Proceedings of the 5th WSEAS Int. Conf. on Instrumentation, Measurement, Circuits and Systems, Hangzhou, China, April 16-18, 2006 (pp268-273)

Conditions of Convergence in Distribution for Random Fuzzy Variables

YUANGUO ZHU
Nanjing University of Science and Technology
Department of Applied Mathematics
Nanjing, Jiangsu 210094
CHINA

Abstract: A random fuzzy variable is a function from a possibility space to the set of random variables. Based on the chance distributions for random fuzzy variables, some sufficient and necessary conditions of convergence of random fuzzy sequence in distribution are investigated.

Key–Words: Fuzzy set, Random fuzzy variable, Chance distribution, Convergence

1 Introduction

Fuzziness plays an essential role in the real world. Fuzzy set theory has been developed very fast since it was introduced by Zadeh (1965) [1]. A fuzzy set was characterized with its membership function by Zadeh. The term fuzzy variable was fist introduced by Kaufmann (1975) [2], and then appeared in Zadeh (1978) [3] and Nahmias (1978) [4] as a fuzzy set of real numbers. In order to establish the mathematics of fuzzy set theory, Nahmias (1978) [4] introduced three axioms to define possibility spaces. A fuzzy variable may be defined as a function from a possibility space to the set of real numbers. In order to define a self-dual measure, Liu and Liu (2002) [5] gave the concept of credibility measure. And Liu (2004) [6] presented an axiomatic foundation of credibility theory dealing with fuzzy variables based on credibility measure.

Fuzzy variable was generalized by bifuzzzy variable, random fuzzy variable, and so on. Bifuzzy variable was introduced by Liu (2002) [7] as a function from a possibility space to the set of fuzzy variables. And random fuzzy variable was defined by Liu (2002) [8] as a function from a possibility space to the set of random variables.

Based on the chance measure and expected value operator in Liu (2002) [8] and Liu and Liu (2003) [9], some mathematical properties of random fuzzy variables were derived by Zhu and Liu (2004) [10] [11]. The concept of chance distribution for random fuzzy variables was introduced and several properties of chance distributions were studied in Zhu and Liu (2004) [10]. For random fuzzy sequences, there are several concepts of convergence, for example, convergence almost surely, convergence in chance, convergence in mean and convergence in distribution. It is useful to deal with the criteria of convergence in distribution for random fuzzy sequences.

In the following, we first recall some useful concepts such as possibility spaces, random fuzzy variables and chance distributions. Then we investigate some sufficient and necessary conditions of convergence in distribution for random fuzzy sequences.

2 Some Concepts

In convenience, we give some useful concepts at first. Let Θ be a nonempty set, and $\mathcal{P}(\Theta)$ the power set of Θ. The triplet $(\Theta, \mathcal{P}(\Theta), \mathrm{Pos})$ is said to be a possibility space if Pos, called possibility measure, is a nonnegative set function defined on $\mathcal{P}(\Theta)$ satisfying that (i) $\mathrm{Pos}\{\emptyset\} = 0$, (ii) $\mathrm{Pos}\{\Theta\} = 1$, (iii) $\mathrm{Pos}\{\cup_k A_k\} = \sup_k \mathrm{Pos}\{A_k\}$ for $A_k \in \mathcal{P}(\Theta)$. Another measure Cr, called credibility measure [5], is defined by $\mathrm{Cr}\{A\} = (\mathrm{Pos}\{A\} + 1 - \mathrm{Pos}\{A^c\})/2$ for any $A \in \mathcal{P}(\Theta)$, where A^c is the complementary set of A. A fuzzy variable is defined as a function from a possibility space to the set of real numbers.

Definition 1. *(Liu [8]) A random fuzzy variable is a function from a possibility space* $(\Theta, \mathcal{P}(\Theta), \mathrm{Pos})$ *to the set of random variables.*

Definition 2. *(Liu [8]) Let ξ be a random fuzzy variable, and B a Borel set of $\mathbb{R}$. Then the chance of random fuzzy event $\{\xi \in B\}$ is a function from $(0,1]$ to $[0,1]$, defined as*

$$\mathrm{Ch}\{\xi \in B\}(\alpha) = \sup_{\mathrm{Cr}\{A\} \geq \alpha} \inf_{\theta \in A} \mathrm{Pr}\{\xi(\theta) \in B\}. \quad (1)$$

Proceedings of the 5th WSEAS Int. Conf. on Instrumentation, Measurement, Circuits and Systems, Hangzhou, China, April 16-18, 2006 (pp268-273)

Definition 3. *(Liu [8]) The chance distribution* $\Phi : (-\infty, +\infty) \times (0, 1] \to [0, 1]$ *of a random fuzzy variable* ξ *is defined by* $\Phi(x, \alpha) = \mathrm{Ch}\{\xi \le x\}(\alpha)$.

