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Adaptive Stabilization of Linear Time-Varying Systems

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Abstract: - This paper addresses the problem of adaptive stabilization of continuous-time and time-varying linear of the form $\dot{x}(t) = \Theta(t)x(t) + Bu(t)$, when the state is accessible and with an input vector of the same dimension as the vector state. The proposed control scheme guarantees that the state of the plant, with bounded time-varying parameters, asymptotically converges to zero. For the general case when there are $2n^2$ unknown parameters (n^2 time-varying and n^2 constant), a controller adjusting n^2+1 parameters (when the control matrix B is unknown) guarantees local stability results, whereas in the case when the control matrix B is known, only one parameter has to be adjusted in the controller and the proposed scheme provides global stability results

Key-Words: Adaptive stabilization, adaptive control, stabilization of time-varying plants, adaptive stabilization of linear time-varying systems.

1 Introduction

Lately, the adaptive control of time-varying linear and nonlinear plants has received considerably attention. Several approaches have been proposed to face this problem making use of different techniques [1-6]. For example, recently, Ge and Wang [6] proposed a robust adaptive tracking method for time-varying nonlinear systems in the strict feedback form with completely unknown time-varying virtual control coefficients, uncertain time-varying parameters and unknown time-varying bounded disturbances. The proposed design method does not require any a priori knowledge of the unknown coefficients except for their bounds.

In the case of time-varying uncertain chaotic systems Li, Chen, Shi and Han proposed a robust adaptive tracking control [7] for a class of nonlinear plants when the control matrix is known and equal to the identity. The desired trajectory and its first time derivative are assumed to be known. The method imposes two assumptions, being the second one very restrictive on the plant to be controlled. This method was simplified in [8] relaxing the second assumption and assuming that a desired trajectory and its first time derivative are known to the designer. Later, in [9, a method where the constraint of the second assumption is moved from the plant to the reference model introduced, is presented. Lately, a further attempt to generalize these results was made in [10] where it is considered only one assumption concerning the boundedness of the time-varying parameters and both cases when a model reference or a desired trajectory and its time derivative are known, were resolved.

In this paper a new effort to generalize these previous results is made, considering the adaptive stabilization of a class of nonlinear plants with arbitrarily fast time-variations, when the control matrix B is unknown but constant and boundedness on the time-varying parameters is the only assumption.

2 Adaptive Stabilization of Linear Time-Varying Plants with State Accessible

Let us consider the case of a time-varying and linear plant with accessible state defined by the following differential equation

$$\dot{x}(t) = \Theta(t)x(t) + Bu(t) \tag{1}$$

where $x(t) \in \Re^n$ is the state of the system, $\Theta(t) \in \Re^{nxn}$ represents the matrix of time-varying and unknown parameters and the matrix $B \in \Re^{nxn}$ is a nonsingular matrix of unknown but constant parameters. $u(t) \in \Re^n$ is the plant input. It is assumed that time-varying elements of the matrix $\Theta(t)$ are bounded, as stated in the following assumption .

Assumption 1: The matrix $\Theta(t) \in \Re^{nxn}$, belongs to a bounded and closed set Ω defined by $\Omega = \left\{\Theta(t) = \left[\theta_{ij}(t)\right] \in \Re^{nxn} / \theta_{ij}(t) \in [\underline{\theta}_{ij}, \overline{\theta}_{ij}]\right\}$, with $\underline{\theta}_{ij}, \overline{\theta}_{ij}$ for $i, j = 1, 2, ..., n$ unknown constant parameters representing the lower and upper bounds respectively on the time-varying parameters $\theta_{ij}(t)$, the elements of matrix $\Theta(t)$. ■

The plant given in (1) can be rewritten as follows

$$\dot{x}(t) = A_m x(t) + (\Theta(t) - A_m)x(t) + Bu(t) \qquad (2)$$

where $A_m \in \Re^{n \times n}$ is any asymptotically stable matrix.

We define the unknown matrix with time-varying parameters $\overline{A}(t) = \left[\overline{a}_{ij}(t)\right] \in \Re^{nxn}$ as

$$\overline{A}(t) = \Theta(t) - A_m \in \Re^{nxn} \qquad (3)$$

From Assumption 1, we can write the following inequality

$$\begin{aligned} \left\|\overline{A}(t)\right\|_F &= \left(Tr\left\{\overline{A}^T\overline{A}\right\}\right)^{1/2} = \left(\sum_{i=1}^{n}\sum_{j=1}^{n}\overline{a}_{ij}^{\,2}(t)\right)^{1/2} = \\ &= \left(\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\theta_{ij}(t) - a_{ij}\right)^2\right)^{1/2} \le \\ &\le \left(\sum_{i=1}^{n}\sum_{j=1}^{n}\max\left\{\left|\underline{\theta}_{ij} - a_{ij}\right|^2, \left|\overline{\theta}_{ij} - a_{iji}\right|^2\right\}\right)^{1/2} \triangleq \beta \end{aligned} \qquad (4)$$

where $\beta \in \Re$ is n unknown constant parameter. $\|\cdot\|_F$ denotes the Frobenius norm of a matrix.
Now we can state the following theorem for adaptively control system (1).

