}u_{n-3}...u_0 \in V(CQ^0_{n-1})$ *and the node* $v = 1v_{n-2}v_{n-3}...v_0 \in V(CQ^1_{n-1})$ *are adjacent in* CQ_n *if and only if*

(1) $u_{n-2} = v_{n-2}$ *if* n *is even, and*

(2) $(u_{2i+1}u_{2i}, v_{2i+1}v_{2i}) \in R$, *for* $0 \leq i < \lfloor \frac{n-1}{2} \rfloor$.

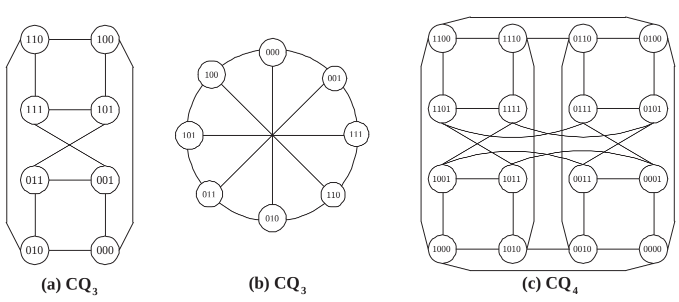

Figure 1: Illustrations of CQ_3 and CQ_4.

A vertex v is the k-dimensional neighbor (abbreviate as k-neighbor) of u, denoted by $v = u^k$, if the left-first different bit of them is k. For a binary bit u_i, $u_i \in \{0, 1\}$, the complement of u_i is denoted by $\overline{u_i} = 1 - u_i$. For convenience, let $u_{(i,j)} = u_iu_{i-1} \ldots u_{j+1}u_j$, $i > j \geq 0$, and let symbol b_i always belong $\{0, 1\}$ throughout this paper.

Let x be an l-bit binary string with $l \leq n$. We use CQ^x_{n-l} to denote the subgraph of CQ_n induced by the set of vertices with the prefix x. It is shown in [10] that CQ^x_{n-l} is isomorphic to CQ_{n-l}. Moreover, for two l-bit binary strings x and y, let $CQ^{\{x,y\}}_{n-l}$ denote the subgraph of CQ_n induced by $CQ^x_{n-l} \bigcup CQ^y_{n-l}$. It is proven in [7] that $CQ^{\{x,y\}}_{n-l}$ is isomorphic to CQ_{n-l+1} if CQ^x_{n-l} and CQ^y_{n-l} are adjacent subgraphs of CQ_n.

Next, Lemmas 1 $\sim$ 3 are useful for verifying the following other results in this paper.

Lemma 1 *[13] Let* u *and* v *be two vertices of* $CQ^{b_1}_{n-1}$, *where* $n \geq 3$. *Then* $d_{CQ_n}(u, v) = d_{CQ^{b_1}_{n-1}}(u, v)$.

Lemma 2 *[6] The diameter of Crossed cube,* CQ_n, *is* $\lceil \frac{n+1}{2} \rceil$.

Lemma 3 *[13] Let* u *and* v *be two vertices of* CQ_n, $n \geq 3$. *Then for every integer* i, $d_{CQ_n}(u, v) + 2 \leq i \leq 2^n - 1$, *the path* $(u, v)_i$ *exists.*

In [4], two reducing strategies of CQ_n were proposed depending on whether n is odd or even. For $n = 2k$, we can can contract those vertices in CQ_{2k} having the same prefix of length two into a vertex and obtain a graph with four vertices. And, this four-vertex graph is isomorphic to CQ_2, as shown in Fig. 2.(a). Similarly, for $n = 2k+1$, we can contract those vertices in CQ_{2k+1} with the same prefix of length three into a vertex and obtain a graph eight vertices. Again, this eight-vertex graph is isomorphic to CQ_3, as shown in Fig. 2.(b). Moreover, for any two vertices u, v in CQ_n, there are some observations on their relative position as the following lemma.