It follows from Zhu and Liu (2004) [10] that the chance distribution $\Phi(x, \alpha)$ of a random fuzzy variable is increasing in x for any α, and decreasing and left-continuous in α for any x.

Definition 4. *Suppose that* $\Phi, \Phi_1, \Phi_2, \ldots$ *are the chance distributions of random fuzzy variables* $\xi, \xi_1, \xi_2, \ldots$*, respectively. We say that* $\{\xi_i\}$ *converges in distribution to* ξ *if* $\Phi_i(x, \alpha) \to \Phi(x; \alpha)$ *for all continuity points* (x, α) *of* Φ.

3 Sufficient and Necessary Conditions of Convergence

Theorem 5. *Let* $\Phi, \Phi_1, \Phi_2, \ldots$ *be chance distributions of random fuzzy variables* $\xi, \xi_1, \xi_2, \ldots$*, and* ξ_i *converge to* ξ *in distribution. Assume that* a *and* b *are two real numbers such that* Φ *is continuous at* $(a, 1)$ *and* $(b, 1)$*. If* $g(\alpha)$ *is a nonnegative continuous function on* $[0, 1]$ *and* $f(x)$ *is a nonnegative continuous function on* $[a, b]$*, then*

$$\lim_{i\to\infty} \int_a^b f(x)\,\mathrm{d}\int_0^1 g(\alpha)\Phi_i(x, \alpha)\mathrm{d}\alpha = \int_a^b f(x)\,\mathrm{d}\int_0^1 g(\alpha)\Phi(x, \alpha)\mathrm{d}\alpha. \quad (2)$$

Conversely, if (2) *holds for any nonnegative continuous functions* $g(\alpha)$ *and* $f(x)$*, then*

$$\lim_{\delta\downarrow 0}\lim_{i\to\infty} [\Phi_i(x, \alpha-\delta) - \Phi_i(a, \alpha-\delta)] = \Phi(x, \alpha) - \Phi(a, \alpha) \quad (3)$$

for any continuity point (x, α) *of* Φ.

Proof. For any positive integer m, n, let $0 = \alpha_0 < \alpha_1 < \ldots < \alpha_m = 1$ be a set of division points of $(0, 1]$, $a = x_0 < x_1 < \ldots < x_n = b$ a set of division points of $[a, b]$, and let these points be continuity points of all $\Phi_i(x, \alpha), \Phi(x, \alpha)$, $i = 1, 2, \ldots$.

Let I_A be the indicator of a set A. Write

$$g_m(x) = \begin{cases} \sum_{j=0}^{m-1} g(\alpha_j) I_{[\alpha_j, \alpha_{j+1})}(\alpha), & 0 \le \alpha < 1, \\ g(\alpha_{m-1}), & \alpha = 1, \end{cases}$$

$$f_n(x) = \begin{cases} \sum_{\nu=0}^{n-1} f(x_\nu) I_{[x_\nu, x_{\nu+1})}(x), & a \le x < b, \\ f(x_{n-1}), & x = b. \end{cases}$$

Then $g_m(\alpha)$ converges uniformly to $g(\alpha)$ as $\max_j(\alpha_{j+1} - \alpha_j) \to 0$, $f_n(x)$ converges uniformly to $f(x)$ as $\max_\nu(x_{\nu+1} - x_\nu) \to 0$. Suppose that the larger the numbers m and n, the smaller the partitions of the interval $[0, 1]$ and the partitions of the interval $[a, b]$, that is, $\max_j(\alpha_{j+1} - \alpha_j) \to 0$ as $m \to \infty$ and $\max_\nu(x_{\nu+1} - x_\nu) \to 0$ as $n \to \infty$. For arbitrary $\varepsilon > 0$, choose m_1 and n_0 such that $|g_m(\alpha) - g(\alpha)| < \varepsilon$ for all $\alpha \in [0, 1]$ when $m \ge m_1$ and $|f_{n_0}(x) - f(x)| < \varepsilon$ for all $x \in [a, b]$. Denote $M = \sup_{\alpha\in[0,1]} |g(\alpha)|$ and $L = \sup_{x\in[a,b]} |f(x)|$. Then

$$\left|\int_a^b (f(x) - f_{n_0}(x))\,\mathrm{d}\int_0^1 g(\alpha)\Phi_i(x, \alpha)\mathrm{d}\alpha\right| \le M\varepsilon \quad (4)$$

uniformly for i,

$$\left|\int_a^b f_{n_0}(x)\,\mathrm{d}\int_0^1 (g(\alpha) - g_m(\alpha))\Phi_i(x, \alpha)\mathrm{d}\alpha\right| \le L\varepsilon \quad (5)$$

uniformly for i when $m \ge m_1$, and

$$\left|\int_a^b (f(x) - f_{n_0}(x))\,\mathrm{d}\int_0^1 g(\alpha)\Phi(x, \alpha)\mathrm{d}\alpha\right| \le M\varepsilon. \quad (6)$$