Theorem 1: *Let us consider the linear time-varying system defined by (1). Let us assume that Assumption 1 is satisfied and define the control law as*

$$u(t) = K(t)\alpha(x, \hat{\beta}) \qquad (5)$$

where $K(t) \in \Re^{nxn}$ *is an adjustable parameter and* $\alpha(x, \hat{\beta}) \in \Re^n$, $\mu(x) \in \Re^{nxn}$ *are given by*

$$\alpha(x, \hat{\beta}) = -\frac{\|x\|^2 Px\hat{\beta}^2}{\|\mu(x)\|_F \hat{\beta} + \varepsilon\|x\|^2} \qquad (6)$$

$$\mu(x) = xx^T P \qquad (7)$$

with $0 < 2\varepsilon < \lambda_{\min}(Q)$ *and* $P = P^T \in \Re^{nxn}$ *solution of* $A_m{}^T P + PA_m = -Q$ *. Let us consider the adaptive law for* $\hat{\beta}(t)$ *given by*

$$\dot{\hat{\beta}}(t) = \gamma\|\mu(x)\|_F \qquad (8)$$

with $\hat{\beta}(t_0) > 0$ *and* $\gamma > 0$ *an adaptive gain, together with the adaptive law for* $K(t) \in \Re^{nxn}$ *(the estimate of* $K^* = B^{-1} \in \Re^{nxn}$ *)*

$$\dot{K}(t) = -K(t)B_m{}^T Pxu^T K(t) \qquad (9)$$

Then, the overall adaptive system is uniformly stable and also the state of the linear time-varying system (1) will asymptotically converge to zero. ■

It is important to notice that using the property $\frac{d}{dt}\left(K^{-1}(t)\right) = -K(t)\left(\frac{d}{dt}K(t)\right)K(t)$, the adaptive law (9) can be written equivalently as follows

$$\dot{\Phi}_K(t) = Pxu^T \qquad (10)$$

where

$$\Phi_K(t) = K^{-1}(t) - K^{*-1} \quad \text{with} \quad K^{*-1} = B \in \Re^{nxn} \qquad (11)$$

Proof:

Replacing *u(t)* from (5) in the equation (1) we get

$$\dot{x}(t) = A_m x(t) + \left(\Theta(t) - A_m\right) x(t) + BK(t)\alpha(x,\hat{\beta})$$

which can be rewritten using definition (3) as

$$\dot{x}(t) = A_m x(t) + \bar{A}(t)x(t) + BK(t)\alpha(x,\hat{\beta}) \qquad (12)$$

Adding and subtracting the term $\alpha(x,\hat{\beta})$ from (6) we obtain

$$\begin{aligned}\dot{x}(t) &= A_m x(t) + \bar{A}(t)x(t) + \alpha(x,\hat{\beta}) - \\ &- \left(I - BK(t)\right)\alpha(x,\hat{\beta})\end{aligned} \qquad (13)$$

We define $K^* \in \Re^{nxn}$ such that the following condition is satisfied

$$\left[I - BK^*\right] = 0 \quad \Rightarrow \quad K^* = B^{-1} \quad \text{or} \quad K^{*-1} = B$$

Then we can write equation (48) as

$$\begin{aligned}\dot{x}(t) &= A_m x(t) + \bar{A}(t)x(t) - \\ &- \left(K^{-1}(t) - K^{*-1}\right)u(t) + \alpha(x,\hat{\beta})\end{aligned} \qquad (14)$$

or equivalently

$$\begin{aligned}\dot{x}(t) &= A_m x(t) + \bar{A}(t)x(t) - \\ &- \Phi_K(t)u(t) + \alpha(x,\hat{\beta})\end{aligned} \qquad (15)$$

where $\Phi_K(t) \in \Re^{nxn}$ is defined in (11).

In order to prove the stability of the overall daptive system, we choose the following Lyapunov function candidate

$$V(x,\Phi_K,\tilde{\beta}) = \frac{1}{2}\left(x^T P x + Trace\left\{\Phi_K \Phi_K^T\right\} + \frac{1}{\gamma}\tilde{\beta}^2 \right) \qquad (16)$$

where $\tilde{\beta}(t) = \hat{\beta}(t) - \beta \in \Re$ ..

