Multiscaling in the distribution of the exchange rates

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Abstract: The paper analyzes the scaling laws of the FX markets by applying a recently introduced distribution-based class of estimators of the self-similarity parameter. Instead of evaluating specific moments, the scaling of the whole distribution is studied by pairwise comparisons of time horizons. The analysis shows that.

Key–Words: Scaling, Self-Similarity, FX markets, fd7

1 Introduction

The analysis of the scaling properties of financial markets is attracting a growing interest for both theoretical (no preferred time scale, continuous time formulation, universality and parsimony) and practical (stability under aggregation, small number of parameters, analytical simplicity) reasons. Since the pioneering work by [18], a huge number of contributions has provided empirical evidence of multi-scaling in finance (see, e.g., [20], [5], [11], [24], [25], [24], [16] and [14] for the FX markets; [9], [12], [3], [4], [22] and [26] for the stock markets and [1] and [21] for the future markets).

A renewed interest towards the scaling properties of volatility has followed the Basel Accords, which have reaffirmed the square–root–of–time rule (correct only under self-similarity with parameter $\frac{1}{2}$) as a proxy for estimating volatility on different time horizons (see [19], [7]) and, *ante litteram*, [8] for an analysis of the drawbacks of this rule of thumb).

Indeed, [10] use scaling analysis to probe the different degree of markets development.

In finance literature, using different techniques, basically two types of scaling behaviours are studied: the scaling of some volatility measures $-$ tipically variance or absolute moments of the returns $-$ as a function of the time interval and, once the time interval has been fixed, the scaling behaviour of the tails of the distribution of returns as a function of the size of the variation [6]. To characterize the scaling properties of financial markets, empirical tests generally use the rescaled range analysis (introduced by [15] and modified by [17]), the multiaffine analysis [23], the more recent Detrended Fluctuation Analysis ([23] and [2]), the ARFIMA estimation by exact maximum likelihood, the moving average-like analysis methods, the Average Wavelet Coefficient Method (see, e.g. [13]).

In this paper we analyse the scaling behaviour of the daily rates of four currencies (Canadian dollar, Japanese yen, Swiss franc and British pound) against U.S. dollar in the period 1972-2006. The original idea underlying this work is the use of a recently introduced distribution-based method which - never used before, at least in the authors' knowledge - provides a very immediate representation of the scaling relation between time horizons.

2 Self-similarity and scaling

2.1 Theoretical background

Let us shortly recall the basic definition of (strong) self-similarity which will be useful in the following.

Definition 1 *The continuous time, real-valued process* $\{X(t),\ t \in T\}$, *with* $X(0) = 0$, *is self-similar with index* $H_0 > 0$ *(concisely,* H_0*-ss) if, for any* $a \in \mathbb{R}^+$ *and any integer* k *such that* $t_1, ..., t_k \in T$, *the following equality holds for its* *finite-dimensional distributions*

$$\{X(at_1), X(at_2), ..., X(at_k)\} \stackrel{d}{=} \\ \stackrel{d}{=} \{a^{H_0}X(t_1), a^{H_0}X(t_2), ..., a^{H_0}X(t_k)\} \quad (1)$$

As equality (1) implies

$$\mathbb{E}(|X(t)|^q) = t^{H_0 q}\mathbb{E}(|X(1)|^q) \quad (2)$$

self-similarity is usually tested by analysing the scaling behaviour of the sample (absolute) moments of $X(t)$ but this approach leads to weak conclusions because the reverse implication (from (2) to (1)) is not necessarily true.

