Proceedings of the 5th WSEAS Int. Conf. on Instrumentation, Measurement, Circuits and Systems, Hangzhou, China, April 16-18, 2006 (pp279-284)

# Generalized Fuzzy Numbers Comparison by Geometric Moments

J.N. SHEEN
Department of Electrical Engineering, Cheng-Shiu University
840 Chengcing Road, Niaosong Township, Kaohsiung County, TAIWAN

*Abstract*: -In the field of engineering and economic studies, mathematical manipulation of fuzzy numbers is cumbersome and does not provide a neatly ordered set of results in the same way that crisp numbers do. Moreover, the generalized fuzzy number (i.e. non-normalized and normalized fuzzy number) have been approved more flexible and more intelligent than the normalized fuzzy number since it takes the degree of confidence of the decision-makers' opinions into account. In this paper the concept of the probability measure of fuzzy events is used to represent the fuzzy alternatives, and a ranking approach based upon their geometric moments are developed to make decision. The approach is computationally simple and its underlying concepts are logically sound. A fuzzy number with a superior geometric mean is ranked above fuzzy numbers having inferior geometric means, and in the case where the geometric means of two numbers happen to be equal, the number with a lower geometric variance is ranked above fuzzy numbers whose geometric variances are higher. A comparative study is conducted on cases used in the previous literatures to examine the performance of the proposed method on rationality and discriminatory ability.

*Key-words:* Decision-making, Probabilistic moments, Fuzzy number, Mellin transform.

## 1. Introduction

Decision-making problems often require a choice to be made between several fuzzy alternatives. In cases of inadequate data, most decision makers rely upon experts' knowledge to carry out simulated modeling of the problem. The generalized fuzzy number (i.e. non-normalized and normalized fuzzy number) have been approved more flexible and more intelligent than the normalized fuzzy number since it takes the degree of confidence of the decision-makers' opinions into account [1-4].

The practical application of fuzzy set theory to engineering and economics problems requires two laborious tasks: (1) fuzzy mathematical operations and (2) comparison or ranking of complex fuzzy numbers generated by mathematical operations. Fuzzy mathematics is based on the extended principle, which applies basic mathematical operations such as addition, subtraction, multiplication and division to fuzzy numbers, or involves their power, logarithmic and exponent manipulation. However, fuzzy mathematics usually tends to be cumbersome, even for simple operations such as addition and multiplication. Following the rating of the resultant fuzzy function, the task of comparing or ranking the complex fuzzy numbers can cause a further problem since fuzzy mathematics operations do not always yield a totally ordered set of results in the same way that crisp numbers do. Many authors have investigated the use of alternative methods to rank fuzzy sets. These methods range from trivial to complex, including one fuzzy number attribute to many fuzzy number attributes. A review and comparison of these existing methods can be found in [5-7]. In [5], ranking methods are classified into four major classes, (1) Preference relation (as in [8-12]), (2) Fuzzy mean and spread (as in [13-15]), (3) Fuzzy scoring (as in [16-22]), and (4) Linguistic expression (as in [23]). Although most fuzzy ranking methods in the previous literatures exhibit satisfactory results for clear-cut problems, they may generate counter-intuitive outcomes or be not discriminatory enough under certain circumstances [5, 7]. In addition, most of them require considerable computational effort. Moreover, some ranking methods assume the membership function to be normal, and this is not adequate in many cases. To overcome these limitations, this paper presents a novel geometric moment method for the ranking of fuzzy numbers, which is not only simple to implement but also conceptually straightforward and suitable even in cases where the membership function is abnormal.

This paper considers a probabilistic ranking method based on *the geometric moments*, which represent the geometric average value of the *domain moments* and the *grade moments*. The geometric moments are easily calculated only based on the vertexes and confidence value of the corresponding fuzzy numbers. A fuzzy number with a superior geometric mean is ranked above fuzzy numbers having inferior geometric means, and in the case where the geometric means of two numbers happen to be equal, the number with a lower coefficient of variation is ranked above fuzzy numbers whose geometric variances are higher. A comparative study is conducted on cases used in the previous literatures to examine the performance of the proposed method on rationality and discriminatory ability.