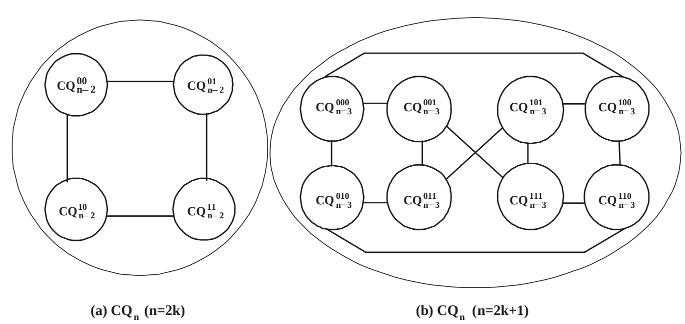

Figure 2: Subgraphs of CQ_{2k} and CQ_{2k+1}.

Lemma 4 *[4] For all* $n \geq 2$, u *and* v *are two vertices of* CQ_n, *then they satisfy either one of the following three conditions:*

(1) u *and* v *belong to a subgraph* CQ_{n-1} *of* CQ_n, *or*

(2) u *and* v *belong to two different* CQ_{n-2} *subgraphs of* CQ_n, *where* $u \in V(CQ_{n-2}^{b_2b_1})$ *and* $v \in V(CQ_{n-2}^{\overline{b_2b_1}})$, *if* n *is even, or*

(3) u *and* v *belong to two different* CQ_{n-3} *subgraphs of* CQ_n, *where* $u \in V(CQ_{n-3}^{b_3b_2b_1})$ *and either* $v \in V(CQ_{n-3}^{\overline{b_3b_2b_1}})$ *or* $v \in V(CQ_{n-3}^{\overline{b_3}b_2\overline{b_1}})$, *if* n *is odd.*

If the relative position of u and v accords with condition 2 of Lemma 4, by the routing algorithm in [6], two shortest paths are described as Lemma 5.

Lemma 5 *Let* n *be even,* $u \in V(CQ_{n-2}^{b_2b_1})$ *and* $v \in V(CQ_{n-2}^{\overline{b_2b_1}})$. *There exist two shortest paths* $P_0^s = \langle u, u^{n-1}, \cdots, v\rangle$ *and* $P_1^s = \langle v, v^{n-1}, \cdots, u\rangle$ *joining* u *and* v *such that all vertices of* P_0^s *(respectively,* P_1^s*) in* CQ_{n-1}^1 *(respectively,* CQ_{n-1}^0*) except* u *(respectively,* v*).*

When u and v conform to condition 3 of Lemma 4, by the symmetry of CQ_3, we only describe a phenomenon for the shortest path with $u \in V(CQ_{n-3}^{000})$ and $v \in V(CQ_{n-3}^{111})$ as Lemma 6.

Lemma 6 *Let* n *be odd (*$n \geq 3$*),* $u \in V(CQ_{n-3}^{000})$ *and* $v \in V(CQ_{n-3}^{111})$. *There exists one shortest path* $\langle u, P_0, v^{n-1}, v\rangle$ *joining* u *and* v *such that all vertices of* P_0 *belong to* $V(CQ_{n-3}^{\{000,001\}})$.

Expending the result of the shortest path in Lemma 6, we have another path joining u and v of length $d_{CQ_{CQ_n}(u,v)+2}$ as the following lemma.

Lemma 7 *Let* n *be odd (*$n \geq 3$*),* $u \in V(CQ_{n-3}^{000})$ *and* $v \in V(CQ_{n-3}^{111})$. *Then there exists the path* $\langle u, u^{n-2}, P_1, (v^{n-2})^{n-1}, v^{n-2}, v\rangle$ *joining* u *and* v *of length* $d_{CQ_n}(u, v) + 2$, *where all vertices of* P_1 *belong to* $V(CQ_{n-3}^{\{010,011\}})$.

3 CQ_n is 4-geodesic pancyclic

This section is dedicated to illustrating the geodesic pancyclic property of crossed cubes. Next, the concepts of geodesic cycle and geodesic k-pancyclic are formally defined and discussed.

