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A simple differential theory for digital curves

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Abstract
In this note we shall propose a simple, effective algorithm to establish a differential theory for digital curves in the 3D Euclidean space. First, we shall follow the ideas in Chen, Chi and Wu to define the derivative of a function or a vector field along a digital curve by the weighted combination method. Then, we shall define the curvature, torsion and the moving frame of a digital curve. The Frenet formulas will also be discussed. Our approach is conceptually simple and natural. Moreover, the results are also very accurate.

Key-words: digital curves, curvature, torsion, the Frenet formulas

1. Introduction

A digital curve c in the 3D Euclidean space R^3 is an ordered set of points $C = \{p_i \in R^3 : i = 1,2,\cdots,k\}$. The digital curves can be obtained by the dicretization of regular curves or from digital images. To understand the geometric and differential properties of the digital curves is an important objects in CAD or CAGD. Especially, the curvature or the torsion of a regular curve c in the 3D Euclidean space are important differential invariants in the theory of space curves and its applications. These curvatures are determined by the differential of the tangent vectors and the normal and binormal vectors of the curve c.

In this note we shall introduce a simple, effective algorithm to establish a differential theory for digital curves in the 3D Euclidean space. We shall follow the ideas in Chen, Chi and Wu [3] to define the derivative of a function or a vector along a digital curve by the weighted combination method. We shall use the centroid weights in our method. These weights are first proposed in Chen and Wu [1] to improve Taubin's method for the estimation of curvatures on a triangular mesh in the 3D Euclidean space. Then, we shall apply this method and discuss the moving frame of a digital curve and their Frenet formulas. The method fits perfectly with the proposal given in Rosenfeld and Klette [5] about the field of digital geometry.

Usually, the accurate estimation of curvatures at vertices of a digital curve plays as the first step for many applications such as simplification, smoothing, subdivision, visualization and image processing, etc. Our estimation is simple and very accurate as we illustrate them in the computational results.

2. The local theory for regular curves

In this section we recall some basic notions and results about the local theory of smooth regular curve in in the 3D Euclidean space. See do Carmo [4] for details. Consider a smooth regular curve $c(s) = (x(s), y(s), z(s))$, $s \in [0, l]$ with arc length parameter s. The tangent vector $c'(s) = (x'(s), y'(s), z'(s))$, denoted by $\vec{t}(s)$, is a unit vector since s is the arc length parameter. The number $\|\vec{t}'(s)\| = \kappa(s)$ is called the curvature of c at s. At points where $\kappa(s) \neq 0$, a unit vector $\vec{n}(s)$ in the direction $\vec{t}'(s)$ is well-defined by the equation $\vec{t}'(s) = \kappa(s)\vec{n}(s)$. Since the tangent vector $\vec{t}(s)$ is a unit vector for all $s \in [0, l]$, $\vec{n}(s)$ is normal to $\vec{t}(s)$. Because by differentiating $\vec{t}(s) \cdot \vec{t}(s) = 1$, we have $\vec{t}(s) \cdot \vec{n}(s) = 0$, $\vec{n}(s)$ is normal to $\vec{t}(s)$ and is called the normal vector of c at s.

The plane determined by the unit tangent and normal vectors, $\vec{n}(s)$ and $\vec{t}(s)$, is called the osculating plane of c at s. At points where $\kappa(s) = 0$, the normal vector and hence the osculating plane are not defined. In what follows, we shall restrict ourselves to curves parametrized by arc length with $\kappa(s) \neq 0$ for all $s \in [0, l]$.

The unit vector $\vec{b}(s)=\vec{t}(s)\times\vec{n}(s)$ is normal to the osculating plane and will be called the binormal vector of c at s s. Since the binormal vector $\vec{b}(s)$, the number $\|\vec{b}'(s)\|$ measures the rate of change of the neighboring osculating planes of c cat s. That is, $\|\vec{b}'(s)\|$ measures how rapidly the curves pulled away from the osculating plane of c at s in a neighborhood of s. From the equation

$$\begin{aligned}\vec{b}'(s) &= \vec{t}'(s)\times\vec{n}(s)+\vec{t}(s)\times\vec{n}'(s)\\ &= \vec{t}(s)\times\vec{n}'(s),\end{aligned}$$