## 2. Fuzzy number

When dealing with uncertainty, decision makers are commonly provided with information which is characterized by vague linguistic descriptions such as "high risk", "low profit", "high annual interest rate" etc. The objective of fuzzy set theory is primarily concerned with the quantification of such vagueness. A fuzzy set is designated as $\forall x \in X, \mu_A(x) \in [0,\ h\ ]$, where $\mu_A(x)$ is the grade of membership of a vague predicate, A, over the universe of objects, X. The more the object fits the vague predicate, the larger its grade of membership will be. The membership function may be viewed as representing an opinion poll of human thought or an expert's opinion. For general purposes, the height of the fuzzy number, $h$, will be considered to possess an arbitrary positive value since this allows both normal and subnormal fuzzy numbers to be considered. Since the membership grade of the fuzzy number lies in the range [0, $h$ ], the membership function can be expressed as:

$$\mu_A(x) = (a_1,\ f_{A_1}(\alpha)/a_2, a_3 / f_{A_2}(\alpha), a_4; h) \qquad (1)$$

where $f_{A_1}(\alpha)$ is a continuous monotone increasing function of the membership grade, α , for $0 \le \alpha \le h$, $f_{A_2}(\alpha)$ is a continuous monotone decreasing function of α for $0 \le \alpha \le h$, $f_{A_1}(0) = a_1$, $f_{A_1}(h) = a_2$, $f_{A_2}(h) = a_3$, $f_{A_2}(0) = a_4$ , $a_1 < a_2 \le a_3 < a_4$ , and $h$ denotes the height of the fuzzy number. The trapezoidal fuzzy number (TrFN) is a particular type of fuzzy number, in which $f_{A_1}$ and $f_{A_2}$ are both straight-line segments. In the case, where $a_2$ equals $a_3$ the trapezoidal fuzzy number becomes a triangular fuzzy number (TFN). The mathematical implementation of the TFN is straightforward and, importantly, it represents a rational basis for the quantification of the vague knowledge that is associated with most decision-making problems. The TFN of $A$ can be expressed simply as $A = (a_1, a_2, a_3; h)$ , where the vertexes $a_1$ , $a_2$ and $a_3$ respectively denote *the smallest possible value, the most promising value* and *the largest possible value* describing a fuzzy event. The membership function and the domain function of the triangular fuzzy number $A = (a_1, a_2, a_3; h)$ are presented in Fig. 1, and can be represented by the linear relations given in Eqs. (2) and (3), respectively.

$$\mu_A(x) = \begin{cases} \mu_{A_1}(x) = \dfrac{h(x-a_1)}{a_2-a_1} & a_1 \le x \le a_2 \\ \mu_{A_2}(x) = \dfrac{h(a_3-x)}{a_3-a_2} & a_2 \le x \le a_3 \end{cases} \qquad (2)$$

$$x = \begin{cases} f_{A_1}(\alpha) = \mu_{A_1}^{-1} = a_1 + \dfrac{(a_2-a_1)\alpha}{h} & 0 \le \alpha \le h \\ f_{A_2}(\alpha) = \mu_{A_2}^{-1} = a_3 - \dfrac{(a_3-a_2)\alpha}{h} & 0 \le \alpha \le h \end{cases} \qquad (3)$$

The membership function given in Eq.(2) represents the mapping of any given value of x to its corresponding grade of membership , $\alpha$ , while the domain function expressed by Eq.(3) is an inverse mapping of any given $\alpha$ to its corresponding x value. The TFN can also be designated in an $\alpha$ -cut form,

$$A = \left[f_{A_1}(\alpha),\ f_{A_2}(\alpha)\right]$$
$$= \left[a_1 + (a_2 - a_1)\alpha / h,\ a_3 - (a_3 - a_2)\alpha / h\right] \qquad (4)$$

The α-cut of a fuzzy set $A$ is a crisp set of numbers that contains all the elements of the universal set $X$ whose membership grades in $A$ are greater or equal to the specified value of α.