Definition 3 *Let* G *be a graph. For two vertices* $u, v \in V(G)$, *a cycle is called a geodesic cycle with* u *and* v *if a shortest path of* G *joining* u *and* v *lies on the cycle. A geodesic* l*-cycle with* u *and* v *in* G, *denoted by* $gC^l(u, v; G)$, *is a geodesic cycle of length* l.

Definition 4 *Let* G *be a graph. For two vertices* $u, v \in V(G)$, *it is called geodesic* k*-pancyclic with* u *and* v *if for every integer* l *satisfying* $2d_G(u, v) + k \leq l \leq |V(G)|$, *the geodesic cycle* $gC^l(u, v; G)$ *exists.*

Let $C = \langle u, P_s, v, P_c, u\rangle$ be a geodesic cycle with vertices u and v, where P_s is the shortest path joining u and v on C. We call P_s and P_c as *s-path* and *c-path* of C, respectively. Let $len(C) = 2d_G(u, v) + k$. Clearly, $len(P_s) = d_G(u, v)$ and $len(P_c) = d_G(u, v) + k$.

Definition 5 *Let* G *be a graph.* G *is called geodesic* k*-pancyclic if any distinct two vertices on* G *are geodesic* k*-pancyclic with them. The geodesic-pancyclicity of* G, *denoted by* $gpc(G)$, *is defined as the minimum integer* k *such that* G *is geodesic* k*-pancyclic.*

We now propose that CQ_3 is 2-geodesic pancyclic.

Lemma 8 CQ_3 *is geodesic 2-pancyclic.*

Proof: Since CQ_3 is vertex-transitive, we assume that $u = 000$ and consider v as the four cases: (1) $v \in \{001, 100\}$, (2) $v = 010$, (3) $v \in \{011, 110\}$, and (4) $v \in \{101, 111\}$. By the symmetry of CQ_3, there is only one vertex discussed for each case and related geodesic cycles are listed as Table 1. ♢

For simplifying the proof of the geodesic pancyclic property of CQ_n, two auxiliary lemmas are present as follows.

Lemma 9 *Let* $u, v \in V(CQ_{n-1}^{b_1})$. *There exist the geodesic cycles* $gC^l(u, v; CQ_n)$ *for all* $2d_{CQ_n}(u, v) + k + 1 + \lceil \frac{n}{2} \rceil \leq l \leq 2^n$ *if the following two conditions are satisfied.*

(a) $2d_{CQ_{n-1}^{b_1}}(u, v) + k + 2 \leq 2^{n-1}$ *and*

(b) *there exist the geodesic cycles* $gC^{l_0}(u, v; CQ_{n-1}^{b_1})$ *for all* $2d_{CQ_{n-1}^{b_1}}(u, v) + k \leq l_0 \leq 2^{n-1}$.

Proof: Without loss of generality, assume $u, v \in V(CQ_{n-1}^0)$. By condition (b), there exist the geodesic cycles $gC^{l_0}(u, v; CQ_{n-1}^0)$ for all $2d_{CQ_{n-1}^0}(u, v) + k \leq l_0 \leq 2^{n-1}$. Let $\langle u, P_s, v, x, P_c', u\rangle$ be the geodesic cycle $gC^{l_0}(u, v; CQ_{n-1}^0)$ where P_s be the s-path and $\langle u, P_c', x, v\rangle$ be the c-path of $gC^{l_0}(u, v; CQ_{n-1}^0)$. By Lemma 2, $d_{CQ_{n-1}^1}(v^{n-1}, x^{n-1}) \leq \lceil \frac{n}{2} \rceil$. By Lemma 3, there exist the paths $(v^{n-1}, x^{n-1})_i$ for all $\lceil \frac{n}{2} \rceil, \lceil \frac{n}{2} \rceil + 2 \leq$

$i \leq 2^{n-1} - 1$. Thus, we can construct a cycle C as $\langle u, P_s, v, v^{n-1}, (v^{n-1}, x^{n-1})_i, x^{n-1}, x, P'_c, u\rangle$. By Lemma 1, $d_{CQ_n}(u,v) = d_{CQ^0_{n-1}}(u,v)$. Then $len(P'_c) = l_0 - d_{CQ^0_{n-1}}(u,v) - 1 = l_0 - d_{CQ_n}(u,v) - 1$. Hence, $len(C) = d_{CQ_n}(u,v) + 1 + i + 1 + (l_0 - d_{CQ_n}(u,v) - 1) = 1 + i + l_0$.