Computing the first derivative of (16) we have

$$\dot{V} = \frac{1}{2}\left(\dot{x}^T P x + x^T P \dot{x} + 2Trace\left\{\dot{\Phi}_K \Phi_K^T\right\} + \frac{2}{\gamma}\tilde{\beta}\dot{\tilde{\beta}} \right) \qquad (17)$$

Evaluating (17) along the trajectory defined by (12) we get

$$\begin{aligned}\dot{V} &= \frac{1}{2} x^T (A_m^T P + P A_m) x + x^T P \bar{A}(t) x(t) - \\ &- x^T P B_m \Phi_K(t) u(t) + x^T P \alpha(e,x,\hat{\beta}) + \\ &+ Trace\left\{\dot{\Phi}_K \Phi_K^T\right\} + \frac{2}{\gamma}\tilde{\beta}\dot{\tilde{\beta}}\end{aligned} \qquad (18)$$

Using $A_m^T P + P A_m = -Q$ and replacing $\alpha(x,\hat{\beta})$ from (6) we can write

$$\begin{aligned}\dot{V} &= -\frac{1}{2} x^T Q x + x^T P \bar{A}(t) x(t) - \\ &- x^T P B_m \Phi_K(t) u(t) - \frac{\left\|x^T P\right\|^2 \left\|x\right\|^2 \hat{\beta}^2}{\left\|\mu(x)\right\|_F \hat{\beta} + \varepsilon \left\|x\right\|^2} + \\ &+ Trace\left\{\dot{\Phi}_K \Phi_K^T\right\} + \frac{2}{\gamma}\tilde{\beta}\dot{\tilde{\beta}}\end{aligned} \qquad (19)$$

Since Assumption 1 is satisfied then we can write the following inequality:

$$\begin{aligned}x^T P \bar{A}(t) x(t) &\le \left\|x^T P \bar{A}(t) x(t)\right\| \le \\ &\le \left\|x^T P\right\| \left\|\bar{A}(t) x(t)\right\|\end{aligned} \qquad (20)$$

We can also write

$$\begin{aligned}\left\|\bar{A}(t)x(t)\right\| &= \left(\sum_{i=1}^{n} \sum_{j=1}^{n} \left(\bar{a}_{ij}(t) x_j(t)\right)^2 \right)^{1/2} \le \\ &\le \left(\sum_{i=1}^{n} \sum_{j=1}^{n} \left(\bar{a}_{ij}^2(t) x_j^2(t)\right) \right)^{1/2} \le \\ &\le \left(\sum_{i=1}^{n} \sum_{j=1}^{n} \bar{a}_{ij}^2(t) \left\|x(t)\right\|^2 \right)^{1/2} \le \\ &\le \left\|x(t)\right\| \left\|\bar{A}(t)\right\|_F \le \left\|x(t)\right\| \beta\end{aligned} \qquad (21)$$

Since we know that

$$\begin{aligned}\left\|ab^T\right\|_F &= \left\|a\right\| \left\|b\right\| \quad \text{and} \quad a^T b = b^T a = \\ &= Trace\left\{ab^T\right\} = Trace\left\{ba^T\right\}\end{aligned} \qquad (22)$$

then we can write

$$x^T P \bar{A}(t) x(t) \le \left\|\mu(x)\right\|_F \beta \qquad (23)$$

On the other hand we have the following inequality

$$\begin{aligned}\frac{\|\mu(x)\|_F^2\,\hat{\beta}^2}{\|\mu(x)\|_F\,\hat{\beta}+\varepsilon\|x\|^2} &= \|\mu(x)\|_F\,\hat{\beta}\left(-1+\frac{\varepsilon\|x\|^2}{\|\mu(x)\|_F\,\hat{\beta}+\varepsilon\|x\|^2}\right) \\ &\le \|\mu(x)\|_F\,\hat{\beta}\left(-1+\frac{\varepsilon\|x\|^2}{\|\mu(x)\|_F\,\hat{\beta}}\right)\end{aligned} \tag{24}$$

Furthermore it is easy to verify that

$$-\frac{1}{2}x^T Q x \le -\frac{\lambda_{\min}(Q)}{2}\|x\|^2 \tag{25}$$

where $\lambda_{\min}(Q)$ is the minimum eigenvalue of the positive definite matrix Q.