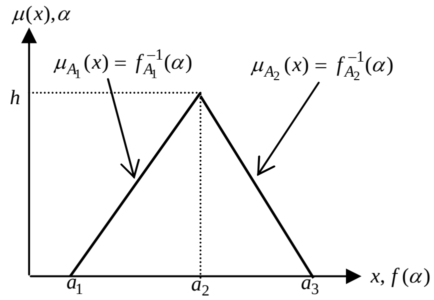


Fig. 1 Membership and domain function of a TFN

## 3. Probabilistic conversion of fuzzy number

Conversion of the membership function of a fuzzy number $\mu(x)$ into an equivalent probability density function can be achieved by using one of two linear transformations [24]: proportional probability density function (*ppdf*): $p(x) = k_p \mu(x)$ , and uniform probability density function (*updf*): $u(x) = \mu(x) + k_u$ , where $k_p$ and $k_u$ are the values of the conversion constants which ensure that the area under the continuous probability function is equal to one.

Figs.2 and 3 show the conversion of a TFN and a TrFN into their corresponding *ppdf* and *updf*, respectively. When the proportional conversion method is used, the height of the resultant *ppdf* is independent of the fuzzy number height, but its domain remains the same as that of the original fuzzy number. When the uniform conversion approach is adopted, the domain and the height of the resultant distribution both reduce (or increase) from their original fuzzy number values. The reduced (or increased) domain indicates the partial ejection (or addition) of some members from (or to) the set. Hence, the uniform distribution reveals certain undesirable properties. Therefore, the application of the proportional density function conversion is recommended in the comparison of fuzzy numbers.