Bianchi (2004) reformulates definition (1) in an equivalent way by introducing a proper metric on the space of the rescaled probability distribution functions (pdf's) as follows. Let $\mathcal{A}$ be any bounded subset of $\mathbb{R}^+$, $\mathfrak{a} = \min(\mathcal{A})$ and $\mathfrak{A} = \max(\mathcal{A}) < \infty$, for any $a \in \mathcal{A}$, consider the k-dimensional distribution Φ of the a-lagged process $X(at)$. Equality (1) becomes

$$\Phi_{\ \ (a)}(\mathbf{x}) = \Phi_{a^{H_0}\ \ (1)}(\mathbf{x}) \quad (3)$$

where $\mathbb{X}(a) = (X(at_1), ..., X(at_k))$ and $\mathbf{x} = (x_1, ..., x_k) \in \mathbb{R}^k$. It follows

$$\begin{aligned} &\Phi_{a^{-H}\ \ (a)}(x) = \\ &= \Pr\left(a^{-H}X(at_1) < x_1, ..., a^{-H}X(at_k) < x_k\right) = \\ &= \text{by } H_0\text{-ss} = \\ &= \Pr\left(a^{H_0-H}X(t_1) < x_1, ..., a^{H_0-H}X(t_k) < x_k\right) = \\ &= \Phi_{a^{H_0-H}\ \ (1)}(x) = \quad (4) \\ &= \Pr\left(X(t_1) < a^{H-H_0}x_1, ..., X(t_k) < a^{H-H_0}x_k\right) = \\ &= \Phi_{\ \ (1)}(a^{H-H_0}x) \end{aligned}$$

So, denoted by ρ the distance function induced by the sup-norm $\|\cdot\|_\infty$ on the space Ψ_H of the (absolutely continuous) k-dimensional pdf's of $\{a^{-H}X(at)\}$ with respect to the set $\mathcal{A}$, the diameter of the metric space (Ψ_H, ρ)

$$\delta^k(\Psi_H) = \sup_{\mathbf{x}\in\ \ ^k}\ \sup_{a_i,a_j\in\mathcal{A}} \left|\Phi_{a_i^{-H}\ \ (a_i)}(\mathbf{x}) - \Phi_{a_j^{-H}\ \ (a_j)}(\mathbf{x})\right| \quad (5)$$

measures the *discrepancy* among the rescaled distributions. If self-similarity holds then, for any $H \neq H_0$[1], by (4), one can trivially notice that

$$\sup_{a_i,a_j\in\mathcal{A}} \left|\Phi_{a_i^{-H}\ \ (a_i)}(\mathbf{x}) - \Phi_{a_j^{-H}\ \ (a_j)}(\mathbf{x})\right| = \\ = \left|\Phi_{\ \ ^{-H}\ (\ )}(\mathbf{x}) - \Phi_{\ \ ^{-H}\ (\ )}(\mathbf{x})\right| \quad (6)$$

[1]For an H_0-ss process, when $H = H_0$ it is trivial to check that $\sup_{a_i,a_j\in\mathcal{A}} \left|\Phi_{a_i^{-H}\ \ (a_i)}(\mathbf{x}) - \Phi_{a_j^{-H}\ \ (a_j)}(\mathbf{x})\right|$ collapses to zero, whatever a_i and a_j.

and relation (5) reduces itself to the well-known statistics of Kolmogorov-Smirnov, enriching the self-similarity analysis with an inferential support.

2.2 Scaling surfaces

In this work we observe that, given two any lags a and b, for a true self-similar process it follows from (3) that

$$\Phi_{a^{-H_0}\ \ (a)}(\mathbf{x}) = \Phi_{b^{-H_0}\ \ (b)}(\mathbf{x})$$

and, combining this with (6), (5) can be written as

$$\delta^k(\Psi_H) = \sup_{\mathbf{x}\in\ \ ^k} \left|\Phi_{b^{-H}\ \ (b)}(\mathbf{x}) - \Phi_{a^{-H}\ \ (a)}(\mathbf{x})\right|. \quad (7)$$

The last relation represents a useful form for testing the scaling properties of a time series by means of the pairwise comparisons of the time horizons (a, b). So, from a geometrical viewpoint, the idea consists in associating at each pair (a, b) with $a < b$ for $a, b \in \mathcal{A}$ the third coordinate given by the estimated self-similarity parameter $H_0(a, b)$ whenever the null hypothesis of the Kolmogorov-Smirnov test (identity of the rescaled distributions $\Phi_{b^{-H}\ \ (b)}(\mathbf{x})$ and $\Phi_{a^{-H}\ \ (a)}(\mathbf{x})$) is not rejected at a given p-level.