Choose a positive number m_2 such that when $m \ge m_2$,

$$\left|\sum_{j=0}^{m-1} g(\alpha_j)\Phi(x_\nu, \alpha_j)(\alpha_{j+1} - \alpha_j) - \int_0^1 g(\alpha)\Phi(x_\nu, \alpha)\mathrm{d}\alpha\right| \le \frac{\varepsilon}{Ln_0}, \quad (7)$$

$$\left|\sum_{j=0}^{m-1} g(\alpha_{j+1})\Phi(x_\nu, \alpha_{j+1})(\alpha_{j+1} - \alpha_j) - \int_0^1 g(\alpha)\Phi(x_\nu, \alpha)\mathrm{d}\alpha\right| \le \frac{\varepsilon}{Ln_0}, \quad (8)$$

for all $\nu = 0, 1, 2, \ldots, n_0$. Choose a positive number m_3 such that when $m \ge m_3$,

$$|g(\alpha_{j+1}) - g(\alpha_j)| \le \frac{\varepsilon}{Ln_0}, \quad (9)$$

for all $j = 0, 1, 2, \ldots, m$. Let $m_0 = \max\{m_1, m_2, m_3\}$. We have

$$J_i = \int_a^b f_{n_0}(x)\,\mathrm{d}\int_0^1 g_{m_0}(\alpha)\Phi_i(x, \alpha)\mathrm{d}\alpha = \sum_{\nu=0}^{n_0-1} f(x_\nu)\int_{x_\nu}^{x_{\nu+1}} \mathrm{d}\int_0^1 g_{m_0}(\alpha)\Phi_i(x, \alpha)\mathrm{d}\alpha$$

$$= \sum_{\nu=0}^{n_0-1} f(x_\nu) \int_0^1 g_{m_0}(\alpha)\left[\Phi_i(x_{\nu+1},\alpha) - \Phi_i(x_\nu,\alpha)\right] \mathrm{d}\alpha$$

$$= \sum_{\nu=0}^{n_0-1} f(x_\nu) \sum_{j=0}^{m_0-1} g(\alpha_j) \int_{\alpha_j}^{\alpha_{j+1}} \left[\Phi_i(x_{\nu+1},\alpha) - \Phi_i(x_\nu,\alpha)\right] \mathrm{d}\alpha.$$

Since $\Phi_i(x,\alpha)$ is decreasing in α, we have

$$\Phi_i(x_\nu,\alpha_{j+1})(\alpha_{j+1}-\alpha_j) \le \int_{\alpha_j}^{\alpha_{j+1}} \Phi_i(x_\nu,\alpha)\mathrm{d}\alpha \le \Phi_i(x_\nu,\alpha_j)(\alpha_{j+1}-\alpha_j), \quad (10)$$

for all $\nu = 0,1,2,\ldots,n_0$. Thus

$$J_i \le J_{1i} \triangleq \sum_{\nu=0}^{n_0-1} f(x_\nu) \sum_{j=0}^{m_0-1} g(\alpha_j)\left[\Phi_i(x_{\nu+1},\alpha_j) - \Phi_i(x_\nu,\alpha_{j+1})\right](\alpha_{j+1}-\alpha_j), \quad (11)$$

and

$$J_i \ge J_{2i} \triangleq \sum_{\nu=0}^{n_0-1} f(x_\nu) \sum_{j=0}^{m_0-1} g(\alpha_j)\left[\Phi_i(x_{\nu+1},\alpha_{j+1}) - \Phi_i(x_\nu,\alpha_j)\right](\alpha_{j+1}-\alpha_j). \quad (12)$$

Letting $i \to \infty$, we get

$$\lim_{i\to\infty} J_{1i} = \sum_{\nu=0}^{n_0-1} f(x_\nu) \sum_{j=0}^{m_0-1} g(\alpha_j)\left[\Phi(x_{\nu+1},\alpha_j) - \Phi(x_\nu,\alpha_{j+1})\right](\alpha_{j+1}-\alpha_j)$$