we know that $\vec{b}'(s)$ is normal to $\vec{t}(s)$. It follows that $\vec{b}'(s)$ is parallel to $\vec{n}(s)$, and we can write

$$\vec{b}'(s)=\tau(s)\vec{n}(s)$$

for some number $\tau(s)$. This number $\tau(s)$ is called the torsion of the curve c at s. Since the normal vector $\vec{n}(s)$ can be written as $\vec{n}(s)=\vec{b}(s)\times\vec{t}(s)$, we have

$$\begin{aligned}\vec{n}'(s) &= \vec{b}'(s)\times\vec{t}(s)+\vec{b}(s)\times\vec{t}'(s)\\ &= -\tau(s)\vec{b}(s)-\kappa(s)\vec{t}(s).\end{aligned}$$

Therefore we have the Frenet formulas:

(1) $\quad \vec{t}'(s)=\kappa(s)\vec{n}(s)$,

(2) $\quad \vec{n}'(s)=-\tau(s)\vec{b}(s)-\kappa(s)\vec{t}(s)$,

(3) $\quad \vec{b}'(s)=\tau(s)\vec{n}(s)$,

The Frenet formulas forms a system of ordinary differential equations for the vectors $\vec{t}(s)$, $\vec{n}(s)$ and $\vec{b}(s)$. Thus the existence and uniqueness theorems for a system of ordinary differential equations quarantee that given smooth functions $\kappa(s)>0$ and $\tau(s)$, $s\in[0,l]$, there exists a regular parametrized curve $c(s)$ with arc length s so that $\kappa(s)$ is the curvature of c at s and $\tau(s)$ the torsion of c at s. Moreover, the regular parametrized curve $c(s)$ with arc length s is unique up to rigid motion.

When the paprameter of a regular curve $c(t)$ is not arc length, the curvature $\kappa(t)$ and the torsion $\tau(t)$ of the curve $c(t)$ at t can also be computed by

$$(4)\qquad \begin{cases}\kappa(t)=\dfrac{\|c'(t)\times c''(t)\|}{\|c'(t)\|^3}\\[2ex] \tau(t)=-\dfrac{(c'(t)\times c''(t))\cdot c'''(t)}{\|c'(t)\times c''(t)\|^2}\end{cases}.$$

3. Our discrete differential theory for digital curves

In this section we shall propose an algorithm to develop a discrete, differential theory for a digital curve. Recall that a digital curve c in the 3D Euclidean space is an ordered set of points $C=\{p_i\in R^3 : i=1,2,\cdots,k\}$. To define the tangent vector $\vec{t}_i$ and the normal vector $\vec{n}_i$ and the binormal vector $\vec{b}_i$ of the digital curve C at the point p_i is the first step to develop the geometric theory for digital curves. To handle this, we need to formulate the concept of the derivative of a function or a vector field defined on a digital curve C. Our idea is to use the weighted combination method as employed in Chen, Wu [1,2] and Chen, Chi and Wu[3].

Consider a point p_i in the digital curve C. We can define the tangent vector $\vec{t}_i$ of C at the point p_i by

$$\vec{t}_i=\frac{(\omega_1\dfrac{p_i-p_{i-1}}{\|p_i-p_{i-1}\|}+\omega_2\dfrac{p_{i+1}-p_i}{\|p_{i+1}-p_i\|})}{\left\|\omega_1\dfrac{p_i-p_{i-1}}{\|p_i-p_{i-1}\|}+\omega_2\dfrac{p_{i+1}-p_i}{\|p_{i+1}-p_i\|}\right\|}$$

where ω_1 and ω_2 are nonnegative weights with $\omega_1+\omega_2=1$. Now the normal vector $\vec{n}_i$ can be computed as follows First we compute the derivative $\vec{t}_i'$ of the tangent field $\vec{t}_i$ of C at the point p_i by

$$\vec{t}_i'=\omega_1\frac{\vec{t}_i-\vec{t}_{i-1}}{\|p_i-p_{i-1}\|}+\omega_2\frac{\vec{t}_{i+1}-\vec{t}_i}{\|p_{i+1}-p_i\|}.$$