In an equivalent way, we can see the result in form of a strictly lower triangular matrix displaying the maximum horizons on the rows and the minimum horizons on the columns and whose elements are the parameters H_0's.

Basically, in our approach the definition of self-similarity is not required to hold for any time horizon; we will be content to determine which (if any) parameter H_0 minimizes the distance (7) to such an extent to be statistically negligible. Therefore, once the set $\mathcal{A}$ has been fixed, the analysis of the shape of the surface drawn by the points (a, b, H_0) will provide information about the nature of the time series: a H_0-high plane will indicate that the process is truly H_0 self-similar (relatively to the set $\mathcal{A}$); on the contrary, a jagged surface is expected to be generated by a multiscaling process. Just as an example, Figure 1 reproduces the scaling surface of the increments of a standard Brownian motion, which is self-similar with parameter $\frac{1}{2}$, and – as expected – the surface is rather flat just around the value $\frac{1}{2}$. Even if with a worse approximation due to the generators, analogous results are obtained with other

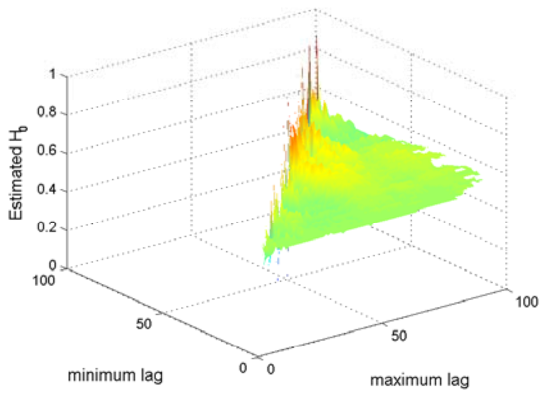


Figure 1: Scaling surface of a sampled Brownian motion

self-similar processes with dependent increments such as the fractional Brownian motion of parameter $H_0 \neq \frac{1}{2}$.

Finally, notice (also in Figure 1) the spurious effect which appears when the two lags assume close high values: in these cases the estimated parameter H_0 tends to polarize at the extreme values of the scale in a very unstable way. If the process has continuous trajectories and self-similar stationary increments, this effect can be easily explained reasoning as follows. Given two sufficiently close and large lags a and $b = a + \epsilon$, for $\epsilon \to 0$, by self-similarity one has

$$\left(1+\frac{\epsilon}{a}\right)^{H_0}[X(t+a)-X(t)] \stackrel{d}{=} X(t+a+\epsilon)-X(t)$$

but, by continuity $\lim_{\epsilon \to 0} X(t+a+\epsilon) = X(t+a)$, hence the distribution of the increments $\{X(t+a) - X(t)\}$ will be very close to that of the increments $\{X(t+a+\epsilon) - X(t)\}$. The larger is a, the closer to 1 is $\left(1+\frac{\epsilon}{a}\right)$ and the exponent H_0 becomes negligible in order to guarantee the identity of the distributions, for the continuity and the large values of a predominate over the values of H_0 itself.

3 Empirical application

3.1 Dataset and methodology

The exchange rate of four currencies against the U.S. dollar (base currency) has been analysed. In detail, the dataset concerns the daily quotes of the Canadian Dollar, the British Pound, the Swiss Franc and the Japanese Yen from January 3th, 1972 to September 6th, 2006, for an amount of 8,728 observations. The values, provided by Prof. Werner Antweiler at UBC's Sauder School of Business, University of British Columbia, refer to the quotes of the noon spot rates (Eastern Time) as determined by trades in the Toronto interbank market and are expressed in volume notation[2]. For each time series the daily log-price variations have been calculated and considered as input for the scaling analysis. The lags a and b have been taken in the set of time horizons $\mathcal{A} = \{1, 2, ..., 100\}$ corresponding to five months of the trading calendar and the null hypothesis has been tested for $\alpha = 0.05$ and $\alpha = 0.01$. The scaling surfaces are obtained by filtering only those estimated H_0's for which the corresponding diameter (7) is below the critical value of the Kolmogorov-Smirnov statistics.