$$= \sum_{\nu=0}^{n_0-1} f(x_\nu)\Bigg[\Bigg(\sum_{j=0}^{m_0-1} g(\alpha_j)\Phi(x_{\nu+1},\alpha_j)(\alpha_{j+1} - \alpha_j) - \int_0^1 g(\alpha)\Phi(x_{\nu+1},\alpha)\mathrm{d}\alpha\Bigg) + \Bigg(\int_0^1 g(\alpha)\Phi(x_\nu,\alpha)\mathrm{d}\alpha - \sum_{j=0}^{m_0-1} g(\alpha_{j+1}) \Phi(x_\nu,\alpha_{j+1})(\alpha_{j+1}-\alpha_j)\Bigg) + \sum_{j=0}^{m_0-1}(g(\alpha_{j+1})-g(\alpha_j))\Phi(x_\nu,\alpha_{j+1})(\alpha_{j+1} - \alpha_j)\Bigg]$$

$$+ \sum_{\nu=0}^{n_0-1} f(x_\nu)\left[\int_0^1 g(\alpha)\Phi(x_{\nu+1},\alpha)\mathrm{d}\alpha - \int_0^1 g(\alpha)\Phi(x_\nu,\alpha)\mathrm{d}\alpha\right]$$

$$\le 3\varepsilon + \sum_{\nu=0}^{n_0-1} f(x_\nu)\left[\int_0^1 g(\alpha)\Phi(x_{\nu+1},\alpha)\mathrm{d}\alpha - \int_0^1 g(\alpha)\Phi(x_\nu,\alpha)\mathrm{d}\alpha\right]$$

$$= \int_a^b f_{n_0}(x)\,\mathrm{d}\int_0^1 g(\alpha)\Phi(x,\alpha)\mathrm{d}\alpha + 3\varepsilon$$

by (7), (8) and (9). Similarly, we can get

$$\lim_{i\to\infty} J_{2i} \ge \int_a^b f_{n_0}(x)\,\mathrm{d}\int_0^1 g(\alpha)\Phi(x,\alpha)\mathrm{d}\alpha - 3\varepsilon.$$

Hence

$$\limsup_{i\to\infty}\left|J_i - \int_a^b f_{n_0}(x)\,\mathrm{d}\int_0^1 g(\alpha)\Phi(x,\alpha)\mathrm{d}\alpha\right| \le 3\varepsilon. \quad (13)$$

We have

$$I = \int_a^b f(x)\,\mathrm{d}\int_0^1 g(\alpha)\Phi_i(x,\alpha)\mathrm{d}\alpha - \int_a^b f(x)\,\mathrm{d}\int_0^1 g(\alpha)\Phi(x,\alpha)\mathrm{d}\alpha$$

$$= \int_a^b [f(x)-f_{n_0}(x)]\,\mathrm{d}\int_0^1 g(\alpha)\Phi_i(x,\alpha)\mathrm{d}\alpha + \int_a^b f_{n_0}(x)\,\mathrm{d}\int_0^1 (g(\alpha)-g_{m_0}(\alpha)\Phi_i(x,\alpha)\mathrm{d}\alpha$$

$$+ \int_a^b f_{n_0}(x)\,\mathrm{d}\int_0^1 g_{m_0}(\alpha)\Phi_i(x,\alpha)\mathrm{d}\alpha - \int_a^b f_{n_0}(x)\,\mathrm{d}\int_0^1 g(\alpha)\Phi(x,\alpha)\mathrm{d}\alpha$$

$$+ \int_a^b [f_{n_0}(x)-f(x)]\,\mathrm{d}\int_0^1 g(\alpha)\Phi(x,\alpha)\mathrm{d}\alpha$$

It follows from (4), (5) and (6) that

$$|I| \le (2M+L)\varepsilon + \left|J_i - \int_a^b f_{n_0}(x)\,\mathrm{d}\int_0^1 g(\alpha)\Phi(x,\alpha)\mathrm{d}\alpha\right|. \quad (14)$$

Letting $i \to \infty$, we get

$$\limsup_{i\to\infty}|I| \le (2M+L+3)\varepsilon$$

by (13). This proves the first part of the theorem.

Conversely, suppose that (2) holds for any non-negative continuous functions $g(\alpha)$ and $f(x)$. Let (x_0,α_0) be a continuity point of $\Phi(x,\alpha)$, $a < x_0 < b, 0 < \alpha_0 \le 1$. Choose $\varepsilon > 0$ such that $x_0 + \varepsilon <$

$b, \alpha_0 - \varepsilon > 0$. Let $g(\alpha)$ and $f(x)$ be two nonnegative continuous functions defined by

$$g(\alpha) = \begin{cases} (n+1)\varepsilon^{-1}(\alpha - \alpha_0 + \varepsilon), & \alpha_0 - \varepsilon \le \alpha \le \alpha_0 - \frac{n}{n+1}\varepsilon, \\ 1, & \alpha_0 - \frac{n}{n+1}\varepsilon \le \alpha \le \alpha_0 - \frac{1}{n+1}\varepsilon, \\ -(n+1)\varepsilon^{-1}(\alpha - \alpha_0), & \alpha_0 - \frac{1}{n+1}\varepsilon \le \alpha \le \alpha_0, \\ 0, & 0 \le \alpha \le \alpha_0 - \varepsilon \text{ or } \alpha_0 \le \alpha \le 1, \end{cases}$$

where n is a positive integer, and

$$f(x) = \begin{cases} 1, & a \le x \le x_0, \\ \varepsilon^{-1}(-x + x_0 + \varepsilon), & x_0 \le x \le x_0 + \varepsilon, \\ 0, & x_0 + \varepsilon \le x \le b. \end{cases}$$