Note that $2d_{CQ^0_{n-1}}(u,v) + k \leq l_0 \leq 2^{n-1}$ and $\lceil \frac{n}{2} \rceil, \lceil \frac{n}{2} \rceil + 2 \leq i \leq 2^{n-1} - 1$. Suppose that $i = \lceil \frac{n}{2} \rceil$. Then $2d_{CQ_n}(u,v) + k + 1 + \lceil \frac{n}{2} \rceil \leq len(C) \leq 2^{n-1} + 1 + \lceil \frac{n}{2} \rceil$. On the other hand, $2d_{CQ^0_{n-1}}(u,v) + k + \lceil \frac{n}{2} \rceil + 3 \leq len(C) \leq 2^n$ if $\lceil \frac{n}{2} \rceil + 2 \leq i \leq 2^{n-1} - 1$. By condition (a), $2d_{CQ^0_{n-1}}(u,v) + k + \lceil \frac{n}{2} \rceil + 3 \leq 2^{n-1} + 1 + \lceil \frac{n}{2} \rceil$. Hence, we can get $2d_{CQ_n}(u,v) + k + 1 + \lceil \frac{n}{2} \rceil \leq len(C) \leq 2^n$ by adjusting the values of l_0 and i. As a result, the geodesic cycles $gC^l(u,v;CQ_n)$ for all $2d_{CQ_n}(u,v) + k + 1 + \lceil \frac{n}{2} \rceil \leq l \leq 2^n$ can be constructed by format C. The proof is complete. ♢

Lemma 10 *Let* $u \in V(CQ^{b_1}_{n-1})$, $v \in V(CQ^{\overline{b_1}}_{n-1})$ *and* $u \neq v^{n-1}$. *There exist the geodesic cycles* $gC^l(u,v;CQ_n)$ *for all* $2d_{CQ_n}(u,v) + k - 1 + \lceil \frac{n}{2} \rceil \leq l \leq 2^n$ *if the following three conditions are satisfied.*

(a) $d_{CQ_n}(u,v) = d_{CQ^{b_1}_{n-1}}(u, v^{n-1}) + 1$,

(b) $2d_{CQ^{b_1}_{n-1}}(u, v^{n-1}) + k + 2 \leq 2^{n-1}$, *and*

(c) *there exist the geodesic cycles* $gC^{l_0}(u, v^{n-1}; CQ^{b_1}_{n-1})$ *for all* $2d_{CQ^{b_1}_{n-1}}(u, v^{n-1}) + k \leq l_0 \leq 2^{n-1}$.

Proof: Without loss of generality, assume $u, v^{n-1} \in V(CQ^0_{n-1})$. By condition (c), there exist the geodesic cycles $gC^{l_0}(u, v^{n-1}; CQ^0_{n-1})$ for all $2d_{CQ^0_{n-1}}(u, v^{n-1}) + k \leq l_0 \leq 2^{n-1}$. Let $\langle u, P_s, v^{n-1}\rangle$ and $\langle u, P'_c, x, v^{n-1}\rangle$ be the s-path and the c-path of $gC^{l_0}(u, v^{n-1}; CQ^0_{n-1})$, respectively. By Lemma 2, $d_{CQ^1_{n-1}}(v, x^{n-1}) \leq \lceil \frac{n}{2} \rceil$. By Lemma 3, there exist the paths $(v, x^{n-1})_i$ for all $\lceil \frac{n}{2} \rceil, \lceil \frac{n}{2} \rceil + 2 \leq i \leq 2^{n-1} - 1$. Thus, we can construct a cycle C as $\langle u, P_s, v^{n-1}, v, (v, x^{n-1})_i, x^{n-1}, x, P'_c, u\rangle$. By condition (a), $d_{CQ_n}(u,v) = d_{CQ^0_{n-1}}(u, v^{n-1}) + 1 = len(P_s) + 1$ and $len(P'_c) = l_0 - d_{CQ^0_{n-1}}(u, v^{n-1}) - 1 = l_0 - d_{CQ_n}(u,v)$. Hence, $len(C) = d_{CQ^0_{n-1}}(u, v^{n-1}) + 1 + i + 1 + len(P'_c) = (d_{CQ_n}(u,v) - 1) + 1 + i + 1 + (l_0 - d_{CQ_n}(u,v)) = i + l_0 + 1$.