Replacing (23), (24) and (25) in (19) we obtain

$$\begin{aligned}\dot{V} \le {} & -\left(\frac{1}{2}\lambda_{\min}(Q)-\varepsilon\right)\|x\|^2-\|\mu(x)\|_F\,\tilde{\beta}- \\ & -Trace\left\{Pxu^T\Phi_K{}^T\right\}+Trace\left\{\dot{\Phi}_K\Phi_K{}^T\right\}+\frac{2}{\gamma}\tilde{\beta}\dot{\tilde{\beta}}\end{aligned} \tag{26}$$

Finally, replacing in (26) the adaptive laws given by (8) and (10) we obtain

$$\dot{V} \le -\left(\frac{1}{2}\lambda_{\min}(Q)-\varepsilon\right)\|x\|^2 \tag{27}$$

Since $0 < 2\varepsilon < \lambda_{\min}(Q)$ then $\dot{V} \le 0$ and the overall system is globally uniformly stable. In particular $x(t), \Phi_K(t)$ and $\tilde{\beta}(t)$ are globally uniformly bounded. From this we can conclude that $\hat{\beta}(t)$ is also globally uniformly bounded. From the definition of $\Phi_K(t)$ given in (11) we can conclude that $K^{-1}(t) \in \Re^{nxn}$ is globally uniformly bounded but $K(t) \in \Re^{nxn}$ is only locally uniformly bounded. From equation (27) we can conclude that $x(t)$ is a signal of square integral ($x(t) \in L^2$). From equations (5), (6) and (7) it follows that the control signal $u(t)$ is locally uniformly bounded. Consequently, from equation (15) we conclude that $\dot{x}(t)$ is locally uniformly bounded ($\dot{x}(t) \in L^\infty$), since it corresponds to a sum and products of locally uniformly bounded functions. Using the Lemma of Barbalat [11], we can conclude that locally, $x(t) \to 0$ when $t \to \infty$. Therefore, the controller given by (5)-(7) and the adaptive laws given by (8) and (9) guarantee that the system (1) is locally uniformly stable. ■

For a more detailed proof of this Theorem the reader is referred to [15].

Remark 1:
Theorem 1, applicable to linear systems with $2n^2$ unknown parameters (n^2 time-varying and n^2 constant) of the form (1), guarantees that locally all the signals remain bounded and the state asymptotically converge to zero by adjusting n^2+1 parameters, providing local uniform stability results for the general case of *B* unknown.

Remark 2:
In the previous analysis, unity adaptive gains were chosen for simplicity in all the adaptive laws (8) and (9) used in the design. It is possible to show that all the results stated in Section 2 will also be valid if constant and positive scalars adaptive gains are used, or constant and positive definite matrices adaptive gains are introduced, or finally, time-varying matrices adaptive gains with a special type of variation are defined [12, 13]. The effect of these adaptive gains will be to improve the transient behavior of the resultant adaptive system.

Remark 3:
The convergence of the controller parameter is not guaranteed in the proposed control scheme. This is achieved only if persistently exciting conditions are met for the vectors and matrices involved in the adaptive laws (8) and (9).

Remark 4:
If the control matrix B has certain particular form, the structure of the proposed control scheme can be simplified and the scope of the method can be enlarged. For example if B is a diagonal matrix, invertible, and the sign of all elements on the diagonal are known, then the resultant controller and adaptive laws have the following form [13, 14]

$$u(t) = K(t)\alpha(x,\hat{\beta})$$

with

$$\dot{K}(t) = sign\{B\}Pxu^T$$

where for notation purposes we have $B = sign\{B\}\left|B\right|$. In this case the results are proven to be global rather that local [13, 14]. Same kind of simplifications can be obtained if the matrix B is positive definite and invertible [13, 14] obtaining again global stability results. Finally, when the matrix B has any general form then we get the results shown in Section 3, which are only local in nature.

Remark 5:
For the case when the matrix *B* is known, it can be shown that the resulting adaptive scheme adjusts only one parameter. In fact, since *B* is known so is $K^* = B^{-1}B_m \in \Re^{nxn}$. Replacing *K(t)* in the control law (5) by the ideal controller parameter K^* the control input becomes

$$u(t) = B^{-1}\alpha(x, \hat{\beta}) \tag{28}$$

with $\alpha(x, \hat{\beta}) \in \Re^n$ and $\mu(x) \in \Re^p$ given by (6) and (7) respectively. Therefore, adaptation for $K(t) \in \Re^{nxn}$ is not needed. Thus, uniform global stability can be achieved for the adaptive system adjusting only the parameter $\hat{\beta} \in \Re$, with the adaptation given by (8).

3 Conclusions

Using Lyapunov's stability theory it was designed a new scheme for adaptive stabilization of time-varying linear plants. This control scheme allows that the state of the plant with bounded time-varying parameters converge asymptotically to zero.

For the linear case given by equation (1) having $2n^2$ unknown parameters (n^2 time-varying and n^2 constant unknown parameters), when the matrix B is unknown the controller has to adjust n^2+1 parameters and the proposed scheme provides only local stability results. On the contrary, when the matrix B is known only one parameter has to be adjusted and the proposed scheme provides global stability results.

In order to verify the behavior of the controller based on Theorem 1 a set of simulations for a second order plant were performed but not shown here for the sake of space, finding that the simulation results are in complete agreement with the theoretically expected results. These simulations will be shown during the presentation of the paper.

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