Then

$$\begin{aligned} &\int_0^1 g(\alpha)[\Phi_i(x_0, \alpha) - \Phi_i(a, \alpha)]\mathrm{d}\alpha \\ &\qquad = \int_a^{x_0} \mathrm{d} \int_0^1 g(\alpha)\Phi_i(x, \alpha)\mathrm{d}\alpha \\ &\qquad \le \int_a^b f(x)\, \mathrm{d} \int_0^1 g(\alpha)\Phi_i(x, \alpha)\mathrm{d}\alpha. \end{aligned} \quad (15)$$

Letting $i \to \infty$, from (2), we get

$$\begin{aligned} &\limsup_{i\to\infty} \int_0^1 g(\alpha)[\Phi_i(x_0, \alpha) - \Phi_i(a, \alpha)]\mathrm{d}\alpha \\ &\le \int_a^b f(x)\, \mathrm{d} \int_0^1 g(\alpha)\Phi(x, \alpha)\mathrm{d}\alpha \\ &\le \int_a^{x_0+\varepsilon} \mathrm{d} \int_0^1 g(\alpha)\Phi(x, \alpha)\mathrm{d}\alpha \\ &= \int_0^1 g(\alpha)\Phi(x_0 + \varepsilon, \alpha)\mathrm{d}\alpha - \int_0^1 g(\alpha)\Phi(a, \alpha)\mathrm{d}\alpha \\ &= \int_{\alpha_0-\varepsilon}^{\alpha_0} g(\alpha)[\Phi(x_0 + \varepsilon, \alpha) - \Phi(a, \alpha)]\mathrm{d}\alpha \\ &\le \int_{\alpha_0-\varepsilon}^{\alpha_0} [\Phi(x_0 + \varepsilon, \alpha) - \Phi(a, \alpha)]\mathrm{d}\alpha \\ &= [\Phi(x_0 + \varepsilon, \alpha') - \Phi(a, \alpha')]\varepsilon, \end{aligned}$$

where $\alpha_0 - \varepsilon < \alpha' < \alpha_0$. Since

$$\begin{aligned} &\int_0^1 g(\alpha)[\Phi_i(x_0, \alpha) - \Phi_i(a, \alpha)]\mathrm{d}\alpha \\ &\ge \int_{\alpha_0-\varepsilon n/(n+1)}^{\alpha_0-\varepsilon/(n+1)} [\Phi_i(x_0, \alpha) - \Phi_i(a, \alpha)]\mathrm{d}\alpha \\ &= [\Phi_i(x_0, \alpha'') - \Phi_i(a, \alpha'')]\frac{n-1}{n+1}\varepsilon, \end{aligned}$$

where $\alpha_0 - \varepsilon n/(n+1) < \alpha'' < \alpha_0 - \varepsilon/(n+1)$. Therefore,

$$\begin{aligned} &\limsup_{i\to\infty} [\Phi_i(x_0, \alpha'') - \Phi_i(a, \alpha'')]\frac{n-1}{n+1} \\ &\qquad \le \Phi(x_0 + \varepsilon, \alpha') - \Phi(a, \alpha'). \end{aligned}$$

Since $\alpha', \alpha'' \uparrow \alpha_0$ as $\varepsilon \to 0$, letting $\varepsilon \to 0$ yields that

$$\begin{aligned} &\limsup_{\varepsilon\to 0} \limsup_{i\to\infty} [\Phi_i(x_0, \alpha'') - \Phi_i(a, \alpha'')]\frac{n-1}{n+1} \\ &\qquad \le \Phi(x_0, \alpha_0) - \Phi(a, \alpha_0). \end{aligned}$$

Letting $n \to \infty$, we have

$$\begin{aligned} &\limsup_{\varepsilon\to 0} \limsup_{i\to\infty} [\Phi_i(x_0, \alpha'') - \Phi_i(a, \alpha'')] \\ &\qquad \le \Phi(x_0, \alpha_0) - \Phi(a, \alpha_0). \end{aligned} \quad (16)$$

Similarly, let $a + \varepsilon < x_0$, and

$$f(x) = \begin{cases} 1, & a \le x < x_0 - \varepsilon, \\ \varepsilon^{-1}(-x + x_0), & x_0 - \varepsilon \le x \le x_0, \\ 0, & x_0 \le x. \end{cases}$$