Note that the vector $\vec{t}_i'$ may not be normal to the tangent vector $\vec{t}_i$. Hence we can define the curvature κ_i and the normal vector $\vec{n}_i$ of the digital curve C at p_i by

$$\begin{cases}\kappa_i=\|\vec{t}_i'-(\vec{t}_i'\cdot\vec{t}_i)\vec{t}_i\|\\[2ex] \vec{n}_i=\dfrac{(\vec{t}_i'-(\vec{t}_i'\cdot\vec{t}_i)\vec{t}_i)}{\|\vec{t}_i'-(\vec{t}_i'\cdot\vec{t}_i)\vec{t}_i\|}\end{cases}.$$

Now the binormal vector $\vec{b}_i$ of the digital curve C at p_i can be defined by $\vec{b}_i=\vec{t}_i\times\vec{n}_i$. Next, we consider the torsion τ_i of the digital curve C at p_i via the derivative of the binormal vector field $\vec{b}_i$. We have

$$\vec{b}_i'=\omega_1\frac{\vec{b}_i-\vec{b}_{i-1}}{\|p_i-p_{i-1}\|}+\omega_2\frac{\vec{b}_{i+1}-\vec{b}_i}{\|p_{i+1}-p_i\|}$$

and the torsion τ_i can be defined by $\tau_i = \vec{b}_i' \cdot \vec{n}_i$. The discrete version of the Frenet formulas will then have the form

(5) $\vec{t}_i' = a_{11}\vec{t}_i + \kappa_i \vec{n}_i$,

(6) $\vec{n}_i' = a_{21}\vec{t}_i + a_{22}\vec{n}_i + a_{23}\vec{b}_i$,

(7) $\vec{b}_i' = a_{31}\vec{t}_i + \tau_i \vec{n}_i + a_{33}\vec{b}_i$.

where the coefficients a_{ij} may not be zero when compared with the Frenet formula. This is due to the digital effect of the digital curve C.

From these discussions, we can define the derivative of a function f or a vector field V on a digital curve C by

$$\begin{cases} f'(p_i) = \omega_1 \dfrac{f(p_i) - f(p_{i-1})}{\|p_i - p_{i-1}\|} + \omega_2 \dfrac{f(p_{i+1}) - f(p_i)}{\|p_{i+1} - p_i\|} \\ V'(p_i) = \omega_1 \dfrac{V(p_i) - V(p_{i-1})}{\|p_i - p_{i-1}\|} + \omega_2 \dfrac{V(p_{i+1}) - V(p_i)}{\|p_{i+1} - p_i\|} \end{cases}$$

Indeed, when we know how to differentiate functions and vector fields on a digital curve C, we can develop a differential theory on C.

From the experience given in Chen, Wu [1,2]and Chen, Chi and Wu[3], We shall use the centroid weights for the weights ω_1 and ω_2. Namely, for the digital curve $C = \{p_i \in R^3 : i = 1,2,\cdots,k\}$, we have at the point p_i

$$\begin{cases} \omega_1 = \dfrac{\dfrac{1}{\|p_i - p_{i-1}\|^2}}{\left(\dfrac{1}{\|p_i - p_{i-1}\|^2} + \dfrac{1}{\|p_{i+1} - p_i\|^2}\right)} \\ \omega_2 = \dfrac{\dfrac{1}{\|p_{i+1} - p_i\|^2}}{\left(\dfrac{1}{\|p_i - p_{i-1}\|^2} + \dfrac{1}{\|p_{i+1} - p_i\|^2}\right)} \end{cases}.$$

4. Computational results

In this section, we will find the Frenet matrix of helix curves :

$$c(t) = (a\cos\frac{s}{c}, a\sin\frac{s}{c}, \frac{b}{c})$$

where a, b are positive number in R and $c = \sqrt{a^2 + b^2}$. and Bezier curves by our discrete differential method. The Frenet matrix of our method is a 3×3 matrix forms as:

$$F = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

where a_{ij} is defined by equations (5), (6) and (7). We will compare the error between the exact Frenet matrix and our results by

$$Error = \frac{\| RF - F \|}{\| RF \|}$$

where RF is the exact Frenet matrix in differential geometry and $\|\bullet\|$ is the norm of matrix. We digitize the curves by different kinds of partitions. Then compute all error of Frenet matrices and observe their average. When we test the helix, we chose 1,000 random values a, b for each kind of partition. In Bezier curves, we test 1,000 random Bezier curves (with different control points) for each partition.