Note that $2d_{CQ^0_{n-1}}(u, v^{n-1}) + k \leq l_0 \leq 2^{n-1}$ and $\lceil \frac{n}{2} \rceil, \lceil \frac{n}{2} \rceil + 2 \leq i \leq 2^{n-1} - 1$. Suppose that $i = \lceil \frac{n}{2} \rceil$. Then $2d_{CQ^0_{n-1}}(u, v^{n-1}) + k + 1 + \lceil \frac{n}{2} \rceil \leq len(C) \leq 2^{n-1} + 1 + \lceil \frac{n}{2} \rceil$. On the other hand, $2d_{CQ^0_{n-1}}(u, v^{n-1}) + k + \lceil \frac{n}{2} \rceil + 3 \leq len(C) \leq 2^n$ if $\lceil \frac{n}{2} \rceil + 2 \leq i \leq 2^{n-1} - 1$. By condition (b), $2d_{CQ^0_{n-1}}(u, v^{n-1}) + k + \lceil \frac{n}{2} \rceil + 3 \leq 2^{n-1} + 1 + \lceil \frac{n}{2} \rceil$. That is, we can get $2d_{CQ^0_{n-1}}(u, v^{n-1}) + k + 1 + \lceil \frac{n}{2} \rceil \leq len(C) \leq 2^n$ by adjusting the values of l_0 and i. Note that $d_{CQ^0_{n-1}}(u, v^{n-1}) = d_{CQ_n}(u,v) - 1$. So the geodesic cycles $gC^l(u,v;CQ_n)$ for all $2d_{CQ_n}(u,v) + k - 1 + \lceil \frac{n}{2} \rceil \leq l \leq 2^n$ can be constructed with format C. The proof is complete. ♢

Then, CQ_4 is demonstrated to be 2-geodesic pancyclic.

Lemma 11 CQ_4 *is geodesic 2-pancyclic.*

Proof: To prove this case is very tedious. With long and detail discussion, we have completed theoretical proof for CQ_4. Nevertheless, we do not present it in this paper for reducing complexity. However, we can also verify this small case directly using computer. ♢

We are now ready to show the geodesic pancyclicity of crossed cubes for $n \geq 3$ as follows.

Theorem 1 CQ_n *is 4-geodesic pancyclic for* $n \geq 3$.

Proof: We show this theorem by induction on n. By Lemmas 8 and 11, the theorem holds for $n = 3, 4$. Assume that the theorem is true for every integer $5 \leq m < n$. Let u and v be two vertices in CQ_n. According to their relative position with Lemma 4, we can divide this proof into three cases: (1)both u and v belong to the same subgraph CQ_{n-1} of CQ_n, (2)$u \in V(CQ^{b_2 b_1}_{n-2})$ and $v \in V(CQ^{\overline{b_2 b_1}}_{n-2})$ if n is even, and (3)$u \in V(CQ^{b_3 b_2 b_1}_{n-3})$ and either $v \in V(CQ^{\overline{b_3 b_2 b_1}}_{n-3})$ or $v \in V(CQ^{\overline{b_3} b_2 \overline{b_1}}_{n-3})$ if n is odd. In addition, let $k = 4$ throughout the proof.