We can get

$$\begin{aligned} &\liminf_{\varepsilon\to 0} \liminf_{i\to\infty} [\Phi_i(x_0, \alpha'') - \Phi_i(a, \alpha'')] \\ &\qquad \ge [\Phi(x_0, \alpha_0) - \Phi(a, \alpha_0)]\frac{n-1}{n+1}. \end{aligned}$$

Letting $n \to \infty$ and combining with (16), we proved

$$\begin{aligned} &\lim_{\delta\downarrow 0} \lim_{i\to\infty} [\Phi_i(x_0, \alpha_0 - \delta) - \Phi_i(a, \alpha_0 - \delta)] \\ &\qquad = \Phi(x_0, \alpha_0) - \Phi(a, \alpha_0). \end{aligned}$$

If $x_0 = a$ or $x_0 = b$, it is easily seen that the above equation holds. Therefore the second part of the theorem is proved.

Theorem 6. *Let $\Phi, \Phi_1, \Phi_2, \ldots$ be chance distributions of random fuzzy variables $\xi, \xi_1, \xi_2, \ldots$, and ξ_i converge to ξ in distribution. Assume that a and b are two real numbers such that Φ is continuous at $(a, 1)$ and $(b, 1)$. If $g(\alpha)$ is a nonnegative continuous function on $[0, 1]$ and $f(x)$ is a nonnegative continuous function on $[a, b]$, then*

$$\begin{aligned} &\lim_{i\to\infty} \int_0^1 g(\alpha)\mathrm{d}\alpha \int_a^b f(x)\mathrm{d}_x\Phi_i(x, \alpha) \\ &\qquad = \int_0^1 g(\alpha)\mathrm{d}\alpha \int_a^b f(x)\mathrm{d}_x\Phi(x, \alpha), \end{aligned} \quad (17)$$

where $\int_a^b f(x)\mathrm{d}_x\Phi(x,\alpha)$ *denotes a Riemann-Stieltjes integral for given* α.

Conversely, if (17) *holds for any nonnegative continuous functions* $g(\alpha)$ *and* $f(x)$*, then*

$$\lim_{\delta\downarrow 0}\lim_{i\to\infty}[\Phi_i(x,\alpha-\delta)-\Phi_i(a,\alpha-\delta)] = \Phi(x,\alpha)-\Phi(a,\alpha) \quad (18)$$

for any continuity point (x,α) *of* Φ.

Proof. The conclusion follows from Theorem 5.

Theorem 7. *Let* $\Phi,\Phi_1,\Phi_2,\ldots$ *be chance distributions of random fuzzy variables* $\xi,\xi_1,\xi_2,\ldots$*, and* ξ_i *converge to* ξ *in distribution. Let* $g(\alpha)$ *be nonnegative continuous on* $(0,1]$ *and* $f(x)$ *be nonnegative continuous on* $\mathbb{R}$ *with* $\lim_{x\to\pm\infty} f(x)=0$*. then*

$$\lim_{i\to\infty}\int_0^1 g(\alpha)\mathrm{d}\alpha\int_{-\infty}^{+\infty} f(x)\mathrm{d}_x\Phi_i(x,\alpha) = \int_0^1 g(\alpha)\mathrm{d}\alpha\int_{-\infty}^{+\infty} f(x)\mathrm{d}_x\Phi(x,\alpha). \quad (19)$$

Conversely, if (19) *holds for any nonnegative continuous function* $f(x)$ *with* $\lim_{x\to\pm\infty} f(x)=0$*, then*

$$\lim_{\delta\downarrow 0}\lim_{i\to\infty}[\Phi_i(x,\alpha-\delta)-\Phi_i(y,\alpha-\delta)] = \Phi(x,\alpha)-\Phi(y,\alpha) \quad (20)$$

for any continuity point $(x,\alpha),(y,\alpha)$ *of* Φ.

Proof. For a given $\varepsilon>0$, choose $A>0$ such that $|f(x)|<\varepsilon$ for $|x|\ge A$ and $(-A,1),(A,1)$ are continuity points of Φ. Then

$$\left|\int_{|x|\ge A} f(x)\mathrm{d}_x\Phi_i(x,\alpha)\right| \le \varepsilon\int_{|x|\ge A}\mathrm{d}_x\Phi_i(x,\alpha)\le\varepsilon,$$

and

$$\left|\int_{|x|\ge A} f(x)\mathrm{d}_x\Phi(x,\alpha)\right| \le \varepsilon\int_{|x|\ge A}\mathrm{d}_x\Phi(x,\alpha)\le\varepsilon.$$