In figure 1, we digital the helix curves by uniform partition and chose the values a, b by random positive number between (0,10] and without any noisy. In this figure, the estimations are very close to the exactly Frenet matrix. In figure 2, we observe the effect of noisy. Although these results didn't converge to the exactly values, but this method is still relatively stability.

Because any curves could be approximated by the Bezier curves, locally. In figure 3 , we test the Bezier curve with control points $b_i = [-5,5]^3 \in R^3$ and degree $n \in \{2,3,4,5\}$. And we show the results of Bezier curves when its degree is more than 10 in figure 4. From this figure, our method is still relatively stable even for curves with higher degrees. Finally, we show the standard derivations and variations of these results. From these results, we come to a conclusion that the discrete differential method is a relatively stable estimation method to find the Frenet matrix and hence the curvatures and torsions.

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degree	Noise	bezier standard deviation	variation
less than 5	noise 0%	0.0017302	2.994E-06
	noise 50%	0.0020139	4.056E-06
More than 10	noise 0%	0.0045716	2.09E-05
	noise 50%	0.0020624	4.254E-06

Table 1. variation of Bezier with different degree.

	helix		bezier	
	standard deviation	variation	standard deviation	variation
noise 0%	2.92966E-06	8.6E-12	0.001730198	3E-06
noise 1%	0.000212896	4.5E-08	0.001533111	2.4E-06
noise 2%	0.000433369	1.9E-07	0.000926984	8.6E-07
noise 3%	0.000639214	4.1E-07	0.001385454	1.9E-06
noise 4%	0.00077753	6E-07	0.000219752	4.8E-08
noise 5%	0.000998233	1E-06	0.000929912	8.6E-07
noise 50%	0.011491019	0.00013		

Table 2. the variation and standard deviation of helix curves and Bezier curves.