Case 1: Both u and v belong to the same subgraph CQ_{n-1} of CQ_n.

Without loss of generality, assume $u, v \in CQ^0_{n-1}$. By Lemma 2, $2d_{CQ^0_{n-1}}(u,v) + (k = 4) + 2 \leq 2\lceil \frac{n}{2} \rceil + 6 \leq 2^{n-1}$ for $n \geq 5$. By induction hypothesis, there exist the geodesic cycles $gC^{l_0}(u,v;CQ^0_{n-1})$ for all $2d_{CQ^0_{n-1}}(u,v) + (k = 4) \leq l_0 \leq 2^{n-1}$. Since conditions (a) and (b) of Lemma 9 both hold, there exist the geodesic cycles $gC^{l_1}(u,v;CQ_n)$ for all $2d_{CQ_n}(u,v) + (k = 4) + 1 + \lceil \frac{n}{2} \rceil \leq l_1 \leq 2^n$. By Lemma 1, $d_{CQ^0_{n-1}}(u,v) = d_{CQ_n}(u,v)$. By

Lemma 2, $2d_{CQ^0_{n-1}}(u,v)+(k=4)+1+\lceil\frac{n}{2}\rceil \leq 3\lceil\frac{n}{2}\rceil+5 \leq 2^{n-1}$ for $n \geq 5$. With the geodesic cycles $gC^{l_0}(u,v;CQ^0_{n-1})$ and $gC^{l_1}(u,v;CQ_n)$, there exist the geodesic cycles $gC^l(u,v;CQ_n)$ for all $2d_{CQ_n}(u,v)+4 \leq l \leq 2^n$.

Case 2: $u \in V(CQ^{b_2b_1}_{n-2})$ and $v \in V(CQ^{\overline{b_2b_1}}_{n-2})$ if n is even.

Without loss of generality, assume that $u \in CQ^{00}_{n-2}$ and $v \in CQ^{11}_{n-2}$. Herein, we prove that there exist the geodesic cycles $gC^l(u,v;CQ_n)$ for all $2d_{CQ_n}(u,v)+2 \leq l \leq 2^n$ and divide the proof into two subcases: (2.1) $2d_{CQ_n}(u,v)+2 \leq l \leq 2^{n-1}+d_{CQ_n}(u,v)$ and (2.2) $2d_{CQ_n}(u,v)+3+\lceil\frac{n}{2}\rceil \leq l \leq 2^n$.

Subcase 2.1: $2d_{CQ_n}(u,v)+2 \leq l \leq 2^{n-1}+d_{CQ_n}(u,v)$.

By Lemma 5, there exist two shortest paths $P^s_0 = \langle v, v^{n-1}, \cdots, u\rangle$ and $P^s_1 = \langle u, u^{n-1}, \cdots, v\rangle$ joining u and v. All vertices on P^s_0 (respectively, P^s_1) belong to $V(CQ^0_{n-1})$ (respectively, $V(CQ^1_{n-1})$) except v (respectively, u). By Lemma 3, there exist the paths $(u,v^{n-1})_i$ for all $d_{CQ^0_{n-1}}(u,v^{n-1})+2 \leq i \leq 2^{n-1}-1$ in CQ^0_{n-1}. Note that $d_{CQ^0_{n-1}}(u,v^{n-1}) = d_{CQ_n}(u,v)-1$. Then $d_{CQ_n}(u,v)+1 \leq i \leq 2^{n-1}-1$. Let $C = \langle u, P^s_1, v, v^{n-1}, (v^{n-1},u)_i, u\rangle$. Then $len(C) = d_{CQ_n}(u,v)+1+i$. Thus, there exist the geodesic cycles $gC^l(u,v;CQ_n)$ for all $2d_{CQ_n}(u,v)+2 \leq l \leq 2^{n-1}+d_{CQ_n}(u,v)$ with format C.

Subcase 2.2: $2d_{CQ_n}(u,v)+3+\lceil\frac{n}{2}\rceil \leq l \leq 2^n$.