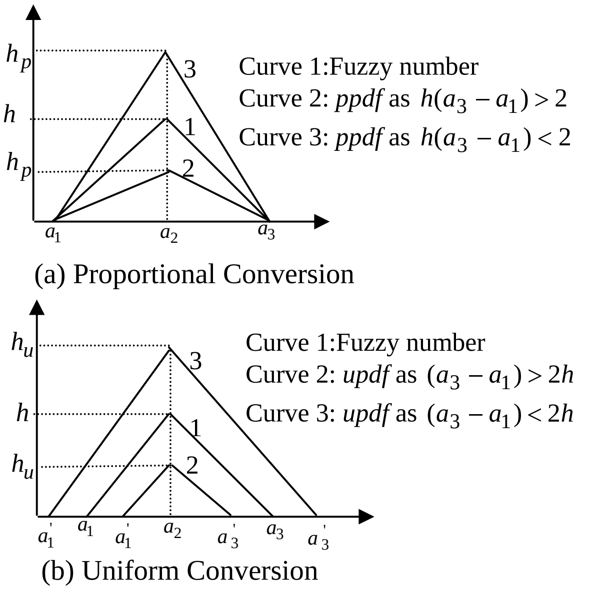


Fig.2 TFN converted into probability density function

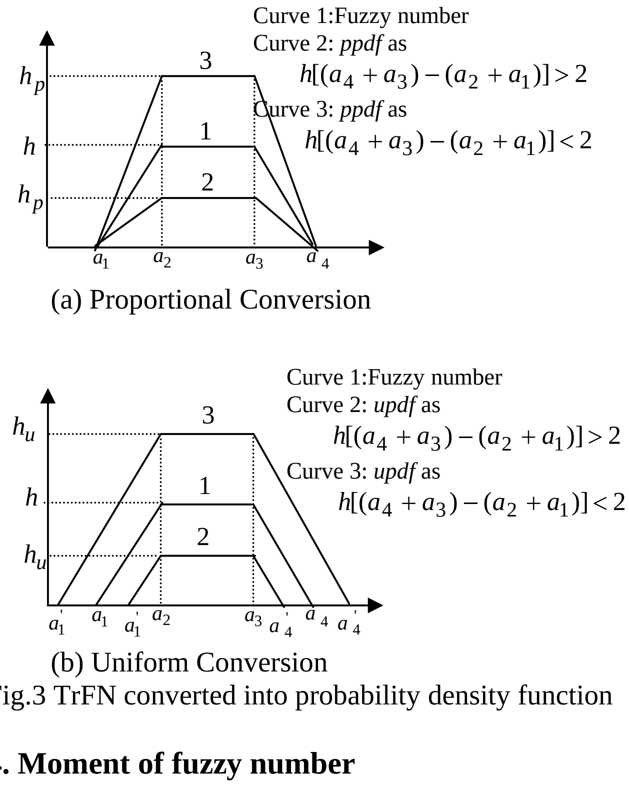


Fig.3 TrFN converted into probability density function

## 4. Moment of fuzzy number

### *4.1. Mellin transform*

Operational calculus techniques are particularly useful when analyzing probabilistic models as part of a decision-making process. In the probabilistic model context, it is often possible to reduce complex operations involving differentiation and integration to simple algebraic manipulations in the transform domain. The Mellin transform is a useful tool for studying the distributions of certain combinations of random variables (r.v.); particularly the random variables associated with products and quotients. The Mellin transform, $M_x(s)$, of a function, $f(x)$ (where x is positive) is defined in [25-26], and is given by:

$$M_X(s) = \int_0^\infty x^{s-1} f(x)dx \quad 0 < x < \infty \tag{5}$$

The Mellin transform has a unique one-to-one correspondence, i.e. $f(x) \leftrightarrow M_X(s)$. The moments of a distribution represent the expected values of the powers of a random variable with $f(x)$ distribution. In general, the ***r****th* moment of a random variable X about a real number c is defined as:

$$M_r(x) = E[(X-c)^r] = \int_X (x-c)^r f(x)dx \tag{6}$$

The moments of interest in an economic analysis are those about the origin ($c = 0$) and those about the mean ($c = \mu$), typically for $r$ =1,2,3 and 4. The ***r*** moments about the origin are denoted by $E[X^r]$. The first moment about the origin represents the mean of the distribution, i.e. $M_X = E[X]$. The second moment taken about the mean represents the variance of the distribution $f(x)$ $V_X = \sigma^2$, while the third and the fourth moments give the skew and the kurtosis of the distribution, respectively. A comparison of Eqs.(5) and Eq.(6) shows that $M_X(s)$ is a special case of $M_r(x)$, where $c = 0$ and $r = s-1$. In other words, if $f(x)$ is viewed as a probability density function, then the Mellin transform $M_X(s) = E[X^{s-1}]$ may be used to determine a series of moments of the distribution. Comparing the first two moments of a distribution with the Mellin transform, the mean and the variance of a distribution are expressed in Eqs.(7) and (8), respectively.

$$M_X = E[X^1] = \int_X xf(x)dx = M_X(2) \tag{7}$$

$$\begin{aligned}\sigma^2 = V_X &= \int_X (x-\mu)^2 f(x)dx \\ &= \int_X x^2 f(x)dx - \mu^2 = M_X(3) - (M_X(2))^2\end{aligned} \tag{8}$$

### *4.2. Domain mean and variance of a fuzzy number*

The close correlation between the Mellin transform and the moments of a distribution makes it simple to establish some important operating properties involving products, quotients and power of random variables. For example, by computing $M_X(s)$ at s=1,2 and 3, the mean and variance of a regular triangular fuzzy number $A(a_1,a_2,a_3;h)$ can be expressed in the forms given by Eqs.(9) and (10), respectively.

$$M_d = M_x(2) = \frac{a_1+a_2+a_3}{3} \quad (9)$$

$$V_d = \sigma_d^2 = \frac{1}{18}(a_1^2+a_2^2+a_3^2-a_1a_2-a_2a_3-a_3a_1) \quad (10)$$

It is found that the mean and variance of a fuzzy number are determined only by its vertexes, i.e. they are independent of the height. The consistent domain property of a fuzzy number with its converted probability density function is another important advantage when calculating the mean and variance of a fuzzy number. The mean and variance of a fuzzy number calculated from the Mellin transform are represented in domain terms as the *domain mean* $M_d$, and the *domain variance* $V_d$, respectively.