Denote $\sup_{\alpha\in[0,1]}|g(\alpha)|=M$. We have

$$\left|\int_0^1 g(\alpha)\mathrm{d}\alpha\int_{|x|\ge A} f(x)\mathrm{d}_x\Phi_i(x,\alpha)\right| \le M\varepsilon\int_0^1\mathrm{d}\alpha\int_{|x|\ge A}\mathrm{d}_x\Phi_i(x,\alpha)\le M\varepsilon, \quad (21)$$

and

$$\left|\int_0^1 g(\alpha)\mathrm{d}\alpha\int_{|x|\ge A} f(x)\mathrm{d}_x\Phi(x,\alpha)\right| \le M\varepsilon\int_0^1\mathrm{d}\alpha\int_{|x|\ge A}\mathrm{d}_x\Phi(x,\alpha)\le M\varepsilon. \quad (22)$$

Thus

$$\begin{aligned}
&\left|\int_0^1 g(\alpha)\mathrm{d}\alpha\int_{-\infty}^{+\infty} f(x)\mathrm{d}_x\Phi_i(x,\alpha) - \int_0^1 g(\alpha)\mathrm{d}\alpha\int_{-\infty}^{+\infty} f(x)\mathrm{d}_x\Phi(x,\alpha)\right| \\
&\le \left|\int_0^1 g(\alpha)\mathrm{d}\alpha\int_{-A}^{A} f(x)\mathrm{d}_x\Phi_i(x,\alpha) - \int_0^1 g(\alpha)\mathrm{d}\alpha\int_{-A}^{A} f(x)\mathrm{d}_x\Phi(x,\alpha)\right| \\
&\quad + \left|\int_0^1 g(\alpha)\mathrm{d}\alpha\int_{|x|\ge A} f(x)\mathrm{d}_x\Phi_i(x,\alpha) - \int_0^1 g(\alpha)\mathrm{d}\alpha\int_{|x|\ge A} f(x)\mathrm{d}_x\Phi(x,\alpha)\right| \\
&\le \left|\int_0^1 g(\alpha)\mathrm{d}\alpha\int_{-A}^{A} f(x)\mathrm{d}_x\Phi_i(x,\alpha) - \int_0^1 g(\alpha)\mathrm{d}\alpha\int_{-A}^{A} f(x)\mathrm{d}_x\Phi(x,\alpha)\right| + 2M\varepsilon
\end{aligned}$$

by (21) and (22). It follows from Theorem 6 that (19) holds.

Conversely, suppose that (19) holds for any nonnegative continuous function $f(x)$ with $\lim_{x\to\pm\infty} f(x)=0$. By the similar procedure to the proof of second part of Theorem 5, we can prove the second part of the theorem.

Remark 8. If $\lim_{x\to\pm\infty} f(x)\ne 0$, the equation (19) may not hold. For example, let

$$\Theta=\{\theta_1,\theta_2,\ldots\},\quad \mathrm{Pos}\{\theta_i\}=1,\ i=1,2,\ldots,$$

$$\Omega=\{\omega_1,\omega_2,\ldots\},\quad \Pr\{\omega_i\}=1/2^i,\ i=1,2,\ldots.$$

Then $(\Theta,\mathcal{P}(\Theta),\mathrm{Pos})$ is a possibility space and $(\Omega,\mathcal{P}(\Omega),\Pr)$ is a probability space. Assume ξ_i are random fuzzy variables defined by

$$\xi_i(\theta_j)=\begin{cases}\eta_i, & \text{if } j\le i\\ 0, & \text{if } j>i,\end{cases}$$

for $i,j=1,2,\ldots$, where η_i are random variables defined by

$$\eta_i(\omega_1)=0,\ \eta_i(\omega_j)=i,$$

for $i = 1, 2, \ldots,\ j = 2, 3, \ldots$, and ξ a random fuzzy variable as $\xi(\theta_i) = \zeta_i$, where ζ_i are random variables as

$$\zeta_1(\omega_j) = 0, \quad \zeta_i(\omega_j) = \begin{cases} 0, & \text{if } j = 1 \\ i, & \text{if } j > 1, \end{cases}$$

for $i = 2, 3, \ldots;\ j = 1, 2, \ldots$. Let $\Phi, \Phi_1, \Phi_2, \ldots$ be chance distributions of random fuzzy variables $\xi, \xi_1, \xi_2, \ldots$. Then when $\alpha > 0.5$, $\Phi_i(x, \alpha) = 0$ if $x < 0$, $\frac{1}{2}$ if $0 \le x < i$, 1 if $x \ge i$, and $\Phi(x, \alpha) = 0$ if $x < 0$, $\frac{1}{2}$ if $x \ge 0$. When $\alpha \le 0.5$, $\Phi_i(x, \alpha) = 0$ if $x < 0$, 1 if $x \ge 0$, and $\Phi(x, \alpha) = 0$ if $x < 0$, 1 if $x \ge 0$. Thus $\Phi_i(x, \alpha) \to \Phi(x, \alpha)$. However if $g(\alpha) = 1,\ f(x) = 1$, we have