Note that $d_{CQ^0_{n-1}}(u,v^{n-1}) = d_{CQ_n}(u,v)-1$. By Lemma 2, $2d_{CQ^0_{n-1}}(u,v^{n-1})+(k=4)+2 \leq 2\lceil\frac{n}{2}\rceil+6 \leq 2^{n-1}$ for $n \geq 5$. By induction hypothesis, there exist the geodesic cycles $gC^{l_0}(u,v^{n-1};CQ^0_{n-1})$ for all $2d_{CQ^0_{n-1}}(u,v^{n-1})+(k=4) \leq l_0 \leq 2^{n-1}$. Since all conditions of Lemma 10 are true, there exist the geodesic cycles $gC^l(u,v;CQ_n)$ for all $2d_{CQ_n}(u,v)+(k=4)-1+\lceil\frac{n}{2}\rceil \leq l \leq 2^n$.

Moreover, $2d_{CQ_n}(u,v)+(k=4)-1+\lceil\frac{n}{2}\rceil \leq 2^{n-1}+d_{CQ_n}(u,v)$ for $n \geq 4$. Then there exist the geodesic cycles $gC^l(u,v;CQ_n)$ for all $2d_{CQ_n}(u,v)+2 \leq l \leq 2^n$ in this case.

Case 3: $u \in V(CQ^{b_3b_2b_1}_{n-3})$ and either $v \in V(CQ^{\overline{b_3b_2b_1}}_{n-3})$ or $v \in V(CQ^{\overline{b_3}b_2\overline{b_1}}_{n-3})$ if n is odd.

With the symmetric property of CQ_3, suppose that $u \in CQ^{000}_{n-3}$ and $v \in CQ^{111}_{n-3}$. Herein, we prove that there exist the geodesic cycles $gC^l(u,v;CQ_n)$ for all $2d_{CQ_n}(u,v)+4 \leq l \leq 2^n$ and divide the proof into two subcases: (3.1) $2d_{CQ_n}(u,v)+4 \leq l \leq 2^{n-1}+2d_{CQ_n}(u,v)$ and (3.2) $2d_{CQ_n}(u,v)+3+\lceil\frac{n}{2}\rceil \leq l \leq 2^n$.

Subcase 3.1: $2d_{CQ_n}(u,v)+4 \leq l \leq 2^{n-1}+2d_{CQ_n}(u,v)$.

By Lemma 6, there exists a shortest path $P_s = \langle u, P_0, v^{n-1}, v\rangle$ joining u and v such that all vertices of $\langle u, P_0, v^{n-1}\rangle$ belong to $V(CQ^{\{000,001\}}_{n-3})$. By Lemma 7, there exists the path $\langle u, u(1) = u^{n-2}, P_1, v(2) = (v^{n-2})^{n-1}, v(1) = v^{n-2}, v\rangle$ joining u and v of length $d_{CQ_n}(u,v)+2$. Herein, $v(1) \in V(CQ^{101}_{n-3})$ and all vertices of path $\langle u(1), P_1, v(2)\rangle$ belong to $V(CQ^{\{010,011\}}_{n-3})$.

By Lemma 3, there exist such paths $(v,v(1))_i$ for all $3 \leq i \leq 2^{n-1}-1$ in CQ^1_{n-1}. Let $C = \langle u, P_s, v, (v,v(1))_i, v(1), v(2), P_1, u(1), u\rangle$. Note that $len(P_s) = d_{CQ_n}(u,v)$ and $len(P_1) = d_{CQ_n}(u,v)-1$. Then $len(C) = d_{CQ_n}(u,v)+i+1+d_{CQ_n}(u,v)-1+1 = 2d_{CQ_n}(u,v)+1+i$. So we can build the geodesic cycles $gC^l(u,v;CQ_n)$ for all $2d_{CQ_n}(u,v)+4 \leq l \leq 2^{n-1}+2d_{CQ_n}(u,v)$ with format C.