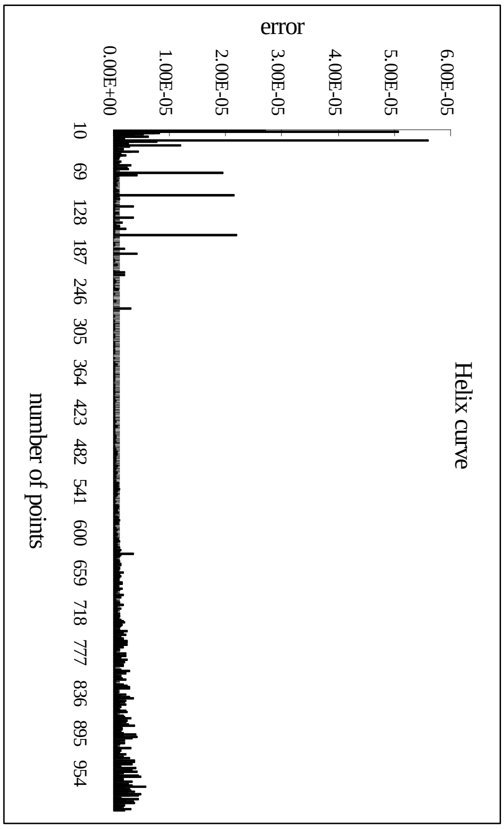

Fig 1. Helix curves without noisy

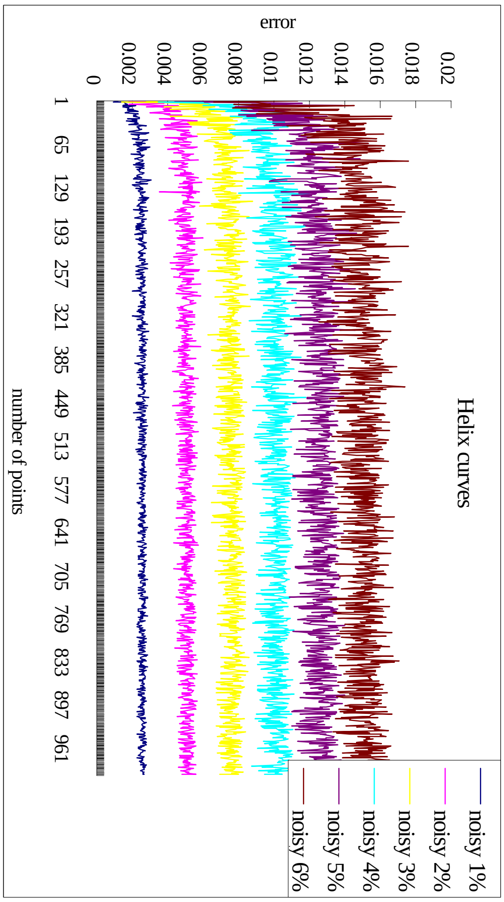

Fig 2. Helix curves with noisy

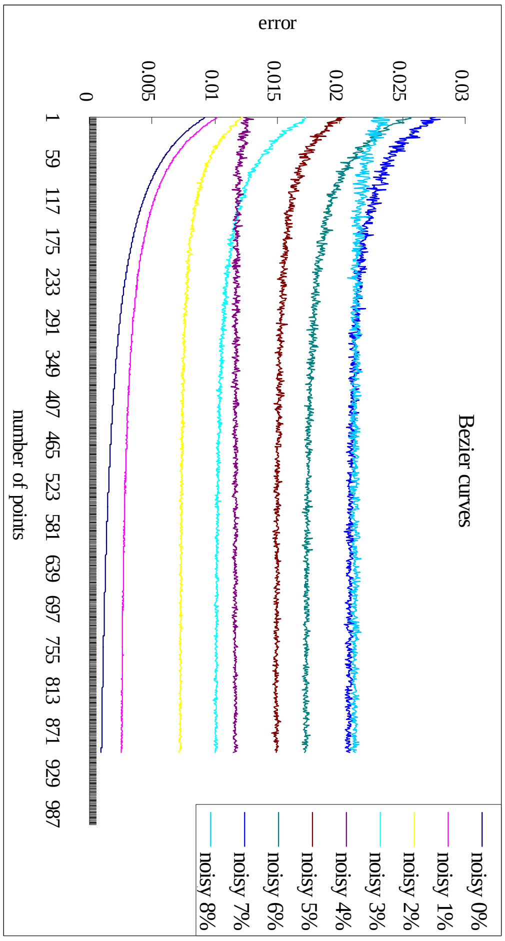

Fig 3. Bezier curves with noisy

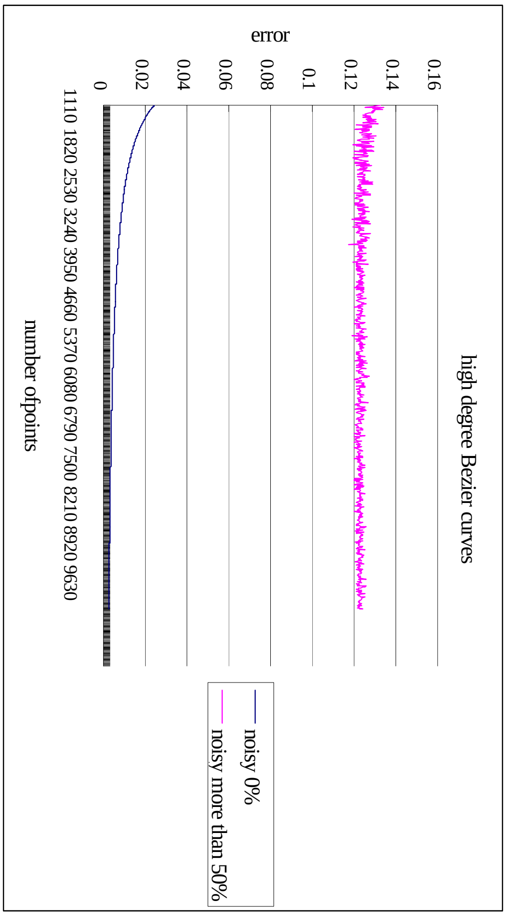

Fig 4. Bezier curves of high degree (more than 10)