### *4.3. Grade mean and variance of a fuzzy number*

From the domain functions of a fuzzy number A, the grade mean $M_g$ and the grade variance $V_g$ can be expressed in the forms given by Eqs.(11) and (12) respectively.

$$M_g = \int_0^h \alpha(f_{A_2}(\alpha)-f_{A_1}(\alpha))d\alpha \quad (11)$$

$$\begin{aligned} V_g &= \int_0^h (\alpha-\mu_g)^2(f_{A_2}(\alpha)-f_{A_1}(\alpha))d\alpha \\ &= \int_0^h \alpha^2(f_{A_2}(\alpha)-f_{A_1}(\alpha))d\alpha-(\mu_g)^2 \end{aligned} \quad (12)$$

It is found that the grade mean and the grade variance are both functions of the height and the vertexes of the fuzzy number. Therefore, a distinction can be made between normal and subnormal fuzzy numbers by comparing their grade moments.

### *4.4. Geometric mean and geometric variance of a fuzzy number*

The geometric mean *M*, geometric variance *V*, and coefficient of variation $C.V.$, of a fuzzy number can be defined by Eqs.(13)-(15), respectively. The standard deviation $\sigma = \sqrt{V}$ .

$$M = \sqrt{M_d^2 + M_g^2} \quad (13)$$

$$V = \sqrt{V_d^2 + V_g^2} \quad (14)$$

$$C.V. = \frac{\sigma}{M} \quad (15)$$

## 5. Fuzzy numbers comparison

Fig. 4 presents a flow chart describing the proposed ranking process of fuzzy numbers. Firstly, the fuzzy numbers are converted to their equivalent probabilistic density functions, and then Eqs.(13)-(15) are used to calculate their geometric moments and coefficients of variation. Fuzzy numbers which share the same most promising values are ranked using Rule 1, while the other fuzzy numbers are ranked using Rule 2. The two rules may be summarized as follows: *Rule 1*: a fuzzy number with a lower coefficient of variation is ranked above fuzzy numbers whose coefficients of variation are higher. *Rule 2*: a fuzzy number with a superior geometric mean is ranked above fuzzy numbers having inferior geometric means.

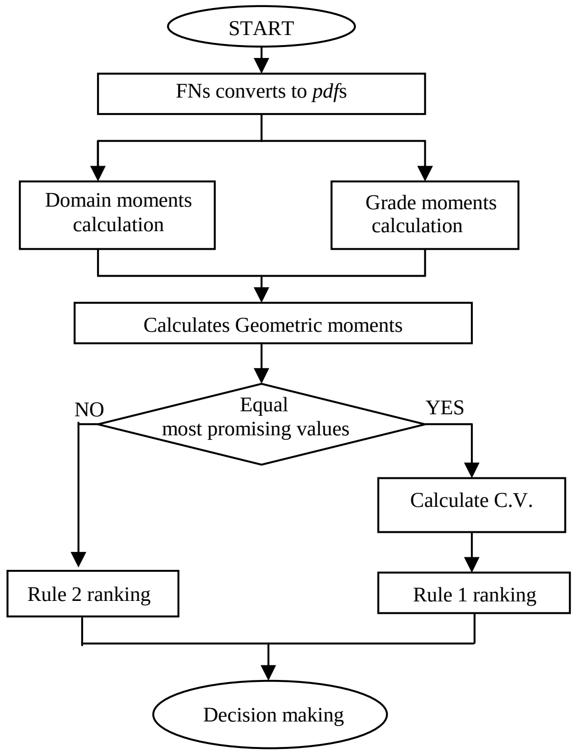


Fig. 4 Flow chart of fuzzy numbers ranking

## 6. Comparative study

In order to verify the robustness of the proposed method, this paper now considers some particular examples, and compares the current results with those provided by previously published methods.