For each currency, the whole time series, the first $4,000$ and the last $4,000$ datapoints have been separately examined in order to check the stability of the results.

3.2 Discussion of result

Figures 2(a)–(d) summarize the results of our analysis. For all the currencies the scaling surfaces are jagged indicating that, when the test has passed, the estimated self-similarity parameter H_0 heavily changes from point to point; the holes of the scaling surfaces denote that the values of the corresponding diameter (7) exceed the critical threshold, given the confidence level α. This behaviour sizes the multiscaling nature of the analysed series, which show significantly different self-similarity parameters when different pairs of time horizons are considered. Nevertheless, in spite of the complexity of this behaviour, the technique we have used is somewhat punctual because it does not limit itself to assess the existence of a plethora of scaling exponents, but attaches its own exponent to each chosen pair of horizons.

[2] According to the rule schemed in Northern America, the values express the number of units of the target currency per unit of the base currency.

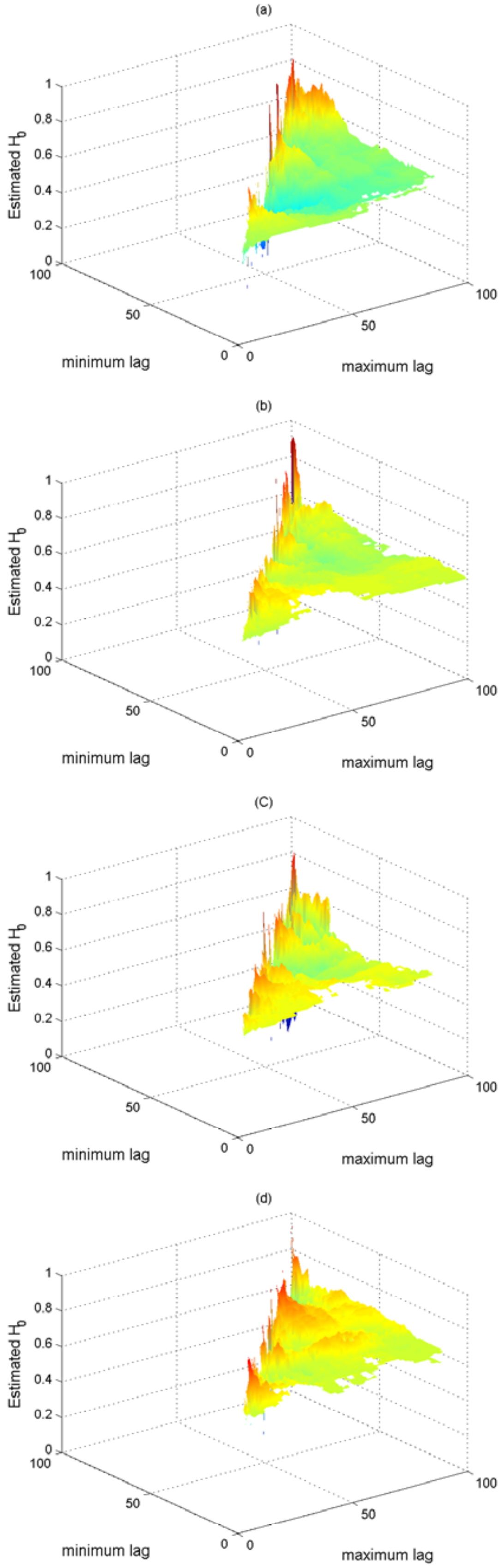
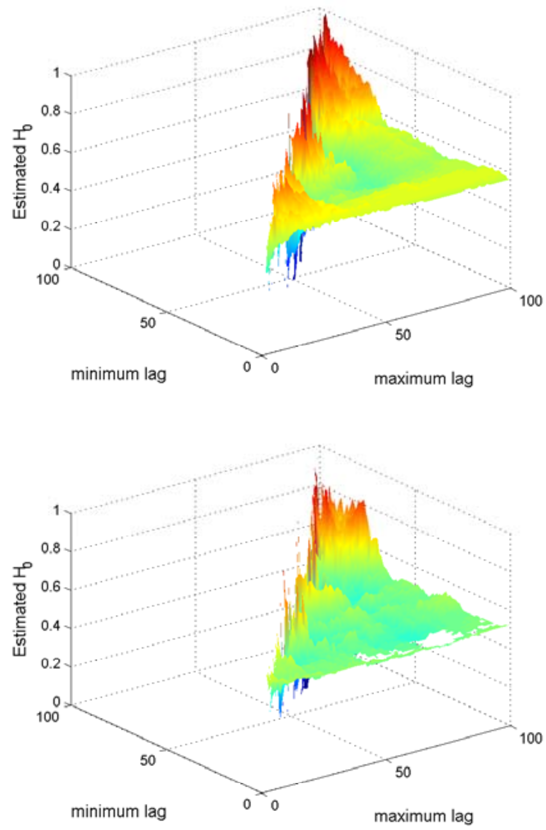
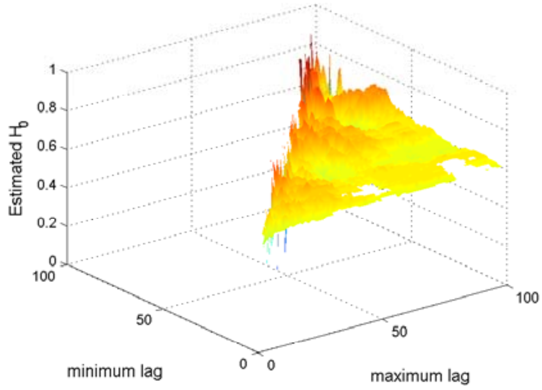