$$\int_0^1 g(\alpha)\mathrm{d}\alpha \int_{-\infty}^{+\infty} f(x)\mathrm{d}_x\Phi_i(x, \alpha) = 1$$
$$\to 1 \ne \frac{3}{4} = \int_0^1 g(\alpha)\mathrm{d}\alpha \int_{-\infty}^{+\infty} f(x)\mathrm{d}_x\Phi(x, \alpha).$$

The following theorem can easily be proved by Theorem 5 and 7

Theorem 9. *Let $\Phi, \Phi_1, \Phi_2, \ldots$ be chance distributions of random fuzzy variables $\xi, \xi_1, \xi_2, \ldots$. Assume that there exists a number $y_0 \in \mathbb{R}$ such that $\Phi_i(y_0, \alpha) = 0$ and $\Phi(y_0, \alpha) = 0$ for each $\alpha \in (0, 1]$, then the following three statements are equivalent:*

(i) ξ_i converges to ξ in distribution;

(ii) for any nonnegative continuous function $g(\alpha)$ on $[0, 1]$ and nonnegative continuous function $f(x)$ on $\mathbb{R}$ with $\lim_{x\to\pm\infty} f(x) = 0$, we have

$$\lim_{i\to\infty} \int_{-\infty}^{+\infty} f(x)\ \mathrm{d} \int_0^1 g(\alpha)\Phi_i(x, \alpha)\mathrm{d}\alpha$$
$$= \int_{-\infty}^{+\infty} f(x)\ \mathrm{d} \int_0^1 g(\alpha)\Phi(x, \alpha)\mathrm{d}\alpha.$$

(iii) for any nonnegative continuous function $g(\alpha)$ on $[0, 1]$ and nonnegative continuous function $f(x)$ on $\mathbb{R}$ with $\lim_{x\to\pm\infty} f(x) = 0$, we have

$$\lim_{i\to\infty} \int_0^1 g(\alpha)\mathrm{d}\alpha \int_{-\infty}^{+\infty} f(x)\mathrm{d}_x\Phi_i(x, \alpha)$$
$$= \int_0^1 g(\alpha)\mathrm{d}\alpha \int_{-\infty}^{+\infty} f(x)\mathrm{d}_x\Phi(x, \alpha).$$

4 Conclusion

In the paper, based on the concept of chance distributions for random fuzzy variables, some sufficient and necessary conditions for convergence of random fuzzy sequences in distribution are given. A sequence of random fuzzy variables converges to a random fuzzy variable in distribution if and only if their corresponding integrals are convergent.

References:

[1] L.A. Zadeh, Fuzzy sets, *Information and Control*, Vol. 8, 338-353, 1965.

[2] A. Kaufmann, *Introduction to the Theory of Fuzzy Subsets*, New York: Academic Press, 1975.

[3] L.A. Zadeh, Fuzzy sets as a basis for a theory of possibility, *Fuzzy Sets and Systems*, Vol. 1, 3-28, 1978.

[4] S. Nahmias, Fuzzy variables, *Fuzzy Sets and Systems*, Vol. 1, 97-110, 1978.

[5] B. Liu and Y.-K. Liu, Expected value of fuzzy variable and fuzzy expected value models, *IEEE Transactions on Fuzzy Systems*, Vol. 10, No. 4, 445-450, 2002.

[6] B. Liu, *Uncertainty Theory: An Introduction to its Axiomatic Foundations*, Berlin: Springer-Verlag, 2004.

[7] B. Liu, Toward fuzzy optimization without mathematical ambiguity, *Fuzzy Optimization and Decision Making*, Vol. 1, No. 1, 43-63, 2002.

[8] B. Liu, *Theory and Practice of Uncertain Programming*, Heidelberg: Physica-Verlag, 2002.

[9] Y.-K. Liu and B. Liu, Expected value operator of random fuzzy variable and random fuzzy expected value models, *International Journal of Uncertainty, Fuzziness & Knowledge-Based Systems*, Vol.11, No.2, 195-215, 2003.

[10] Y. Zhu and B. Liu, Continuity theorems and chance distribution of random fuzzy variables, *Proceedings of the Royal Society of Lodon Series A*, Vol. 460, 2505-2519, 2004.

[11] Y. Zhu and B. Liu, Some inequalities of random fuzzy variables with application to moment convergence, *Computers & Mathematics with Applications*, Vol. 50, No. 5-6, 719-727, 2005.