**Example 1:** Two alternatives share the same means, but have different variances.

Two alternatives' attributes are represented by two triangular numbers, i.e. A(0,1,2;1) and B(0.2,1,1.75;1), are taken from Cheng's study of the Distance method [19]. Table 1 presents the ranking of A and B as determined by Rule 1. The results indicate that the current approach provides a ranking of A and B which is consistent with that obtained from [19]. However, Lee and Li's decision criterion is base on higher mean value and at the same time lower spread. Clearly, we cannot intuitively compare its orders by Lee and Li's method [6].

**Example 2:** Complex comparison due to the partial overlap between the supports of the fuzzy numbers.

Three triangular numbers, i.e. A(0.2,0.3,0.5;1.0), B(0.17,0.32,0.58;1.0), C(0.25,0.4,0.7;1.0), are taken from [12]. In this case, an intuitive judgment is nearly impossible since the partial overlap between the supports of the fuzzy numbers. Table 2 summarizes the ranking outcome provided by the current geometric moment method and by other published approaches. It is noted that the current method yields a ranking of $A \prec B \prec C$ (bottom low of Table 2), which is consistent with the other methods.

Table 1 Fuzzy numbers ranking for Example 1

| Ranking method | Distance method[19] | | | Proposed method | | |
|---|---|---|---|---|---|---|
| | $x_0$ | $\sigma$ | C.V. | $M$ | $\sigma$ | C.V. |
| A | 1.000 | 0.167 | 0.167 | 1.060 | 0.176 | 0.166 |
| B | 0.983 | 0.100 | 0.102 | 1.017 | 0.118 | 0.116 |
| Ranking | $B \succ A$ | | | $B \succ A$ | | |

Table 2 Fuzzy numbers ranking for Example 2

| Ranking method | A | B | C | Ranking |
|---|---|---|---|---|
| Distance method[19] | 0.590 | 0.604 | 0.662 | $C \succ B \succ A$ |
| Possibilistic mean[14] | 0.217 | 0.232 | 0.292 | $C \succ B \succ A$ |
| Convex combined method[22] | $\phi(A,B)=.5$ $\phi(A,C)=.2$ | $\phi(B,A)=.5$ $\phi(B,C)=.3$ | $\phi(C,A)=.8$ $\phi(C,B)=.7$ | $C \succ B \succ A$ |
| D&P's method[13] | 0.857 | 0.927 | 1.000 | $C \succ B \succ A$ |
| Jain's method[17] | 0.452 | 0.519 | 0.639 | $C \succ B \succ A$ |
| Ordinary number[20] | 0.325 | 0.348 | 0.438 | $C \succ B \succ A$ |
| Math. Expectation [21] | 0.050 | 0.073 | 0.101 | $C \succ B \succ A$ |
| Proposed method | 0.337 | 0.361 | 0.456 | $C \succ B \succ A$ |

**Example 3:** Fuzzy numbers have the same supports and differ only in the most promising values.

Three triangular numbers, i.e. A(0.4,0.5,1;1), B(0.4,0.7,1;1), C(0.4,0.9,1;1), taken from Cheng [19]. These three fuzzy numbers have common supports and differ only in the most promising value of the triplet. From Table 3, it can be seen that the current method gives a ranking of the fuzzy numbers as $C \succ B \succ A$, which is consistent with the ranking results provided by the other listed methods.

**Example 4**: Comparison among different fuzzy distribution shapes, and generalized fuzzy numbers.

This example considers normal and subnormal triangular fuzzy numbers, and normal and subnormal trapezoidal fuzzy numbers with different supports and heights. The numbers are taken from [19] and are presented in Fig. 9, which shows a normal triangular fuzzy number $A_1$(3,5,7;1), a subnormal triangular fuzzy number.$A_2$ (3,5,7;0.8), a normal trapezoid number $B_1$(5,7,9,10;1), two subnormal trapezoid numbers $B_2$ (6,7,9,10;0.6) and $B_3$ (7,8,9,10;0.4).