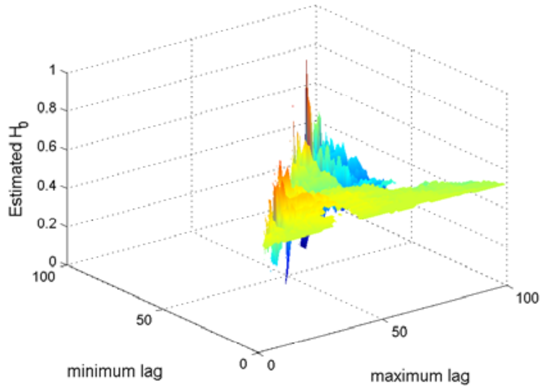
Figure 2. Scaling surfaces of (a) CAN/USD, (b) CHF/USD, (c) GPB/USD, (d) YEN/USD

Dividing the whole time series into two separate subperiods, observations $1-4,000$ (01/04/72;12/17/87) and $4,728-8727$ (11/08/90;09/06/06), reveals interesting features of the scaling surfaces: in all the analysed series the estimated H_0's are in the first subperiod higher than in the second one, for which asymptotically (for large a and b) they seem to converge to $\frac{1}{2}$ (Figures 3-?). As high values of H indicate more regular behaviours that can generate arbitrage and values close to $\frac{1}{2}$ are associated to pure random processes that exclude arbitrage opportunities, from a financial viewpoint the different shape of the scaling surfaces in the two subperiods can be interpreted as a signal of improvement of the efficiency of the FX markets in the last years.