Subcase 3.2: $2d_{CQ_n}(u,v)+3+\lceil\frac{n}{2}\rceil \leq l \leq 2^n$.

Note that $d_{CQ^0_{n-1}}(u,v^{n-1}) = d_{CQ_n}(u,v)-1$. By Lemma 2, $2d_{CQ^0_{n-1}}(u,v^{n-1})+(k=4)+2 \leq 2\lceil\frac{n}{2}\rceil+6 \leq 2^{n-1}$ for $n \geq 5$. By induction hypothesis, there exist the geodesic cycles $gC^{l_0}(u,v^{n-1};CQ^0_{n-1})$ for all $2d_{CQ^0_{n-1}}(u,v^{n-1})+4 \leq l_0 \leq 2^{n-1}$. Since the three conditions of Lemma 10 are all true, there exist the geodesic cycles $gC^l(u,v;CQ_n)$ for all $2d_{CQ_n}(u,v)+(k=4)-1+\lceil\frac{n}{2}\rceil \leq l \leq 2^n$.

Moreover, $2d_{CQ_n}(u,v)+(k=4)-1+\lceil\frac{n}{2}\rceil \leq 2^{n-1}+2d_{CQ_n}(u,v)$ for $n \geq 4$. Therefore, there exist the geodesic cycles $gC^l(u,v;CQ_n)$ for all $2d_{CQ_n}(u,v)+4 \leq l \leq 2^n$ in this case.

This completes the proof. $\diamondsuit$

Furthermore, the geodesic-pancyclicity of Crossed cube CQ_n is stated as follows:

Corollary 1 $2 \leq gpc(CQ_n) \leq 4$ *for* $n \geq 3$.

4 Conclusions

In this paper, we study the existence of cycles with some requirements in CQ_n. For the given two vertices u and v, the cycle is called a geodesic cycle if it contains the shortest path between u and v. Clearly, a geodesic cycle can minimize the transmission delay from u to v. Herein, we prove that CQ_n is geodesic 4-pancyclic.

The question of embedding geodesic cycles in other important networks, like the star graphs, Twisted cubes and M$\ddot{o}$bius cubes still remains open.

Table 1: Summary of the geodesic cycles with $u = 000$ and v in CQ_3.

v	$\langle geodesic\ cycle \rangle$
001	$\langle 000, \underline{001}, 011, 010, 000 \rangle$
001	$\langle 000, \underline{001}, 111, 110, 010, 000 \rangle$
001	$\langle 000, \underline{001}, 011, 010, 110, 100, 000 \rangle$
001	$\langle 000, \underline{001}, 011, 101, 111, 110, 010, 000 \rangle$
001	$\langle 000, \underline{001}, 111, 110, 010, 011, 101, 100, 000 \rangle$
010	$\langle 000, \underline{010}, 011, 001, 000 \rangle$
010	$\langle 000, \underline{010}, 110, 111, 001, 000 \rangle$
010	$\langle 000, \underline{010}, 110, 111, 101, 100, 000 \rangle$
010	$\langle 000, \underline{010}, 011, 101, 111, 110, 100, 000 \rangle$
010	$\langle 000, \underline{010}, 011, 101, 100, 110, 111, 001, 000 \rangle$
110	$\langle 000, 010, \underline{110}, 100, 000 \rangle$
110	$\langle 000, 010, \underline{110}, 111, 001, 000 \rangle$
110	$\langle 000, 010, \underline{110}, 111, 101, 100, 000 \rangle$
110	$\langle 000, 010, \underline{110}, 111, 101, 011, 001, 000 \rangle$
110	$\langle 000, 010, \underline{110}, 111, 001, 011, 101, 100, 000 \rangle$
111	$\langle 000, 001, \underline{111}, 110, 010, 000 \rangle$
111	$\langle 000, 001, \underline{111}, 101, 011, 010, 000 \rangle$
111	$\langle 000, 001, \underline{111}, 101, 100, 110, 010, 000 \rangle$
111	$\langle 000, 001, \underline{111}, 110, 100, 101, 011, 010, 000 \rangle$

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