From Table 4, it can be seen that the current ranking approach gives a classification of $B_3 \succ B_2 \succ B_1 \succ A_1 \succ A_2$, which is consistent with Cheng's Distance method [19]. It should be noted that, in this more complex case, the comparison can be performed only by the currently proposed method and by the approach developed in [19].

Table 3 Fuzzy numbers ranking for Example 3

| Ranking method | A | B | C | Ranking |
|---|---|---|---|---|
| Distance method[19] | 0.790 | 0.860 | 0.927 | $C \succ B \succ A$ |
| Possibilistic mean[14] | 0.400 | 0.467 | 0.533 | $C \succ B \succ A$ |
| Convex combined method[22] | $\phi(A,B)=.3$ $\phi(A,C)=.2$ | $\phi(B,A)=.7$ $\phi(B,C)=.3$ | $\phi(C,A)=.8$ $\phi(C,B)=.7$ | $C \succ B \succ A$ |
| D&P's method[13] | 0.600 | 0.750 | 1.000 | $C \succ B \succ A$ |
| Jain's method[17] | 0.545 | 0.667 | 0.857 | $C \succ B \succ A$ |
| Ordinary number[20] | 0.600 | 0.700 | 0.800 | $C \succ B \succ A$ |
| Math. expectation[21] | 0.190 | 0.210 | 0.230 | $C \succ B \succ A$ |
| Proposed method | 0.641 | 0.707 | 0.773 | $C \succ B \succ A$ |

Table 4 Fuzzy numbers ranking for Example 4

| Ranking method | Distance method[19] | | | Proposed method | | |
|---|---|---|---|---|---|---|
| | $x_0$ | $y_0$ | R(.) | domain mean | grade mean | geometric mean |
| $A_1$ | 5.000 | 0.500 | 5.025 | 5.000 | 0.667 | 5.044 |
| $A_2$ | 5.000 | 0.400 | 5.016 | 5.000 | 0.427 | 5.018 |
| $B_1$ | 7.714 | 0.505 | 7.731 | 7.714 | 1.500 | 7.859 |
| $B_2$ | 8.000 | 0.300 | 8.006 | 8.000 | 0.480 | 8.014 |
| $B_3$ | 8.500 | 0.200 | 8.502 | 8.500 | 0.133 | 8.501 |
| Ranking | $B_3 \succ B_2 \succ B_1 \succ A_1 \succ A_2$ | | | $B_3 \succ B_2 \succ B_1 \succ A_1 \succ A_2$ | | |

## 7. Conclusions

This paper has proposed a probabilistic approach to the ranking of generalized fuzzy numbers that is based upon their geometric mean and variance as derived from their domain and membership grade. The currently proposed approach is computationally simple and its underlying concepts are logically sound. A comparative study is conducted on cases used in the previous literatures to examine its performance on rationality and discriminatory ability. It should be noted that the proposed approach is suitable for problems that contain both normal and non-normal fuzzy numbers. However, the other previously proposed methods could be taken to apply in the cases that contain only normal fuzzy numbers.

## 8. Reference

[1] S.H. Chen, "Operations on fuzzy numbers with function principle", *Tamkang Journal of Management Sciences*, Vol.6, No.10, pp.13-25, 1985.

[2] S.J. Chen and S.M. Chen, "Fuzzy risk analysis based on similarity measures of generalized fuzzy numbers", *IEEE Trans. on Fuzzy Systems*, Vol.11, No.1, pp.45-56, 2003.

[3] S.H. Chen, "Operations of fuzzy numbers with step form membership function using function principle", *Information Sciences*, Vol.108, pp.149-155, 1998.

[4] C.H. Hsieh, "A model and algorithm of fuzzy product positioning", *Information Sciences*, Vol.121, No.1-2, pp.61-82, 1999.