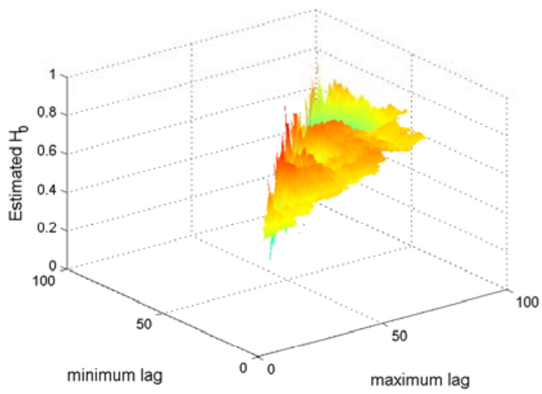


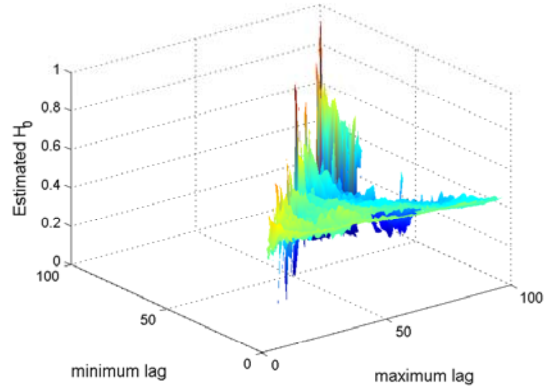
Scaling surface at $\alpha = 0.01$ of the CAN/USD. (above) first 4,000 datapoints (bottom) last 4,000 datapoints



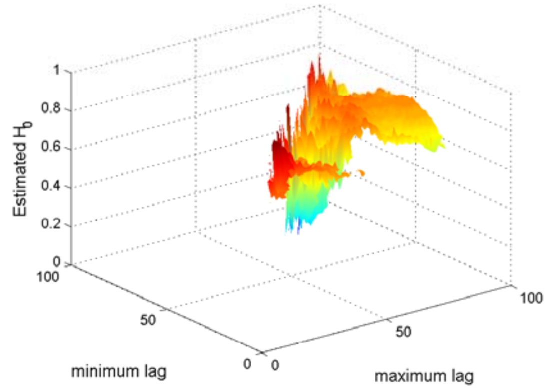


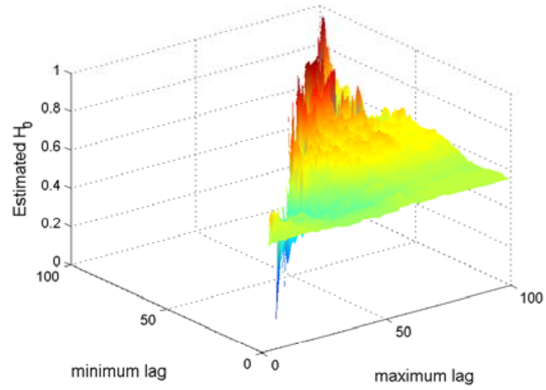
Scaling surface at $\alpha = 0.01$ of the CHF/USD. (above) first 4,000 datapoints (bottom) last 4,000 datapoints





Scaling surface at $\alpha = 0.01$ of the GPB/USD. (above) first 4,000 datapoints (bottom) last 4,000 datapoints





Scaling surface at $\alpha = 0.01$ of the YEN/USD. (above) first 4,000 datapoints (bottom) last 4,000 datapoints

4 Conclusion

In this paper we have proposed to test the scaling properties of financial time series by using a recently introduced estimator of the self-similarity parameter based on the whole distribution in place of the traditional approaches which generally refer to specific moments.

As shown in the previous section, the technique allows in a very simple way to assign at each pair of fixed time horizons the exponent H_0 (if any) which makes the two rescaled distributions identical at a given p-value. In spite of its simplicity, the method looks very useful in many circumstances: for example, when investors have to scale the volatility between two time horizons a and b (e.g. from daily volatility to weekly volatility), the rule of thumb consists in multiplying the base volatility a (1 day) by the square root of b (5, number of trading days in a week). This rule, based on the assumption of self-similarity with parameter $\frac{1}{2}$ of the price process, is well known to be misleading (see e.g. [7], [8] or [19]) and can be easily replaced by the following $\sigma_b \cong \left(\frac{b}{a}\right)^{H_0(a,b)} \sigma_a$, where $H_0(a, b)$ is the proper scaling index that can be calculated using the above procedure.

Possible further development will concern how to manage the spurious effect which produces the unstable values of the estimated H_0 on the subdiagonals.

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