[5] S.J. Chen and C.L. Hwang, Fuzzy multiple attribute decision making methods and applications, *in Lecture Notes in Economics and Mathematical Systems*, Springer, Berlin, 1992.

[6] E.S. Lee and R.L. Li, Comparison of fuzzy numbers based on the probability measure of fuzzy events, *Computers and mathematics with Applications*, 15(10) (1988) 887-896.

[7] G. Bortolan and R. Degani, "A Review of Some Methods for Ranking Fuzzy Subsets", *Fuzzy Sets and Systems*, 15 (1985) 1-19.

[8] J.M. Adamo, Fuzzy decision trees, *Fuzzy Sets and Systems*, 4 (1980) 207-220.

[9] S.M. Bass and H. Kwakernaak, Rating and ranking of multiple aspect alternative using fuzzy sets, *Automatica*, 13 (1977) 47-58.

[10] M. Delgado, J.L. Verdegay and M.A. Villa, A procedure for ranking fuzzy numbers using fuzzy relations, *Fuzzy Sets and Systems*, 26 (1988) 207-220.

[11] K. Nakamura, Preference relation on a set of fuzzy utilities as a basis for decision making, *Fuzzy Sets and Systems*, 20 (1986) 147-162.

[12] W. Kolodziejczyk, Orlovsky's concept of decision-making with fuzzy preference relation-further results, *Fuzzy Sets and Systems*, 19 (1986) 11-20.

[13] D. Dubois and H. Prade, "Ranking fuzzy numbers in the setting of possibility theory", *Information Sciences*, 30 (1983) 183-224.

[14] C. Carlson, R. Fuller, "On possibilistic mean value and variance of fuzzy numbers", *Fuzzy sets and Systems*, 122 (2001) 315-326.

[15] R.R. Yager, M. Detyniecki and B.B. Meunier, "A Context-dependent method for ordering fuzzy numbers using probabilities", *Information Sciences*, 138 (2001) 237-255.

[16] S.H. Chen, ranking fuzzy numbers with maximizing set and minimizing set, *Fuzzy sets and Systems*, 17 (1985) 113-129.

[17] R. Jain, "Decision-marking in the presence of fuzzy variables", *IEEE Trans. On Systems, Man, Cybernet*, 6 (1976) 698-703.

[18] T.S. Liou, M.J. Wang, Ranking fuzzy numbers with integral value, *Fuzzy sets and Systems*, 50 (1992) 247-255.

[19] C.H. Cheng, A new approach for ranking fuzzy numbers by distance method, *Fuzzy sets and Systems*, 95 (1998) 307-317.

[20] A. Kaufmann and M.M. Gupta, *Fuzzy Mathematical Models in Engineering and Management Science*, Elsevier Science Publishers B.V. 1988.

[21] W. Chang, "Ranking of fuzzy utilities with triangular membership function", Proc. Int. Conf. Of Policy Anal. and Inf. Systems, (1981)263-272.

[22] G. Facchinetti, R.G. Ricci and S. Muzzioli, Note on ranking fuzzy triangular numbers, *International Journal of Intelligent Systems*, 13 (1998) 613-662.

[23] R.M. Tong and P.P. Bonissone, Linguistic solution to fuzzy decision problems, in: Zimmermann Ed., *TIMS/Studies in the Management Science*, 20 (1984 323-334, Elsevier, Amsterdam.

[24] A. Kaufmann and M.M. Gupta, *Introduction to Fuzzy Arithmetic: Theory and Applications*, Van Nostrand Reinhold, New York, 1985.

[25] W.C. Giffin, Transform Techniques for Probability Modeling, Academic Press, New York, 1975.

[26] L. Debnath, *Integral Transform and Their Application*, CRC Press, New York, 1995.

[27] C.S. Park, The Mellin transform in probabilistic cash flow modeling, *The Engineering Economist*, 32 (2) (1987) 115-134.